Journal of Mathematical Inequalities
Volume 14, Number 4 (2020), 1151–1163 doi:10.7153/jmi-2020-14-75

ALGORITHMS FOR SPLIT COMMON NULL POINT PROBLEM WITHOUT PRE–EXISTING ESTIMATION OF OPERATOR NORM

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(*Communicated by J. Kyu Kim*)

Abstract. The purpose of this paper is to present iterative methods to solve a split common null point problem in real Hilbert spaces such that the implementation of proposed iterative schemes do not require any pre-existing estimation of the norm of bounded linear operator. We give the weak and strong convergence of the proposed algorithms under some mild and standard assumptions in Hilbert spaces. A numerical example is also constructed to illustrate the algorithm for strong convergence.

1. Introduction

The theory of variational inequalities have played an important role in the development of mathematical models arising in economics, optimizations, physics, networking structural analysis, and medical images. The split feasible problem also has an important role in optimization theory and nonlinear analysis. In 1994, Censor and Elfving [3], first presented it for modeling in medical image reconstruction. Now a days, the split fractional problem has been implemented widely in intensity-modulation therapy treatment planning. In [4], Censor et al. combined the variational inequality problem and split feasibility problem and presented a new type of variational inequality problem called split variational inequality problem (in short, S_pVIP) as follows:

$$Find\ x^* \in C\ \ such\ \ that\ \langle f(x^*), x - x^*\rangle \geqslant 0,\ for\ all\ x \in C \tag{1.1}$$

$$such\ that\ y^* = Ax^* \in Q\ \ solves\ \ \langle g(y^*), y - y^*\rangle \geqslant 0,\ for\ all\ y \in Q, \tag{1.2}$$

where C and Q are closed, convex subsets of Hilbert spaces H_1 and H_2, respectively; $A : H_1 \to H_2$ is a bounded linear operator; $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are two operators.

Censor et al. [5] investigated S_pVIP as a prototypical split inverse problem (in short, S_pIP), which is the combination of two inverse problems denoted by IP_1 and IP_2 defined as follows:

$$Find\ x^* \in X\ \ \ that\ solves\ \ IP_1, \tag{1.3}$$

$$such\ that\ y^* = Ax^* \in Y\ \ solves\ \ IP_2, \tag{1.4}$$

Mathematics subject classification (2010): 49J40, 49J53, 47J20, 58A05.
Keywords and phrases: Split common null point problem, algorithm, operator norm, resolvent operator, convergence.

where X, Y are two vector spaces and $A : X \to Y$ is a bounded linear operator. S_pVIP is quite general and enables split minimization between two spaces so that the image of a solution point of one minimization problem under a given bounded linear operator is a solution point of another minimization problem. Another special case of the S_pVIP is the split feasibility problem (in short, S_pFP), which is a combination of an inverse problem as IP_1 and a feasibility problem as IP_2.

Let H_1 and H_2 be two real Hilbert spaces, $C \subseteq H_1$ and $Q \subseteq H_2$ be two nonempty, closed and convex sets, and $A : H_1 \to H_2$ be a bounded linear operator. The split feasibility problem (S_pFP) is to find:

$$x^* \in C \text{ such that } y^* = Ax^* \in Q, \tag{1.5}$$

which was discussed and used in practice as a model in intensity-modulated radiation therapy (IMRT) treatment planning; see, [6, 7]. S_pFP has many real life applications such as multi-domain adaptive filtering (MDAF) [24] and navigation on the Pareto frontier in multiobjective optimization; see, [9]. Moudafi [14] generalized split variational inequality problem to split monotone variational inclusion problem (in short, S_pMVIP) as follows:

$$Find\ x^* \in H_1 \ \ such\ that\ 0 \in f(x^*) + B_1(x^*), \tag{1.6}$$

$$such\ that\ y^* = Ax^* \in H_2 \ \ solves\ \ 0 \in g(y^*) + B_2(y^*), \tag{1.7}$$

where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are set-valued mappings on Hilbert spaces H_1 and H_2, respectively, $A : H_1 \to H_2$ is a bounded linear operator; $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are two given single-valued operators.

Moudafi [14] formulated the following iterative algorithm to find the solution of S_pMVIP (1.6)–(1.7). Let $\lambda > 0$, select an arbitrary starting point $x_0 \in H_1$. Compute

$$x_{n+1} = U[x_n + \gamma A^*(T - I)Ax_n], \tag{1.8}$$

where $\gamma \in (0, 1/L)$ with L being spectral radius of operator A^*A, A^* is the adjoint operator of A, $U = J_\lambda^{B_1}(I - \lambda f)$ and $T = J_\lambda^{B_2}(I - \lambda g)$.

If $B_1 = N_C$ and $B_2 = N_Q$ to be the normal cones of two closed and convex sets C and Q, respectively, then S_pMVIP reduces to S_pVIP. If $f = g = 0$, then S_pMVIP reduces to split common null point problem (in short, S_pCNPP) for set-valued maximal monotone mappings, introduced and studied by Byrne et al. [1]:

$$Find\ x^* \in H_1 \ \ such\ that\ 0 \in B_1(x^*), \tag{1.9}$$

$$such\ that\ y^* = Ax^* \in H_2 \ \ solves\ \ 0 \in B_2(y^*). \tag{1.10}$$

Based on CQ-algorithm, Byrne et al. [1] presented the following iterative algorithm to find the solution of S_pCNPP (1.9)–(1.10). Let $\lambda > 0$, select starting point $x_0 \in H_1$. Compute

$$x_{n+1} = J_\lambda^{B_1}[x_n + \gamma A^*(I - J_\lambda^{B_2})Ax_n], \tag{1.11}$$

where A^* is the adjoint operator of A, $L = \|A^*A\|$, and $\gamma \in (0, 2/L)$. After that Kazmi and Rizvi [10], considered S_pCNPP and a fixed point problem. They find the common solution of S_pCNPP and a fixed point of nonexpansive mapping using following algorithm:

$$\begin{aligned} y_n &= J_\lambda^{B_1}[x_n + \gamma A^*(I - J_\lambda^{B_2})Ax_n], \\ x_{n+1} &= \alpha_n f(x_n) + \alpha_n S y_n, \end{aligned}$$

where f is contraction mapping and S is a nonexpansive mapping. Later, Sitthithakerngkiet et al. [18] studied the common solution of S_pCNPP and a fixed point of an infinite family of nonexpansive mappings using following algorithm:

$$\begin{aligned} y_n &= J_\lambda^{B_1}[x_n + \gamma A^*(I - J_\lambda^{B_2})Ax_n], \\ x_{n+1} &= \alpha_n \xi u + \beta_n x_n + [(1-\beta_n)I - \alpha_n D]W_n y_n, \quad \forall n \geqslant 1, \end{aligned}$$

where $u \in H_1$ is a given point, Wn is a W-mapping which is generated by an infinite family of nonexpansive mappings. Later, many authors studied number of split variational inequalities and variational inclusion problems using different innovative techniques; see, for example, [11, 12, 17, 19, 22, 23, 25] but most of the problems solved are based on the formulation of algorithm (1.8) or (1.11); see, for example, [8, 10, 16] and references therein.

We noticed that the implementation of algorithms in all the methods mentioned above required the pre-existing calculation or estimation of the norm of bounded linear operator A.

Lopez et al. [13] solved the split feasibility problem without knowledge of matrix norm. They introduced and studied the following iterative algorithm:

$$x_{k+1} = P_C[I - \tau_k A^*(I - P_Q)A]x_k,$$

where P_C and P_Q are orthogonal projections on the closed convex sets C and Q, respectively and the step size τ_k is computed as:

$$\tau_k = \frac{\rho_k f(x_k)}{\|\nabla f(x_k)\|^2}$$

and

$$f(x) = \frac{1}{2}\|(I - PQ)Ax\|^2, \quad \nabla f(x) = A^*(I - P_Q)Ax, \quad k \geqslant 0,$$

with $0 < \rho_k < 4$ and $\inf_{\rho_k}(4 - \rho_k) > 0$. Wang [20] obtained the split common fixed point problem which is a generalization of split feasibility problem. He also constructed an iterative algorithm and discussed the strong convergence to study the solution of split common fixed point problem without the prior calculation of the norm of operator A.

Motivated by the work of Censor et al. [5], Moudafi [14], Byrne et al. [1], Kazmi and Rizvi [10], Sitthithakerngkiet et al. [18], Lopez et al. [13] and Wang[20], we propose two iterative algorithms for solving split common null point problem (1.9)–(1.10) so that the choice of step size does not need any pre-existing estimation of the operator norm $\|A\|$. Finally, weak and strong convergence of proposed algorithms are presented under some mild and standard assumptions. Furthermore, iterative algorithm is illustrated by a non-trivial example.

Let H be a real Hilbert space. The strong convergence and weak convergence of a sequence $\{x_n\}$ to x are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let $T : H \to H$ be an operator. The set of all fixed point of T is denoted as $Fix(T) = \{x : Tx = x\}$. The operator T is said to be nonexpansive if for all $x, y \in H$, $\|T(x) - T(y)\| \leqslant \|x - y\|$; firmly nonexpansive if for all $x, y \in H$, $\|T(x) - T(y)\|^2 \leqslant \langle x - y, Tx - Ty \rangle$. T is called directed, if

$$\|T(x) - z\|^2 \leqslant \|x - z\|^2 - \|(I - T)x\|^2, \ \forall\, z \in Fix(T), \ \ x \in H,$$

or

$$\langle x - z, Tx - x \rangle \leqslant \|x - T(x)\|^2, \ \forall\, z \in Fix(T), \ \ x \in H.$$

Let $B : H \to 2^H$ be a set-valued operator. The graph of B is defined by $\{(x, y) : y \in B(x)\}$ and inverse of B is denoted by $B^{-1} = \{(y, x) : y \in B(x)\}$. A set-valued mapping B is said to be monotone if $\langle u - v, x - y \rangle \geqslant 0,$ for all $u \in B(x), v \in B(y)$. A monotone operator B is called a maximal monotone if there exits no other monotone operator such that its graph properly contains the graph of B. The resolvent of a maximal monotone operator B is defined by $J_\lambda^B = (I + \lambda B)^{-1}$, where λ is a positive real number. A resolvent operator of maximal monotone operator is single-valued and firmly nonexpansive. The class of directed operator includes the resolvents of maximal monotone operators.

DEFINITION 1.1. [15] (Demiclosedness Principle): Let C be a nonempty closed convex subset of a Hilbert space H and $T : H \to H$ be an operator with $Fix(T) \neq \phi$. If the sequence $\{x_n\}$ in C converges weakly to an element $x \in C$ and the sequence $\{x_n - Tx_n\}_{n=1}^{\infty}$ converges strongly to zero, then x is a fixed point of the operator T.

REMARK 1.1. It is well known that if T is a nonexpansive operator, then $I - T$ is demiclosed at zero. This property is also shared by firmly nonexpansive operators and averaged nonexpansive operators.

DEFINITION 1.2. A sequence $\{x_n\}$ in H_1 is said to be Féjer monotone with respect to a nonempty closed convex subset C of H_1, if

$$\|x_{n+1} - p\| \leqslant \|x_n - p\|, \ \ \forall\, n \geqslant 0, \ \ \forall\, p \in C.$$

LEMMA 1.1. [2] *Let C be a nonempty closed convex subset of H_1. If the sequence $\{x_n\}$ is Féjer monotone with respect to C, then the following hold:*

(i) $x_n \rightharpoonup x^* \in C$ if and only if the weak limit set, $\omega_W(x_n) \subseteq C$,

(ii) the sequence $\{P_C x_n\}$ converges strongly,

(iii) if $x_n \rightharpoonup x^* \in C$, then $x^* = \lim_{n\to\infty} P_C x_n$.

LEMMA 1.2. [21] *Assume that* $\{a_n\}$ *is a sequence of non-negative real numbers such that*

$$a_{n+1} \leqslant (1-t_n)a_n + t_n b_n, \ \ n \geqslant 0,$$

where $\{t_n\}$ *is a sequence in* $(0,1)$ *and* $\{b_n\}$ *is a sequence in* $\mathbb{R}$ *such that*

(i) $\sum_{n=0}^{\infty} t_n = \infty$;

(ii) $\lim_{n\to\infty} t_n \leqslant 0$ *or* $\sum_{n=0}^{\infty} \|t_n b_n\| < \infty$;

Then $\lim_{n\to\infty} a_n = 0$.

2. Main results

In this section, we present the existence of solution for S_pCNPP (1.9)–(1.10). We assume that the problem S_pCNPP (1.9)–(1.10) is consistent and solution set is denoted by $\prod = \{x^* \in H_1 : 0 \in B_1(x^*) \text{ and } 0 \in B_2(Ax^*)\}$.

First, we prove following lemmas, which will be used in the proof of our main results.

LEMMA 2.1. x^* *solves* S_pCNPP $(1.9)-(1.10)$ *if and only if*

$$\|x^* - J_\lambda^{B_1}x^* + A^*(I - J_\lambda^{B_2})Ax^*\| = 0.$$

Proof. Let x^* solves (1.9)–(1.10). Then $J_\lambda^{B_1}x^* = x^*$ and $J_\lambda^{B_2}Ax^* = Ax^*$. Therefore

$$\|x^* - J_\lambda^{B_1}x^* + A^*(I - J_\lambda^{B_2})Ax^*\| = 0.$$

Conversely, let $\|x^* - J_\lambda^{B_1}x^* + A^*(I - J_\lambda^{B_2})Ax^*\| = 0$ and for $p \in \prod$. Then we have

$$\begin{aligned} 0 &= \|x^* - J_\lambda^{B_1}x^* + A^*(I - J_\lambda^{B_2})Ax^*\|\|x^* - p\| \\ &\geqslant \langle x^* - J_\lambda^{B_1}x^* + A^*(I - J_\lambda^{B_2})Ax^*, x^* - p\rangle \\ &= \langle x^* - J_\lambda^{B_1}x^*,\ x^* - p\rangle + \langle A^*(I - J_\lambda^{B_2})Ax^*,\ x^* - p\rangle \\ &= \langle x^* - J_\lambda^{B_1}x^*,\ x^* - p\rangle + \langle (I - J_\lambda^{B_2})Ax^*,\ Ax^* - Ap\rangle. \end{aligned} \tag{2.1}$$

Since the resolvent of maximal monotone operator is nonexpansive and hence directed, that is,

$$0 \geqslant \|x^* - J_\lambda^{B_1}x^*\|^2 + \|(I - J_\lambda^{B_2})Ax_n\|^2 \geqslant 0.$$

Therefore $x^* = J_\lambda^{B_1}x^*$ and $Ax^* = J_\lambda^{B_2}Ax^*$. This completes the proof. □

LEMMA 2.2. *Let* $\{x_n\}$ *be a bounded sequence. If*

$$\|x_n - J_\lambda^{B_1}x_n + A^*(I - J_\lambda^{B_2})Ax_n\| = 0, \tag{2.2}$$

then

$$\lim_{n\to\infty} \|x_n - J_\lambda^{B_1}x_n\| = \lim_{n\to\infty} \|(I - J_\lambda^{B_2})Ax_n\| = 0.$$

Proof. For $p \in \prod$ and using the fact that $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ are directed, we have

$$\begin{aligned}
&\|x_n - J_\lambda^{B_1}x_n\|^2 + \|A^*(I - J_\lambda^{B_2})Ax_n\|^2 \\
&\leqslant \langle x_n - J_\lambda^{B_1}x_n,\, x_n - p\rangle + \langle A^*(I - J_\lambda^{B_2})Ax_n,\, x_n - p\rangle \\
&= \langle x_n - J_\lambda^{B_1}x_n + A^*(I - J_\lambda^{B_2})Ax_n,\, x_n - p\rangle \\
&\leqslant \|x_n - J_\lambda^{B_1}x_n + A^*(I - J_\lambda^{B_2})Ax_n\|\|x_n - p\|.
\end{aligned}$$

Using boundedness of $\{x_n\}$ and (2.2), we conclude the desired result. □

ALGORITHM 2.1. *Choose arbitrary* $x_0 \in H_1$.

Step 1. *Given* x_n, *compute the next iteration by*

$$u_n = x_n - J_\lambda^{B_1}x_n + A^*(I - J_\lambda^{B_2})Ax_n,$$

$$x_{n+1} = x_n - \gamma_n u_n,$$

where $\gamma_n = \dfrac{\|x_n - J_\lambda^{B_1}x_n\|^2 + \|A^*(I - J_\lambda^{B_2})Ax_n\|^2}{\|x_n - J_\lambda^{B_1}x_n + A^*(I - J_\lambda^{B_2})Ax_n\|^2}$.

Step 2. *If the following equality*

$$\|x_{n+1} - J_\lambda^{B_1}x_{n+1} + A^*(I - J_\lambda^{B_2})Ax_{n+1}\|^2 = 0$$

holds, then stop; otherwise go to Step 1.

LEMMA 2.3. *If* x_n *satisfies*

$$\lim_{n\to\infty} \frac{(\|x_n - J_\lambda^{B_1}x_n\|^2 + \|(I - J_\lambda^{B_2})Ax_n\|^2)^2}{\|x_n - J_\lambda^{B_1}x_n + A^*(I - J_\lambda^{B_2})Ax_n\|^2} = 0,$$

then

$$\lim_{n\to\infty} \|x_n - J_\lambda^{B_1}x_n\| = \lim_{n\to\infty} \|(I - J_\lambda^{B_2})Ax_n\| = 0.$$

Proof. Note that

$$\begin{aligned}
&\frac{(\|x_n - J_\lambda^{B_1}x_n\|^2 + \|(I - J_\lambda^{B_2})Ax_n\|^2)^2}{\|x_n - J_\lambda^{B_1}x_n + A^*(I - J_\lambda^{B_2})Ax_n\|^2} \\
&\geqslant \frac{(\|x_n - J_\lambda^{B_1}x_n\|^2 + \|(I - J_\lambda^{B_2})Ax_n\|^2)^2}{2(\|x_n - J_\lambda^{B_1}x_n\|^2 + \|A\|^2\|(I - J_\lambda^{B_2})Ax_n\|^2)^2} \\
&\geqslant \frac{(\|x_n - J_\lambda^{B_1}x_n\|^2 + \|(I - J_\lambda^{B_2})Ax_n\|^2)^2}{2\max(1, \|A\|^2)\|x_n - J_\lambda^{B_1}x_n\|^2 + \|(I - J_\lambda^{B_2})Ax_n\|^2} \\
&= \frac{\|x_n - J_\lambda^{B_1}x_n\|^2 + \|(I - J_\lambda^{B_2})Ax_n\|^2}{2\max(1, \|A\|^2)}.
\end{aligned} \tag{2.3}$$

Taking limit on both sides, we have

$$\lim_{n\to\infty}\|x_n - J_\lambda^{B_1}x_n\| = \lim_{n\to\infty}\|(I-J_\lambda^{B_2})Ax_n\| = 0. \quad \square$$

THEOREM 2.1. *Let H_1, H_2 be Hilbert spaces, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be set-valued maximal monotone operators and $A : H_1 \to H_2$ be a bounded linear operator. Then the sequence $\{x_n\}$ generated by Algorithm 2.1 converges weakly to a solution x^* of S_pCNPP (1.9)–(1.10), where $x^* = \lim_{n\to\infty} P_{\prod} x_n$.*

Proof. Let $p \in \prod$ and $u_n = x_n - J_\lambda^{B_1}x_n + A^*(I-J_\lambda^{B_2})Ax_n$. Then

$$\begin{aligned}\langle u_n,\, x_n - p\rangle &= \langle x_n - J_\lambda^{B_1}x_n + A^*(I-J_\lambda^{B_2})Ax_n,\, x_n - p\rangle \\ &= \langle x_n - J_\lambda^{B_1}x_n,\, x_n - p\rangle + \langle (I-J_\lambda^{B_2})Ax_n,\, Ax_n - Ap\rangle \\ &\geqslant \|x_n - J_\lambda^{B_1}x_n\|^2 + \|(I-J_\lambda^{B_2})Ax_n\|^2. \end{aligned} \tag{2.4}$$

Now, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|x_n - \tau u_n\|^2 \\ &= \|x_n - p\|^2 - 2\tau\langle u_n,\, x_n - p\rangle + \tau^2\|u_n\|^2 \\ &= \|x_n - p\|^2 - 2\frac{(\|x_n - J_\lambda^{B_1}x_n\|^2 + \|(I-J_\lambda^{B_2})Ax_n\|^2)^2}{\|x_n - J_\lambda^{B_1}x_n + A^*(I-J_\lambda^{B_2})Ax_n\|^2} \\ &\leqslant \|x_n - p\|^2 - \frac{(\|x_n - J_\lambda^{B_1}x_n\|^2 + \|(I-J_\lambda^{B_2})Ax_n\|^2)^2}{\|x_n - J_\lambda^{B_1}x_n + A^*(I-J_\lambda^{B_2})Ax_n\|^2} \\ &\leqslant \|x_n - p\|. \end{aligned} \tag{2.5}$$

From (2.4), we deduce that $\{x_n\}$ is a Féjer-monotone. It follows that $\{x_n\}$ is bounded. Thus by (2.5), we have

$$\frac{(\|x_n - J_\lambda^{B_1}x_n\|^2 + \|(I-J_\lambda^{B_2})Ax_n\|^2)^2}{\|x_n - J_\lambda^{B_1}x_n + A^*(I-J_\lambda^{B_2})Ax_n\|^2} \leqslant \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

Since, $\|x_n - p\|$ is bounded, we have by induction

$$\sum_{n=1}^{\infty} \frac{(\|x_n - J_\lambda^{B_1}x_n\|^2 + \|(I-J_\lambda^{B_2})Ax_n\|^2)^2}{\|x_n - J_\lambda^{B_1}x_n + A^*(I-J_\lambda^{B_2})Ax_n\|^2} < \infty.$$

Using the property of convergent series, we have

$$\lim_{n\to\infty} \frac{(\|x_n - J_\lambda^{B_1}x_n\|^2 + \|(I-J_\lambda^{B_2})Ax_n\|^2)^2}{\|x_n - J_\lambda^{B_1}x_n + A^*(I-J_\lambda^{B_2})Ax_n\|^2} = 0.$$

Thus, by Lemma 2.3, it follows that

$$\lim_{n\to\infty}\|x_n - J_\lambda^{B_1}x_n\| = \lim_{n\to\infty}\|(I-J_\lambda^{B_2})Ax_n\| = 0.$$

Since $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ are firmly nonexpansive and demiclosed at zero, from Lemma 1.1, we deduce that the sequence $\{x_n\}$ converges weakly to a solution x^* of S_pCNPP (1.9)–(1.10). □

Now, we propose the another iterative algorithm for solving problem (1.9)–(1.10) and analyze its strong convergence.

ALGORITHM 2.2. *Choose arbitrary $x_0 \in H_1$ and some fixed $u \in H_1$.*

Step 1. *Given x_n, compute the next iteration by*

$$u_n = x_n - J_\lambda^{B_1} x_n + A^*(I - J_\lambda^{B_2})Ax_n,$$

$$x_{n+1} = \alpha_n u + (1-\alpha_n)(x_n - \gamma_n u_n),$$

where $\gamma_n = \dfrac{\|x_n - J_\lambda^{B_1} x_n\|^2 + \|A^*(I - J_\lambda^{B_2})Ax_n\|^2}{\|x^* - J_\lambda^{B_1} x_n + A^*(I - J_\lambda^{B_2})Ax_n\|^2}$ *and $\{\alpha_n\}$ be a sequence in $(0,1)$.*

Step 2. *If the following equality*

$$\|x_{n+1} - J_\lambda^{B_1} x_{n+1} + A^*(I - J_\lambda^{B_2})Ax_{n+1}\|^2 = 0$$

holds, then stop; otherwise go to Step 1.

THEOREM 2.2. *Let H_1, H_2 be Hilbert spaces, $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be set-valued maximal monotone operators and $A : H_1 \to H_2$ be a bounded linear operator. If $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, then the sequence $\{x_n\}$ generated by Algorithm 2.2 converges strongly to a solution $p = P_{\prod}(u)$ of S_pCNPP (1.9)–(1.10) for some fixed $u \in H_1$.*

Proof. Let $p \in P_{\prod} u$. Then from (2.4) and the definition of γ_n, we have

$$\begin{aligned}
\|x_n - p - \gamma_n u_n\|^2 &= \|x_n - p\|^2 - 2\gamma_n \langle u_n, x_n - p\rangle + \tau_n^2 \|u_n\|^2 \\
&= \|x_n - p\|^2 - 2\frac{(\|x_n - J_\lambda^{B_1} x_n\|^2 + \|(I - J_\lambda^{B_2})Ax_n\|^2)^2}{\|x_n - J_\lambda^{B_1} x_n + A^*(I - J_\lambda^{B_2})Ax_n\|^2} \\
&\quad + \frac{(\|x_n - J_\lambda^{B_1} x_n\|^2 + \|(I - J_\lambda^{B_2})Ax_n\|^2)^2}{\|x_n - J_\lambda^{B_1} x_n + A^*(I - J_\lambda^{B_2})Ax_n\|^2} \\
&= \|x_n - p\|^2 - \frac{(\|x_n - J_\lambda^{B_1} x_n\|^2 + \|(I - J_\lambda^{B_2})Ax_n\|^2)^2}{\|x_n - J_\lambda^{B_1} x_n + A^*(I - J_\lambda^{B_2})Ax_n\|^2} \\
&\leqslant \|x_n - p\|^2. \qquad (2.6)
\end{aligned}$$

It follows from (2.6) and Algorithm 2.2 that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n u+(1-\alpha_n)(x_n-\gamma_n u_n)-p\|\\ &\leqslant \alpha_n\|u-p\|+(1-\alpha_n)\|x_n-p-\gamma_n u_n\|\\ &\leqslant \alpha_n\|u-p\|+(1-\alpha_n)\|x_n-p\|\\ &\quad\vdots\\ &\leqslant \max\{\|u-p\|,\|x_n-p\|\}.\end{aligned} \tag{2.7}$$

Hence $\{x_n\}$ is bounded. Again by Algorithm 2.2 and (2.6), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n u+(1-\alpha_n)(x_n-\gamma_n u_n)-p\|^2\\ &\leqslant (1-\alpha_n)\|x_n-\gamma_n u_n-p\|+2\alpha_n\langle u-p,\, x_{n+1}-p\rangle\\ &\leqslant (1-\alpha_n)\|x_n-p\|^2+2\alpha_n\langle u-p,\, x_{n+1}-p\rangle+\\ &\quad -(1-\alpha_n)\frac{(\|x_n-J_\lambda^{B_1}x_n\|^2+\|(I-J_\lambda^{B_2})Ax_n\|^2)^2}{\|x_n-J_\lambda^{B_1}x_n+A^*(I-J_\lambda^{B_2})Ax_n\|^2}\\ &\leqslant (1-\alpha_n)\|x_n-p\|^2+\alpha_n\Big[2\langle u-p,\, x_{n+1}-p\rangle\\ &\quad -\frac{(1-\alpha_n)}{\alpha_n}\frac{(\|x_n-J_\lambda^{B_1}x_n\|^2+\|(I-J_\lambda^{B_2})Ax_n\|^2)^2}{\|x_n-J_\lambda^{B_1}x_n+A^*(I-J_\lambda^{B_2})Ax_n\|^2}\Big].\end{aligned}$$

That is,

$$a_{n+1}\leqslant(1-\alpha_n)a_n+\alpha_n b_n,$$

where $a_n=\|x_n-p\|^2$ and $b_n=2\langle u-p,\, x_{n+1}-p\rangle$

$$-\frac{(1-\alpha_n)}{\alpha_n}\frac{(\|x_n-J_\lambda^{B_1}x_n\|^2+\|(I-J_\lambda^{B_2})Ax_n\|^2)^2}{\|x_n-J_\lambda^{B_1}x_n+A^*(I-J_\lambda^{B_2})Ax_n\|^2}.$$

Since $\{x_n\}$ is bounded and $b_n\leqslant 2\langle u-p,\ x_{n+1}-p\rangle\leqslant 2\|u-p\|\|x_{n+1}-p\|$, it follows that $\limsup_{n\to\infty} b_n<+\infty$.

Now, we show that $\limsup\limits_{n\to\infty} b_n\geqslant -1$.

Suppose that $\limsup\limits_{n\to\infty} b_n\ngeqslant -1$, i.e., $\limsup\limits_{n\to\infty} b_n<-1$. Then there exists n_0 such that $b_n\leqslant -1$ for all $n\geqslant n_0$. So, we have

$$\begin{aligned}a_{n+1} &\leqslant (1-\alpha_n)a_n+\alpha_n b_n\\ &\leqslant (1-\alpha_n)a_n-\alpha_n\\ &\leqslant a_n-\alpha_n(a_n+1)\\ &\leqslant a_n-\alpha_n.\end{aligned} \tag{2.8}$$

For $n\geqslant n_0$, we have

$$a_{n+1}\leqslant a_{n_0}-\sum_{i=n_0}^{n}\alpha_i,$$

taking $\limsup_{n\to\infty}$ in the above inequality, we get

$$\limsup_{n\to\infty} a_{n+1} \leqslant a_{n_0} - \lim_{n\to\infty} \sum_{i=n_0}^{n} \alpha_i = -\infty,$$

which contradicts the fact that $a_n = \|x_n - p\|$ is sequence of nonnegative real numbers. Consequently, we have $-1 \leqslant \limsup_{n\to\infty} b_n < \infty$. Hence, we can have a subsequence $\{b_{n_k}\}$ satisfying

$$\begin{aligned} \limsup_{n\to\infty} b_n &= \lim_{k\to\infty} b_{n_k} \\ &= \lim_{k\to\infty} 2\langle u-p,\, x_{n_k+1}-p\rangle \\ &\quad - \lim_{k\to\infty}\Big[\frac{(1-\alpha_{n_k})}{\alpha_n}\frac{(\|x_{n_k}-J_\lambda^{B_1}x_{n_k}\|^2+\|(I-J_\lambda^{B_2})Ax_{n_k}\|^2)^2}{\|x_{n_k}-J_\lambda^{B_1}x_{n_k}+A^*(I-J_\lambda^{B_2})Ax_{n_k}\|^2}\Big]. \end{aligned} \tag{2.9}$$

Since $\langle u-p,\, x_{n_k+1}-p\rangle$ is bounded, we may assume $\lim_{k\to\infty}\langle u-p,\, x_{n_k+1}-p\rangle$ exists. Consequently, one has

$$\lim_{k\to\infty}\Big[\frac{(1-\alpha_{n_k})}{\alpha_{n_k}}\frac{(\|x_{n_k}-J_\lambda^{B_1}x_n\|^2+\|(I-J_\lambda^{B_2})Ax_{n_k}\|^2)^2}{\|x_{n_k}-J_\lambda^{B_1}x_{n_k}+A^*(I-J_\lambda^{B_2})Ax_{n_k}\|^2}\Big],$$

also exists. Therefore

$$\lim_{k\to\infty}\Big[\frac{(\|x_{n_k}-J_\lambda^{B_1}x_{n_k}\|^2+\|(I-J_\lambda^{B_2})Ax_{n_k}\|^2)^2}{\|x_{n_k}-J_\lambda^{B_1}x_{n_k}+A^*(I-J_\lambda^{B_2})Ax_{n_k}\|^2}\Big] = 0.$$

From Lemma 2.3, we have

$$\lim_{k\to\infty}\|x_{n_k}-J_\lambda^{B_1}x_{n_k}\| = \lim_{k\to\infty}\|(I-J_\lambda^{B_2})Ax_{n_k}\| = 0.$$

It follows that the weak cluster point of $\{x_{n_k}\}$ belongs to $\prod$.

Note that

$$\lim_{k\to\infty}\|x_{n_k}-u_{n_k}\| = \lim_{k\to\infty}\gamma_{n_k}\|(x_{n_k}-J_\lambda^{B_1}x_{n_k})+A^*(I-J_\lambda^{B_2})Ax_{n_k}\| \tag{2.10}$$

$$= \lim_{k\to\infty}\frac{\|(x_{n_k}-J_\lambda^{B_1}x_{n_k}\|^2+\|(I-J_\lambda^{B_2}Ax_{n_k}\|^2}{\|(x_{n_k}-J_\lambda^{B_1}x_{n_k})+A^*(I-J_\lambda^{B_2})Ax_{n_k}\|} \tag{2.11}$$

Also,

$$\begin{aligned} x_{n_k+1} &= \alpha_n u + (1-\alpha_{n_k})(x_{n_k}-\gamma_{n_k}u_{n_k}) \\ &= \alpha_{n_k}u + x_{n_k} - \gamma_{n_k}u_{n_k} - \alpha_{n_k}x_{n_k} + \alpha_{n_k}\gamma_{n_k}u_{n_k}, \end{aligned}$$

$$\|x_{n_k+1}-x_{n_k}\| = \alpha_{n_k}\|u-x_{n_k}\| + (1-\alpha_{n_k})\gamma_{n_k}\|u_{n_k}-x_{n_k}\| \tag{2.12}$$

$$\leqslant \alpha_{n_k}\|u-x_{n_k}\| + \|u_{n_k}-x_{n_k}\| \to 0. \tag{2.13}$$

Therefore

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0.$$

This implies that weak cluster point of $\{x_{n_k+1}\}$ also belongs $\prod$. We assume that $\{x_{n_k+1}\}$ converges weakly to x^*. Therefore

$$\begin{aligned}\limsup_{n\to\infty} b_n &\leqslant \lim_{k\to\infty} 2\langle u-p, x_{n_{k+1}} - p\rangle \\ &= 2\langle u-p, x^*-p\rangle \leqslant 0.\end{aligned}$$

Since $p = P_{\prod} u$ and using Lemma 1.2, we conclude that $\|x_n - p\| \to 0$. This completes the proof. □

Now, we illustrate the Algorithm 2.2 and the convergence analysis of the sequences in Theorem 2.2.

EXAMPLE 2.1. Let $H_1 = H_2 = \mathbb{R}$. B_1 be defined by $B_1(x) = x-1$, and B_2 be defined by $B_2(x) = 2(x+1)$. Let $A : H_1 \to H_2$ be defined by $A(x) = -x$ be a bounded linear operator. For $\lambda = 1$ the resolvents of operators B_1 and B_2 are respectively given by

$$\begin{aligned}J_\lambda^{B_1}(x) &= \frac{x}{2} + \frac{1}{2}, \\ J_\lambda^{B_2}(x) &= \frac{x}{3} - \frac{2}{3}.\end{aligned}$$

The iterative sequences $\{u_n\}$ and $\{x_n\}$ are computed by the following iterative method:

$$\begin{aligned}u_n &= x_n - J_\lambda^{B_1}(x_n) + A^*(I - J_\lambda^{B_2}(x)Ax_n, \\ x_{n+1} &= \alpha_n u + (1-\alpha_n)(x_n - \gamma_n u_n).\end{aligned}$$

x_{n+1}	$x_0 = -5$	$x_0 = 5$	$x_0 = 10$	$x_0 = 15$
x_1	-0.8215	2.2143	3.7322	5.2501
x_2	0.4119	1.3932	1.8847	2.3762
x_3	0.8011	1.1326	1.2984	1.4642
x_4	0.9309	1.0460	1.1035	1.1611
x_5	0.9756	1.0162	1.0367	1.0570
x_6	0.9912	1.0058	1.0132	1.0205
x_7	0.9968	1.0021	1.0048	1.0075
x_8	0.9988	1.0008	1.0018	1.0028
x_9	0.9996	1.0003	1.0007	1.0010
x_{10}	0.9998	1.0001	1.0002	1.0004
x_{11}	0.9999	1.0000	1.0000	1.0001
x_{12}	0.9999	1.0000	1.0000	1.0000

Table 1: *Values of x_n for $x_0 = -5$, $x_0 = 5$, $x_0 = 10$ and $x_0 = 15$.*

For $u = 1$ and $\alpha_n = \frac{1}{n+4}$, we compute the values of sequence $\{x_n\}$ upto four decimal places with some different initial guess in Table 1.

We observe the following points in the above example:

(i) The sequence $\{x_n\}$ in the Table 1 converge to $x = 1$ for different values of initial guess and it can be easily seen that $x = 1$ is a solution of our problem.

(ii) We do not impose the condition $\sum_{n=}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ which guarantee the convergence of the sequences in some existing iterative methods.

(iii) It is also to be noted that, we do not require the norm of operator A to compute the solution of the problem.

3. Conclusions

In this paper, we have focused on a split common null point problem for the set-valued maximal monotone operators. Two iterative algorithms for solving split common null point problem are presented. We have investigated the weak and strong convergence of proposed algorithms such that the implementation of the algorithms do not require the pre-existing estimation of norm of the operator.

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(Received May 16, 2019)

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