Journal of
Mathematical
Inequalities
Volume 14, Number 4 (2020), 1135–1149
doi:10.7153/jmi-2020-14-74

ON A PARAMETRIC MORE ACCURATE HILBERT–TYPE INEQUALITY

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(*Communicated by Y. Sawano*)

Abstract. By the use of the weight functions, the idea of introducing parameters and Hermite-Hadamard's inequality, a more accurate discrete Hilbert-type inequality with the general homogeneous kernel and the equivalent form are given. The equivalent statements of the best possible constant factor related to some parameters, the operator expressions and some particular cases are considered.

1. Introduction

Suppose that $0 < \sum_{m=1}^{\infty} a_m^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$. We have the following discrete Hilbert's inequality with the best possible constant factor π (cf [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{2}}. \tag{1}$$

We still have the following more accurate Hilbert's inequality with the same best possible constant factor π (cf. [1], Theorem 323):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n-1} < \pi\left(\sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{2}}, \tag{2}$$

Assuming that $0 < \int_0^{\infty} f^2(x)dx < \infty$ and $0 < \int_0^{\infty} g^2(y)dy < \infty$, we have the following Hilbert's integral inequality:

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dxdy < \pi\left(\int_0^{\infty} f^2(x)dx \int_0^{\infty} g^2(y)dy\right)^{\frac{1}{2}}, \tag{3}$$

with the best possible constant factor π (cf. [1], Theorem 316).

Inequalities (1), (2), (3) and their extensions with the conjugate exponents (p,q) $(p>1, \frac{1}{p}+\frac{1}{q}=1)$ are important in analysis and its applications (cf. [2]-[11]).

Mathematics subject classification (2010): 26D15.
Keywords and phrases: Weight function, Hilbert-type inequality, equivalent statement, Hermite-Hadamard's inequality, operator expression.
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Paper JMI-14-74

The following half-discrete Hilbert-type inequality was provided (cf. [1], Theorem 351): If $K(x)(x>0)$ is decreasing, $p>1, \frac{1}{p}+\frac{1}{q}=1, 0<\phi(s)=\int_0^\infty K(x)x^{s-1}dx<\infty$, then

$$\int_0^\infty x^{p-2}(\sum_{n=1}^\infty K(nx)a_n)^p dx < \phi^p(\frac{1}{p})\sum_{n=1}^\infty a_n^p. \tag{4}$$

Some new extensions of (4) with their applications were provided by [12]-[17].

In 2016, by the use of the technique of real analysis, Hong [18] considered some equivalent statements of the extensions of (1) with the best possible constant factor related to a few parameters. The other similar works on the extensions of (3) and (4) were given by [19]-[23].

In this paper, following the way of [18], by the use of the weight functions, the idea of introducing parameters and Hermite-Hadamard's inequality, a more accrate discrete Hilbert-type inequality with the general homogeneous kernel and the equivalent form are given, which are extension of (2). The equivalent statements of the best possible constant factor related to a few parameters, the operator expressions and some particular examples are also considered.

2. Some lemmas

In what follows, we suppose that $p>1, \frac{1}{p}+\frac{1}{q}=1, \alpha,\beta\in(0,1], \xi,\eta\in[0,\frac{1}{2}]. \lambda\in\mathbf{R}$, $\lambda_1, \lambda-\lambda_2 \leqslant \frac{1}{\alpha}$, λ_2 $, \lambda-\lambda_1 \leqslant \frac{1}{\beta}, k_\lambda(x,y)$ is a positive homogeneous function of degree $-\lambda$, satisfying for any $u,x,y>0$,

$$k_\lambda(ux,uy) = u^{-\lambda}k_\lambda(x,y).$$

Also, $k_\lambda(x,y)$ is strictly convex and strictly decreasing with respect to $x,y>0$, such that

$$(-1)^i\frac{\partial^i}{\partial x^i}k_\lambda(x,y), (-1)^i\frac{\partial^i}{\partial y^i}k_\lambda(x,y)>0 (i=1,2),$$

and for any $\gamma=\lambda_1, \lambda-\lambda_2$,

$$k_\lambda(\gamma) \triangleq \int_0^\infty k_\lambda(u,1)u^{\gamma-1}du \in \mathbf{R}_+ = (0,\infty). \tag{5}$$

We still assume that $a_m, b_n \geqslant 0$ $(m,n\in\mathbf{N}=\{1,2,\cdots\})$, such that

$$0<\sum_{m=1}^\infty (m-\xi)^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p<\infty, \text{ and}$$
$$0<\sum_{n=1}^\infty (n-\eta)^{q[1-\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}b_n^q<\infty.$$

DEFINITION 1. We set function

$$K(x,y) \triangleq k_\lambda((x-\xi)^\alpha,(y-\eta)^\beta)(x,y>\frac{1}{2}),$$

and define the following weight functions:

$$\omega_\lambda(\lambda_2,m) \triangleq (m-\xi)^{\alpha(\lambda-\lambda_2)}\sum_{n=1}^{\infty}K(m,n)(n-\eta)^{\beta\lambda_2-1}\ (m\in\mathbf{N}), \tag{6}$$

$$\varpi_\lambda(\lambda_1,n) \triangleq (n-\eta)^{\beta(\lambda-\lambda_1)}\sum_{m=1}^{\infty}K(m,n)(m-\xi)^{\alpha\lambda_1-1}\ (n\in\mathbf{N}). \tag{7}$$

LEMMA 2. *We have the following inequalities:*

$$\omega_\lambda(\lambda_2,m) < \frac{1}{\beta}k_\lambda(\lambda-\lambda_2)\ (m\in\mathbf{N}), \tag{8}$$

$$\varpi_\lambda(\lambda_1,n) < \frac{1}{\alpha}k_\lambda(\lambda_1)\ (n\in\mathbf{N}). \tag{9}$$

Proof. For $\beta\lambda_2-1\leqslant 0$, it is evident that

$$(-1)^i\frac{d^i}{dy^i}(y-\eta)^{\beta\lambda_2-1}\geqslant 0(i=1,2)$$

and

$$K(m,y)(y-\eta)^{\beta\lambda_2-1}=k_\lambda((m-\xi)^\alpha,(y-\eta)^\beta)(y-\eta)^{\beta\lambda_2-1}$$

is a strictly convex and strictly decreasing function with respect to $y>\frac{1}{2}$. By Hermite-Hadamard's inequality (cf. [24]), setting $u=\frac{(m-\xi)^\alpha}{(y-\eta)^\beta}$, we find that

$$\begin{aligned}\omega_\lambda(\lambda_2,m) &< (m-\xi)^{\alpha(\lambda-\lambda_2)}\int_{\frac{1}{2}}^{\infty}K(m,y)y^{\beta\lambda_2-1}dy\\ &= \frac{1}{\beta}\int_0^{\frac{(m-\xi)^\alpha}{(\frac{1}{2}-\eta)^\beta}}k_\lambda(u,1)u^{(\lambda-\lambda_2)-1}du\\ &\leqslant \frac{1}{\beta}\int_0^{\infty}k_\lambda(u,1)u^{(\lambda-\lambda_2)-1}du=\frac{1}{\beta}k_\lambda(\lambda-\lambda_2).\end{aligned}$$

Hence, we have (8).

For $\alpha\lambda_1-1\leqslant 0$, it is evident that $K(x,n)(x-\xi)^{\alpha\lambda_1-1}$ is a strictly convex and strictly decreasing function with respect to $x>\frac{1}{2}$. By Hermite- Hadamard's inequality, setting $u=\frac{(x-\xi)^\alpha}{(n-\eta)^\beta}$, we find that

$$\begin{aligned}\varpi_\lambda(\lambda_1,n) &< (n-\eta)^{\beta(\lambda-\lambda_1)}\int_{\frac{1}{2}}^{\infty}k_\lambda((x-\xi)^\alpha,(n-\eta)^\beta)(x-\xi)^{\alpha\lambda_1-1}dx\\ &= \frac{1}{\alpha}\int_{\frac{(\frac{1}{2}-\xi)^\alpha}{(n-\eta)^\beta}}^{\infty}k_\lambda(u,1)u^{\lambda_1-1}du\leqslant\frac{1}{\alpha}k_\lambda(\lambda_1).\end{aligned}$$

Hence, we have (9). □

LEMMA 3. *We have the following inequality:*

$$I \triangleq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} K(m,n)a_m b_n$$
$$< \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \left\{\sum_{m=1}^{\infty}(m-\xi)^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p\right\}^{\frac{1}{p}}$$
$$\times \left\{\sum_{n=1}^{\infty}(n-\eta)^{q[1-\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1} b_n^q\right\}^{\frac{1}{q}}. \qquad (10)$$

Proof. By Hölder's inequality with weight (cf. [24]), we obtain

$$I = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} K(m,n)\left[\frac{(n-\eta)^{(\beta\lambda_2-1)/p}}{(m-\xi)^{(\alpha\lambda_1-1)/q}} a_m\right]\left[\frac{(m-\xi)^{(\alpha\lambda_1-1)/q}}{(n-\eta)^{(\beta\lambda_2-1)/p}} b_n\right]$$
$$\leqslant \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} K(m,n)\frac{(n-\eta)^{\beta\lambda_2-1}}{(m-\xi)^{(\alpha\lambda_1-1)(p-1)}} a_m^p\right\}^{\frac{1}{p}}$$
$$\times \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} K(m,n)\frac{(m-\xi)^{\alpha\lambda_1-1}}{(n-\eta)^{(\beta\lambda_2-1)(q-1)}} b_n^q\right\}^{\frac{1}{q}}$$
$$= \left\{\sum_{m=1}^{\infty}\omega_\lambda(\lambda_2,m)(m-\xi)^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p\right\}^{\frac{1}{p}}$$
$$\times \left\{\sum_{n=1}^{\infty}\varpi_\lambda(\lambda_1,n)(n-\eta)^{q[1-\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1} b_n^q\right\}^{\frac{1}{q}}.$$

Then by (8) and (9), we have (10). □

REMARK 4. By (10), for $\lambda_1+\lambda_2=\lambda$, we find

$$0 < \sum_{m=1}^{\infty}(m-\xi)^{p(1-\alpha\lambda_1)-1} a_m^p < \infty \text{ and}$$
$$0 < \sum_{n=1}^{\infty}(n-\eta)^{q(1-\beta\lambda_2)-1} b_n^q < \infty,$$

and the following inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_\lambda((m-\xi)^\alpha,(n-\eta)^\beta)a_m b_n$$
$$< \frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}\left[\sum_{m=1}^{\infty}(m-\xi)^{p(1-\alpha\lambda_1)-1} a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\eta)^{q(1-\beta\lambda_2)-1} b_n^q\right]^{\frac{1}{q}}. \qquad (11)$$

In particular, for $\xi=\eta=0$, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k_{\lambda}(m^{\alpha},n^{\beta})a_m b_n < \frac{k_{\lambda}(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}\left[\sum_{m=1}^{\infty}m^{p(1-\alpha\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\beta\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}. \tag{12}$$

For $p=q=2,\alpha=\beta=\lambda=1,\xi=\eta=\lambda_1=\lambda_2=\frac{1}{2}$, $k_1(x-\frac{1}{2},y-\frac{1}{2})=\frac{1}{x+y-1}$, (11) reduces to (2). Hence, inequality (11) is a more accurate extension of (12), and an extension of (2).

LEMMA 5. *The constant factor* $\frac{k_{\lambda}(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}$ *in (11) is the best possible.*

Proof. For any $\varepsilon>0$, we set

$$\widetilde{a}_m \triangleq (m-\xi)^{\alpha(\lambda_1-\frac{\varepsilon}{p})-1}, \widetilde{b}_n \triangleq (n-\eta)^{\beta(\lambda_2-\frac{\varepsilon}{q})-1}\ (m,n\in\mathbf{N}).$$

If there exists a constant $M\leqslant\frac{k_{\lambda}(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}$, such that (11) is valid when replacing $\frac{k_{\lambda}(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}$ by M, then in particular, we have

$$\widetilde{I} \triangleq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k_{\lambda}((m-\xi)^{\alpha},(n-\eta)^{\beta})\widetilde{a}_m\widetilde{b}_n < M\left[\sum_{m=1}^{\infty}(m-\xi)^{p(1-\alpha\lambda_1)-1}\widetilde{a}_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\eta)^{q(1-\beta\lambda_2)-1}\widetilde{b}_n^q\right]^{\frac{1}{q}}.$$

We obtain

$$\begin{aligned}
\widetilde{I} &< M\left[\sum_{m=1}^{\infty}(m-\xi)^{p(1-\alpha\lambda_1)-1}(m-\xi)^{p\alpha(\lambda_1-\frac{\varepsilon}{p})-p}\right]^{\frac{1}{p}}\\
&\quad\times\left[\sum_{n=1}^{\infty}(n-\eta)^{q(1-\beta\lambda_2)-1}(n-\eta)^{q\beta(\lambda_2-\frac{\varepsilon}{q})-q}\right]^{\frac{1}{q}}\\
&= M\left[(1-\xi)^{-\alpha\varepsilon-1}+\sum_{m=2}^{\infty}(m-\xi)^{-\alpha\varepsilon-1}\right]^{\frac{1}{p}}\times\left[(1-\eta)^{-\beta\varepsilon-1}+\sum_{n=2}^{\infty}(n-\eta)^{-\beta\varepsilon-1}\right]^{\frac{1}{q}}\\
&< M\left[(1-\xi)^{-\alpha\varepsilon-1}+\int_1^{\infty}(t-\xi)^{-\alpha\varepsilon-1}dt\right]^{\frac{1}{p}}\times\left[(1-\eta)^{-\beta\varepsilon-1}+\int_1^{\infty}(t-\eta)^{-\beta\varepsilon-1}dt\right]^{\frac{1}{q}}\\
&= \frac{M}{\varepsilon}\left[\varepsilon(1-\xi)^{-\alpha\varepsilon-1}+\frac{(1-\xi)^{-\alpha\varepsilon}}{\alpha}\right]^{\frac{1}{p}}\left[\varepsilon(1-\eta)^{-\beta\varepsilon-1}+\frac{(1-\eta)^{-\beta\varepsilon}}{\beta}\right]^{\frac{1}{q}}.
\end{aligned}$$

By the decreasingness property and Fubini theorem (cf. [25]), for $u = \frac{(x-\xi)^{\alpha}}{(y-\eta)^{\beta}}$, we find

$$\begin{aligned}
\widetilde{I} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} K(m,n)\frac{(m-\xi)^{\alpha\lambda_1-1}}{(m-\xi)^{\frac{\alpha\varepsilon}{p}}}\frac{(n-\eta)^{\beta\lambda_2-1}}{(n-\eta)^{\frac{\beta\varepsilon}{q}}} \\
&\geqslant \int_1^{\infty}\left[\int_1^{\infty} k_{\lambda}((x-\xi)^{\alpha}x,(y-\eta)^{\beta})\frac{(x-\xi)^{\alpha\lambda_1-1}}{(x-\xi)^{\frac{\alpha\varepsilon}{p}}}\frac{(y-\eta)^{\beta\lambda_2-1}}{(y-\eta)^{\frac{\beta\varepsilon}{q}}}dx\right]dy \\
&= \frac{1}{\alpha}\int_1^{\infty}(y-\eta)^{-\beta\varepsilon-1}\left[\int_{\frac{(1-\xi)^{\alpha}}{(y-\eta)^{\beta}}}^{\infty} k_{\lambda}(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\right]dy \\
&= \frac{1}{\alpha}\int_1^{\infty}(y-\eta)^{-\beta\varepsilon-1}\left[\int_{\frac{(1-\xi)^{\alpha}}{(y-\eta)^{\beta}}}^{\frac{(1-\xi)^{\alpha}}{(1-\eta)^{\beta}}} k_{\lambda}(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\right]dy \\
&\quad+\frac{1}{\alpha}\int_1^{\infty}(y-\eta)^{-\beta\varepsilon-1}\left[\int_{\frac{(1-\xi)^{\alpha}}{(1-\eta)^{\beta}}}^{\infty} k_{\lambda}(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\right]dy \\
&= \frac{1}{\alpha}\int_0^{\frac{(1-\xi)^{\alpha}}{(1-\eta)^{\beta}}}\left[\int_{\eta+(1-\xi)u^{-1/\beta}}^{\infty}(y-\eta)^{-\beta\varepsilon-1}dy\right]k_{\lambda}(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du \\
&\quad+\frac{(1-\eta)^{-\beta\varepsilon}}{\alpha\beta\varepsilon}\int_{\frac{(1-\xi)^{\alpha}}{(1-\eta)^{\beta}}}^{\infty} k_{\lambda}(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du \\
&= \frac{1}{\alpha\beta\varepsilon}\left[(1-\xi)^{-\beta\varepsilon}\int_0^{\frac{(1-\xi)^{\alpha}}{(1-\eta)^{\beta}}} k_{\lambda}(u,1)u^{\lambda_1+\frac{\varepsilon}{q}-1}du\right. \\
&\quad\left.+(1-\eta)^{-\beta\varepsilon}\int_{\frac{(1-\xi)^{\alpha}}{(1-\eta)^{\beta}}}^{\infty} k_{\lambda}(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\right].
\end{aligned}$$

Then we have

$$\begin{aligned}
&\frac{1}{\alpha\beta}\left[(1-\xi)^{-\beta\varepsilon}\int_0^{\frac{(1-\xi)^{\alpha}}{(1-\eta)^{\beta}}} k_{\lambda}(u,1)u^{\lambda_1+\frac{\varepsilon}{q}-1}du\right. \\
&\left.+(1-\eta)^{-\beta\varepsilon}\int_{\frac{(1-\xi)^{\alpha}}{(1-\eta)^{\beta}}}^{\infty} k_{\lambda}(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\right] \\
&< M\left[\varepsilon(1-\xi)^{-\alpha\varepsilon-1}+\frac{(1-\xi)^{-\alpha\varepsilon}}{\alpha}\right]^{\frac{1}{p}}\left[\varepsilon(1-\eta)^{-\beta\varepsilon-1}+\frac{(1-\eta)^{-\beta\varepsilon}}{\beta}\right]^{\frac{1}{q}}.
\end{aligned}$$

For $\varepsilon \to 0^{+}$, by Fatou lemma (cf. [25]), we find

$$\frac{1}{\alpha\beta}k_{\lambda}(\lambda_1) = \frac{1}{\alpha\beta}\left[\underline{\lim}_{\varepsilon\to 0^{+}}(1-\xi)^{-\beta\varepsilon}\int_0^{\frac{(1-\xi)^{\alpha}}{(1-\eta)^{\beta}}}\underline{\lim}_{\varepsilon\to 0^{+}} k_{\lambda}(u,1)u^{\lambda_1+\frac{\varepsilon}{q}-1}du\right.$$

$$+\lim_{\varepsilon\to0^+}(1-\eta)^{-\beta\varepsilon}\int_{\frac{(1-\xi)^\alpha}{(1-\eta)^\beta}}^{\infty}\lim_{\varepsilon\to0^+}k_\lambda(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\Bigg]$$

$$\leqslant\frac{1}{\alpha\beta}\lim_{\varepsilon\to0^+}\Bigg[(1-\xi)^{-\beta\varepsilon}\int_0^{\frac{(1-\xi)^\alpha}{(1-\eta)^\beta}}k_\lambda(u,1)u^{\lambda_1+\frac{\varepsilon}{q}-1}du$$

$$+(1-\eta)^{-\beta\varepsilon}\int_{\frac{(1-\xi)^\alpha}{(1-\eta)^\beta}}^{\infty}k_\lambda(u,1)u^{\lambda_1-\frac{\varepsilon}{p}-1}du\Bigg]$$

$$\leqslant M\left(\frac{1}{\alpha}\right)^{\frac{1}{p}}\left(\frac{1}{\beta}\right)^{\frac{1}{q}},$$

namely, $\frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}\leqslant M$. Hence, $M=\frac{k_\lambda(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}$ is the best possible constant factor of (11). □

REMARK 6. Setting $\widehat{\lambda}_1\triangleq\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\widehat{\lambda}_2\triangleq\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$, we find

$$\widehat{\lambda}_1+\widehat{\lambda}_2=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}+\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}=\frac{\lambda}{p}+\frac{\lambda}{q}=\lambda,$$

$$\widehat{\lambda}_1\leqslant\frac{1}{p\alpha}+\frac{1}{q\alpha}=\frac{1}{\alpha},\widehat{\lambda}_2\leqslant\frac{1}{q\beta}+\frac{1}{p\beta}=\frac{1}{\beta},$$

and by Hölder's inequality (cf. [24]), we obtain

$$\begin{aligned}0&<k_\lambda(\lambda-\widehat{\lambda}_2)=k_\lambda(\widehat{\lambda}_1)=k_\lambda(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})=\int_0^\infty k_\lambda(u,1)u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}du\\&=\int_0^\infty k_\lambda(u,1)(u^{\frac{\lambda-\lambda_2-1}{p}})(u^{\frac{\lambda_1-1}{q}})du\\&\leqslant\left(\int_0^\infty k_\lambda(u,1)u^{\lambda-\lambda_2-1}du\right)^{\frac{1}{p}}\left(\int_0^\infty k_\lambda(u,1)u^{\lambda_1-1}du\right)^{\frac{1}{q}}\\&=k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)<\infty.\end{aligned}\tag{13}$$

We can rewrite (10) as follows:

$$\begin{aligned}I<\frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)\left[\sum_{m=1}^\infty(m-\xi)^{p(1-\alpha\widehat{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}}\\\times\left[\sum_{n=1}^\infty(n-\eta)^{q(1-\beta\widehat{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}.\end{aligned}\tag{14}$$

LEMMA 7. *If the constant factor* $\frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ *in (10) is the best possible, then* $\lambda_1+\lambda_2=\lambda$.

Proof. If the constant factor $\frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ in (10) is the best possible, then by (14) and (11), the unique best possible constant factor must be $\frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda(\widehat{\lambda}_1)$ $(\in \mathbf{R}_+)$, namely,

$$k_\lambda(\widehat{\lambda}_1) = k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1).$$

We find that (13) keeps the form of equality if and only if there exist constants A and B, such that they are not all zero and (cf. [24])

$$Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1} \text{ a.e. in } \mathbf{R}_+.$$

Assuming that $A \neq 0$, it follows that $u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A}$ a.e. in $\mathbf{R}_+$, and $\lambda-\lambda_2-\lambda_1 = 0$, namely, $\lambda_1+\lambda_2 = \lambda$. □

3. Main results

THEOREM 8. *Inequality (10) is equivalent to*

$$J \triangleq \left[\sum_{n=1}^{\infty}(n-\eta)^{p\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})-1}\left(\sum_{m=1}^{\infty}K(m,n)a_m\right)^p\right]^{\frac{1}{p}}$$
$$< \frac{k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}\left\{\sum_{m=1}^{\infty}(m-\xi)^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\right\}^{\frac{1}{p}}. \tag{15}$$

If the constant factor in (10) is the best possible, then so is the constant factor in (15).

Proof. Suppose that (15) is valid. By Hölder's inequality (cf. [24]), we find

$$I = \sum_{n=1}^{\infty}\left[(n-\eta)^{\frac{-1}{p}+\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})}\sum_{m=1}^{\infty}K(m,n)a_m\right]\times\left[(n-\eta)^{\frac{1}{p}-\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})}b_n\right]$$
$$\leqslant J\left\{\sum_{n=1}^{\infty}(n-\eta)^{q[1-\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}b_n^q\right\}^{\frac{1}{q}}. \tag{16}$$

Then by (15), we obtain (10).

On the other hand, assuming that (10) is valid, we set

$$b_n := (n-\eta)^{p\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})-1}\left(\sum_{m=1}^{\infty}K(m,n)a_m\right)^{p-1}, n \in \mathbf{N}.$$

If $J = 0$, then (15) is naturally valid; if $J = \infty$, then it is impossible that makes (15) valid, namely, $J < \infty$. Suppose that $0 < J < \infty$. By (10), it follows that

$$\sum_{n=1}^{\infty}(n-\eta)^{q[1-\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1}b_n^q = J^p = I$$

$$< \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \left\{ \sum_{m=1}^{\infty} (m-\xi)^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p \right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_{n=1}^{\infty} (n-\eta)^{q[1-\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1} b_n^q \right\}^{\frac{1}{q}},$$

$$J = \left\{ \sum_{n=1}^{\infty} (n-\eta)^{q[1-\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1} b_n^q \right\}^{\frac{1}{p}}$$

$$< \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) \left\{ \sum_{m=1}^{\infty} (m-\xi)^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1} a_m^p \right\}^{\frac{1}{p}},$$

namely, (15) follows, which is equivalent to (10).

If the constant factor in (10) is the best possible, then so is constant factor in (15). Otherwise, by (16), we would reach a contradiction that the constant factor in (10) is not the best possible. □

THEOREM 9. *The following statements (i), (ii), (iii) and (iv) are equivalent:*

(i) $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ *is independent of* p,q*;*

(ii) $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ *is expressible as a single integral;*

(iii) $\frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ *is the best possible constant factor of (10);*

(iv) $\lambda_1+\lambda_2=\lambda$.

If the statement (iv) follows, namely, $\lambda_1+\lambda_2=\lambda$*, then we have (11) and the following equivalent inequality with the best possible constant factor* $\frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda(\lambda_1)$*:*

$$\left\{ \sum_{n=1}^{\infty} (n-\eta)^{p\beta\lambda_2-1} \left[\sum_{m=1}^{\infty} k_\lambda((m-\xi)^\alpha,(n-\eta)^\beta) a_m \right]^p \right\}^{\frac{1}{p}}$$

$$< \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda(\lambda_1) \left[\sum_{m=1}^{\infty} (m-\xi)^{p(1-\alpha\lambda_1)-1} a_m^p \right]^{\frac{1}{p}}. \tag{17}$$

Proof. $(i) \Rightarrow (ii)$. Since $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$ is independent of p,q, we find

$$k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) = \lim_{p\to\infty} \lim_{q\to 1^+} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1) = k_\lambda(\lambda_1),$$

namely, $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ is expressible as a single integral

$$k_\lambda(\lambda_1)=\int_0^\infty k_\lambda(u,1)u^{\lambda_1-1}du.$$

$(ii)\Rightarrow(iv)$. In (13), if $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ is expressible as a single integral $k_\lambda(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$, then (13) keeps the form of equality, which follows that $\lambda_1+\lambda_2=\lambda$.

$(iv)\Rightarrow(i)$. If $\lambda_1+\lambda_2=\lambda$, then $k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)=k_\lambda(\lambda_1)$, which is independent of p,q. Hence, we have $(i)\Leftrightarrow(ii)\Leftrightarrow(iv)$.

$(iii)\Rightarrow(iv)$. By Lemma 4, we have $\lambda_1+\lambda_2=\lambda$.

$(iv)\Rightarrow(iii)$. By Lemma 3, for $\lambda_1+\lambda_2=\lambda$,

$$\frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)(=\frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda(\lambda_1))$$

is the best possible constant factor of (10). Therefore, we have $(iii)\Leftrightarrow(iv)$.

Hence, the statements (i), (ii), (iii) and (iv) are equivalent. □

REMARK 10. (i) For $\alpha=\beta=\lambda=1,\lambda_1=\frac{1}{q},\lambda_2=\frac{1}{p}$ in (11) and (17), we have the following equivalent inequalities with the best possible constant factor $k_1(\frac{1}{q})$:

$$\sum_{n=1}^\infty\sum_{m=1}^\infty k_1(m-\xi,n-\eta)a_mb_n<k_1(\frac{1}{q})\left(\sum_{m=1}^\infty a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty b_n^q\right)^{\frac{1}{q}},\tag{18}$$

$$\left[\sum_{n=1}^\infty\left(\sum_{m=1}^\infty k_1(m-\xi,n-\eta)a_m\right)^p\right]^{\frac{1}{p}}<k_\lambda(\frac{1}{q})\left(\sum_{m=1}^\infty a_m^p\right)^{\frac{1}{p}}.\tag{19}$$

(ii) For $\alpha=\beta=\lambda=1,\lambda_1=\frac{1}{p},\lambda_2=\frac{1}{q}$ in (11) and (17), we have the following equivalent inequalities with the best possible constant factor $k_1(\frac{1}{p})$:

$$\sum_{n=1}^\infty\sum_{m=1}^\infty k_1(m-\xi,n-\eta)a_mb_n$$
$$<k_1(\frac{1}{p})\left[\sum_{m=1}^\infty(m-\xi)^{p-2}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^\infty(n-\eta)^{q-2}b_n^q\right]^{\frac{1}{q}},\tag{20}$$

$$\left[\sum_{n=1}^\infty(n-\eta)^{p-2}\left(\sum_{m=1}^\infty k_1(m-\xi,n-\eta)a_m\right)^p\right]^{\frac{1}{p}}$$
$$<k_1(\frac{1}{p})\left[\sum_{m=1}^\infty(m-\xi)^{p-2}a_m^p\right]^{\frac{1}{p}}.\tag{21}$$

(iii) For $p = q = 2$, both (18) and (20) reduce to

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_1(m-\xi, n-\eta)a_m b_n < k_1(\frac{1}{2})\left(\sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{2}}, \tag{22}$$

and both (19) and (21) reduce to the equivalent form of (22) as follows:

$$\left[\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} k_1(m-\xi, n-\eta)a_m\right)^2\right]^{\frac{1}{2}} < k_1(\frac{1}{2})\left(\sum_{m=1}^{\infty} a_m^2\right)^{\frac{1}{2}}. \tag{23}$$

4. Operator expressions and some particular examples

We set functions

$$\varphi(m) \triangleq (m-\xi)^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}, \psi(n) \triangleq (n-\eta)^{q[1-\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})]-1},$$

wherefrom,

$$\psi^{1-p}(n) = (n-\eta)^{p\beta(\frac{\lambda_2}{p}+\frac{\lambda-\lambda_1}{q})-1} \; (m, n \in \mathbf{N}).$$

Define the following real normed spaces:

$$l_{p,\varphi} \triangleq \left\{a = \{a_m\}_{m=1}^{\infty}; ||a||_{p,\varphi} := (\sum_{m=1}^{\infty} \varphi(m)|a_m|^p)^{\frac{1}{p}} < \infty\right\},$$

$$l_{q,\psi} \triangleq \left\{b = \{b_n\}_{n=1}^{\infty}; ||b||_{q,\psi} := (\sum_{n=1}^{\infty} \psi(n)|b_n|^q)^{\frac{1}{q}} < \infty\right\},$$

$$l_{p,\psi^{1-p}} \triangleq \left\{c = \{c_n\}_{n=1}^{\infty}; ||c||_{p,\psi^{1-p}} := (\sum_{n=1}^{\infty} \psi^{1-p}(n)|c_n|^p)^{\frac{1}{p}} < \infty\right\}.$$

Assuming that $a \in l_{p,\varphi}$, setting

$$c = \{c_n\}_{n=1}^{\infty}, c_n \triangleq \sum_{m=1}^{\infty} K(m,n)a_m, n \in \mathbf{N},$$

we can rewrite (15) as follows:

$$||c||_{p,\psi^{1-p}} < \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)||a||_{p,\varphi} < \infty,$$

namely, $c \in l_{p,\psi^{1-p}}$.

DEFINITION 11. Define a Hilbert-type operator $T : l_{p,\varphi} \to l_{p,\psi^{1-p}}$ as follows: For any $a \in l_{p,\varphi}$, there exists a unique representation $c \in l_{p,\psi^{1-p}}$. Define the formal inner product of Ta and $b \in l_{q,\psi}$, and the norm of T as follows:

$$(Ta,b) \triangleq \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} K(m,n)a_m\right) b_n,$$

$$||T|| \triangleq \sup_{a(\neq\theta)\in l_{p,\varphi}} \frac{||Ta||_{p,\psi^{1-p}}}{||a||_{p,\varphi}}.$$

By Theorem 1 and Theorem 2, we have

THEOREM 12. *If $a \in l_{p,\varphi}, b \in l_{q,\psi}, ||a||_{p,\varphi}, ||b||_{q,\psi} > 0$, then we have the following equivalent inequalities:*

$$(Ta,b) < \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)||a||_{p,\varphi}||b||_{q,\psi}, \tag{24}$$

$$||Ta||_{p,\psi^{1-p}} < \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)||a||_{p,\varphi}. \tag{25}$$

Moreover, $\lambda_1+\lambda_2=\lambda$ if and only if the constant factor

$$\frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda^{\frac{1}{p}}(\lambda-\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1)$$

in (24) and (25) is the best possible, namely,

$$||T|| = \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda(\lambda_1). \tag{26}$$

EXAMPLE 13. We set $k_\lambda(x,y) = \frac{1}{(cx+y)^\lambda}(c,\lambda>0;x,y>0)$. Then we find

$$K(m,n) = k_\lambda((m-\xi)^\alpha,(n-\eta)^\beta) = \frac{1}{[c(m-\xi)^\alpha+(n-\eta)^\beta]^\lambda}.$$

For $0<\lambda_1,\lambda-\lambda_2 \leqslant \frac{1}{\alpha}, 0<\lambda_2,\ \lambda-\lambda_1 \leqslant \frac{1}{\beta}, k_\lambda(x,y)$ is a positive homogeneous function of degree $-\lambda$, such that $k_\lambda(x,y)$ is strictly convex and strictly decreasing with respect to $x,y>0$, and for $\gamma=\lambda_1,\lambda-\lambda_2$,

$$k_\lambda(\gamma) = \int_0^\infty \frac{u^{\gamma-1}}{(cu+1)^\lambda}du = \frac{1}{c^\gamma}\int_0^\infty \frac{v^{\gamma-1}}{(v+1)^\lambda}dv = \frac{1}{c^\gamma}B(\gamma,\lambda-\gamma) \in \mathbf{R}_+.$$

In view of Theorem 3, it follows that $\lambda_1+\lambda_2=\lambda$ if and only if

$$||T|| = \frac{1}{\beta^{1/p}\alpha^{1/q}} k_\lambda(\lambda_1) = \frac{1}{\beta^{1/p}\alpha^{1/q}}\frac{1}{c^{\lambda_1}}B(\lambda_1,\lambda_2).$$

EXAMPLE 14. We set $k_\lambda(x,y) = \frac{\ln(cx/y)}{(cx)^\lambda - y^\lambda}(c > 0, 0 < \lambda \leqslant 1; x,y > 0)$. Then we find

$$K(m,n) = k_\lambda((m-\xi)^\alpha, (n-\eta)^\beta) = \frac{\ln[c(m-\xi)^\alpha/(n-\eta)^\beta]}{c^\lambda(m-\xi)^{\alpha\lambda} - (n-\eta)^{\beta\lambda}}.$$

For $0 < \lambda_1, \lambda-\lambda_2 \leqslant \frac{1}{\alpha}, 0 < \lambda_2,\ \lambda-\lambda_1 \leqslant \frac{1}{\beta}, k_\lambda(x,y)$ is a positive homogeneous function of degree $-\lambda$, such that $k_\lambda(x,y)$ is strictly convex and strictly decreasing with respect to $x,y > 0$ (cf. [2], Example 2.2.1), and for $\gamma = \lambda_1, \lambda-\lambda_2$,

$$\begin{aligned} k_\lambda(\gamma) &= \int_0^\infty \frac{u^{\gamma-1}\ln(cu)}{(cu)^\lambda - 1}du \\ &= \frac{1}{c^\gamma\lambda^2}\int_0^\infty \frac{v^{(\gamma/\lambda)-1}\ln v}{v-1}dv = \frac{1}{c^\gamma}[\frac{\pi}{\lambda\sin(\pi\gamma/\lambda)}]^2 \in \mathbf{R}_+. \end{aligned}$$

In view of Theorem 3, it follows that $\lambda_1 + \lambda_2 = \lambda$ if and only if

$$||T|| = \frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda(\lambda_1) = \frac{1}{\beta^{1/p}\alpha^{1/q}}\frac{1}{c^{\lambda_1}}[\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}]^2.$$

EXAMPLE 15. We set $k_\lambda(x,y) = \prod_{k=1}^s \frac{1}{(x^{\lambda/s} + c_k y^{\lambda/s})}(0 < c_1 \leqslant \cdots c_s, 0 < \lambda \leqslant s; x,y > 0)$. Then we find

$$K(m,n) = k_\lambda((m-\xi)^\alpha, (n-\eta)^\beta) = \prod_{k=1}^s \frac{1}{(m-\xi)^{\alpha\lambda/s} + c_k(n-\eta)^{\beta\lambda/s}}.$$

For $0 < \lambda_1, \lambda-\lambda_2 \leqslant \frac{1}{\alpha}, 0 < \lambda_2,\ \lambda-\lambda_1 \leqslant \frac{1}{\beta}, k_\lambda(x,y)$ is a positive homogeneous function of degree $-\lambda$, such that $k_\lambda(x,y)$ is strictly convex and strictly decreasing with respect to $x,y > 0$, and for $\gamma = \lambda_1, \lambda-\lambda_2$, by Example 1 of [26], it follows that

$$k_\lambda^{(s)}(\gamma) = \int_0^\infty \prod_{k=1}^s \frac{1}{u^{\lambda/s} + c_k}u^{\gamma-1}du = \frac{\pi s}{\lambda\sin(\frac{\pi s\gamma}{\lambda})}\sum_{k=1}^s c_k^{\frac{s\gamma}{\lambda}} \prod_{j=1(j\neq k)}^s \frac{1}{c_j - c_k} \in \mathbf{R}_+.$$

In view of Theorem 3, it follows that $\lambda_1 + \lambda_2 = \lambda$ if and only if

$$||T|| = \frac{k_\lambda^{(s)}(\lambda_1)}{\beta^{1/p}\alpha^{1/q}} = \frac{1}{\beta^{1/p}\alpha^{1/q}}\frac{\pi s}{\lambda\sin(\frac{\pi s\lambda_1}{\lambda})}\sum_{k=1}^s c_k^{\frac{s\lambda_1}{\lambda}} \prod_{j=1(j\neq k)}^s \frac{1}{c_j - c_k}.$$

In particular, for $c_1 = \cdots = c_s = c$, we have $k_\lambda(x,y) = \frac{1}{(x^{\lambda/s} + c_k y^{\lambda/s})^s}$ and

$$\begin{aligned} \widetilde{k}_\lambda^{(s)}(\lambda_1) &:= \int_0^\infty \frac{u^{\lambda_1 - 1}du}{(u^{\lambda/s} + c)^s} = \frac{s}{\lambda c^{(1-\frac{\lambda_1}{\lambda})s}}\int_0^\infty \frac{v^{(s\lambda_1/\lambda)-1}}{(v+1)^s}dv \\ &= \frac{s}{\lambda c^{(1-\frac{\lambda_1}{\lambda})s}}B(\frac{s\lambda_1}{\lambda}, \frac{s\lambda_2}{\lambda}) \in \mathbf{R}_+. \end{aligned}$$

If $s=1$, then we have $k_\lambda(x,y)=\frac{1}{x^\lambda+cy^\lambda}$ and

$$||T||=\frac{\widetilde{k}_\lambda^{(1)}(\lambda_1)}{\beta^{1/p}\alpha^{1/q}}=\frac{1}{\beta^{1/p}\alpha^{1/q}}\frac{s}{\lambda c^{\frac{\lambda_2}{\lambda}}}\frac{\pi}{\sin(\frac{\pi\lambda_1}{\lambda})}.$$

5. Conclusions

In this paper, by the use of the weight functions, the idea of introducing parameters and Hermite-Hadamard's inequality, a more accurate discrete Hilber-type inequality with the general homogeneous kernel and the equivalent form are given in Theorem 1. The equivalent statements of the best possible constant factor related to some parameters are considered in Theorem 2. The operator expressions and some particular examples are given in Theorem 3, Remark 1, Remark 3 and Example 1-3. The lemmas and theorems provide an extensive account of this type of inequalities.

Acknowledgement. This work is supported by the National Natural Science Foundation (No. 61772140), and Science and Technology Planning Project Item of Guangzhou City (No. 201707010229). We are grateful for this help.

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(Received May 11, 2019)

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