INITIAL SUCCESSIVE COEFFICIENTS FOR CERTAIN CLASSES OF UNIVALENT FUNCTIONS INVOLVING THE EXPONENTIAL FUNCTION

LEI SHI, ZHI-GANG WANG*, REN-LI SU AND MUHAMMAD ARIF

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Abstract. Let $\mathscr{S}$ denote the family of all functions that are analytic and univalent in the unit disk $\mathbb{D} := \{z : |z| < 1\}$ and satisfy $f(0) = f'(0) - 1 = 0$. In the present paper, we consider certain subclasses of univalent functions associated with the exponential function, and obtain the sharp upper bounds on the initial coefficients and the difference of initial successive coefficients for functions belonging to these classes.

1. Introduction

Let $\mathscr{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the unit disk $\mathbb{D} := \{z : |z| < 1\}$ and satisfy the conditions $f(0) = f'(0) - 1 = 0$. Let $\mathscr{S}$ be the set of all functions $f \in \mathscr{A}$ that are univalent in $\mathbb{D}$. Let $\mathscr{S}^*$ and $\mathscr{K}$ denote the subclasses of $\mathscr{S}$ consisting of starlike functions and convex functions, respectively.

Let $\mathscr{P}$ denote the class of all functions $\mathfrak{p}(z)$ analytic and having positive real part in $\mathbb{D}$, with the form

$$\mathfrak{p}(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{1.2}$$

For two functions f and g, analytic in $\mathbb{D}$, we say that the function f is subordinate to g in $\mathbb{D}$, and write

$$f(z) \prec g(z) \quad (z \in \mathbb{D}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{D}$ with

$$\omega(0) = 0 \text{ and } |\omega(z)| < 1 \ (z \in \mathbb{D})$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{D}).$$

Using the subordination relationship, Ma and Minda [10] introduced the class of Ma-Minda type of starlike functions $\mathscr{S}^*(\phi)$, which is defined by

$$\mathscr{S}^*(\phi) := \left\{ f \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \ (z \in \mathbb{D}) \right\}, \tag{1.3}$$

where $\phi(z)$ is analytic and univalent in $\mathbb{D}$ and for which $\phi(\mathbb{D})$ is convex with $\phi(z) \in \mathscr{P}$ for $z \in \mathbb{D}$.

For a constant λ with $0 < \lambda \leqslant \frac{\pi}{2}$, by setting

$$\phi(z) = e^{\lambda z} \quad (z \in \mathbb{D}), \tag{1.4}$$

we have the class $\mathscr{S}^*_{\lambda e}$ which is defined by the condition

$$\mathscr{S}^*_{\lambda e} := \left\{ z \in \mathscr{A} : \frac{zf'(z)}{f(z)} \prec e^{\lambda z} \ (z \in \mathbb{D}) \right\}. \tag{1.5}$$

It can be seen that the condition (1.5) is equivalent to

$$\left| \log \frac{zf'(z)}{f(z)} \right| \leqslant \lambda \quad (z \in \mathbb{D}). \tag{1.6}$$

Also, we denote by $\mathscr{K}_{\lambda e}$ the class of functions $f \in \mathscr{A}$ satisfying the condition

$$1 + \frac{zf''(z)}{f'(z)} \prec e^{\lambda z} \quad (z \in \mathbb{D}). \tag{1.7}$$

Let $z = re^{i\theta}$, $r \in [0, 1)$, $\theta \in [0, 2\pi]$, we have $\Re(e^{\lambda z}) = e^{\lambda r \cos\theta} \cos(\lambda r \sin\theta)$. It is clear that $\cos(\lambda r \sin\theta) > 0$ for $\lambda \in (0, \frac{\pi}{2}]$ and thus $\Re(e^{\lambda z}) > 0$ $(z \in \mathbb{D})$. Indeed, the class $\mathscr{S}^*_{\lambda e}$ is a subclass of starlike functions $\mathscr{S}^*$ and $\mathscr{K}_{\lambda e}$ is a subclass of convex functions $\mathscr{K}$.

By choosing $\lambda = 1$, we obtain the families $\mathscr{S}^*_e$ and $\mathscr{K}^*_e$ which were introduced and investigated by Mediratta *et al.* [11] and were later studied by many authors, see [4, 5, 12, 17, 18, 22, 23] and the references cited therein. Clearly, for $0 < \zeta \leqslant 1$ and $1 \leqslant \eta \leqslant \frac{\pi}{2}$, we have

$$\mathscr{S}^*_{\zeta e} \subseteq \mathscr{S}^*_e \subseteq \mathscr{S}^*_{\eta e}.$$

In recent years, the difference of the moduli of successive coefficients of a function $f \in \mathscr{S}$ has attracted many researchers' attention (see [3, 6, 20, 21]). Because of the triangle inequality $||a_{n+1}| - |a_n|| \leqslant |a_{n+1} - a_n|$, sometimes it maybe useful to study the upper bounds of $|a_{n+1} - a_n|$ for some refined subclasses of starlike and convex functions to obtain the upper bound of $||a_{n+1}| - |a_n||$. In [15], Robertson proved that $|a_{n+1} - a_n| \leqslant \frac{2n+1}{3}|a_2 - 1|$ for all $f \in \mathscr{K}$. Recently, Li and Sugawa [7] studied the related problem of maximizing the functional $||a_{n+1}| - |a_n||$ with the help of $|a_{n+1} - a_n|$ for convex functions f with $f''(0) = p$ for a prescribed $p \in [0, 2]$. For some subclasses of analytic univalent functions, the sharp upper bounds of $|a_3 - a_2|$ and $|a_4 - a_3|$ were obtained by Peng and Obradović [13].

Motivated essentially by the above work, in the present paper, we aim at proving some results on the upper bounds of the initial coefficients and the difference of initial successive coefficients for f belonging to the classes $\mathscr{S}^*_{\lambda e}$ and $\mathscr{K}_{\lambda e}$.

2. Preliminary results

To derive our main results, we need the following lemmas.

LEMMA 1. (See [14]) *Let* $\omega(z) = \sum_{n=1}^{\infty} c_k z^k$ *be a Schwarz function. Then, for any real number* μ *and* ν *the following sharp estimate holds*

$$\Psi(\omega) = \left| c_3 + \mu c_1 c_2 + \nu c_1^3 \right| \leqslant \Phi(\mu, \nu), \tag{2.1}$$

where $\Phi(\mu, \nu)$ *is given in complete form in* [14, Lemma 2]*, and here we will only use*

$$\Phi(\mu, \nu) \leqslant \begin{cases} 1 & ((\mu, \nu) \in D_1 \bigcup D_2), \\[2mm] \frac{2}{3}(|\mu|+1)\sqrt{\frac{|\mu|+1}{3(|\mu|+1+\nu)}} & ((\mu, \nu) \in D_3 \bigcup D_4), \\[2mm] \frac{1}{3}\nu\left(\frac{\mu^2-4}{\mu^2-4\nu}\right)\sqrt{\frac{\mu^2-4}{3(\nu-1)}} & ((\mu, \nu) \in D_5), \\[2mm] |\nu| & ((\mu, \nu) \in D_6), \end{cases} \tag{2.2}$$

with

$$D_1 = \left\{ (\mu, \nu) : |\mu| \leqslant \frac{1}{2}, \ -1 \leqslant \nu \leqslant 1 \right\},$$

$$D_2 = \left\{ (\mu, \nu) : \frac{1}{2} \leqslant |\mu| \leqslant 2, \ \frac{4}{27}(|\mu|+1)^3 - (|\mu|+1) \leqslant \nu \leqslant 1 \right\},$$

$$D_3 = \left\{ (\mu, \nu) : \frac{1}{2} \leqslant |\mu| \leqslant 2, \ -\frac{2}{3}(|\mu|+1) \leqslant \nu \leqslant \frac{4}{27}(|\mu|+1)^3 - (|\mu|+1) \right\},$$

$$D_4 = \left\{ (\mu, \nu) : |\mu| \geqslant 2, \ -\frac{2}{3}(|\mu|+1) \leqslant \nu \leqslant \frac{2|\mu|(|\mu|+1)}{\mu^2 + 2|\mu| + 4} \right\},$$

$$D_5 = \left\{ (\mu, \nu) : 2 \leqslant |\mu| \leqslant 4, \ \frac{2|\mu|(|\mu|+1)}{\mu^2 + 2|\mu| + 4} \leqslant \nu \leqslant \frac{1}{12}(\mu^2 + 8) \right\} \setminus \{(2,1)\},$$

$$D_6 = \left\{ (\mu, \nu) : 2 \leqslant |\mu| \leqslant 4, \ \nu \geqslant \frac{1}{12}(\mu^2 + 8) \right\}.$$

LEMMA 2. (See [8, 9]) *Let* $-2 \leqslant p_1 \leqslant 2$ *and* $p_2, p_3 \in \mathbb{C}$. *Then there exists a function* $\mathfrak{p} \in \mathscr{P}$ *with*

$$\mathfrak{p}(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{2.3}$$

if and only if

$$2p_2 = p_1^2 + (4 - p_1^2)x \tag{2.4}$$

and

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - (4 - p_1^2)p_1 x^2 + 2(4 - p_1^2)(1 - |x|^2)y \tag{2.5}$$

for some $x, y \in \mathbb{C}$ *with* $|x| \leqslant 1$ *and* $|y| \leqslant 1$.

LEMMA 3. (See [7]) *For given real numbers a,b,c, let*

$$Y(a,b,c) = \max_{z \in \mathbb{D}} \left(|a+bz+cz^2| + 1 - |z|^2 \right). \tag{2.6}$$

If $a \geqslant 0$ and $c \geqslant 0$, then

$$Y(a,b,c) = \begin{cases} a + |b| + c & (|b| \geqslant 2(1-c)), \\[2mm] 1 + a + \frac{b^2}{4(1-c)} & (|b| \leqslant 2(1-c)). \end{cases} \tag{2.7}$$

The maximum in the definition of $Y(a,b,c)$ is attained at $z = \pm 1$ in the first case according as $b = \pm|b|$.

LEMMA 4. (See [16]) *If $\mu(z) = 1 + \sum_{k=1}^{\infty} \mu_k z^k$ is subordinate to $\nu = 1 + \sum_{k=1}^{\infty} \nu_k z^k$ in $\mathbb{D}$, where $\nu(z)$ is univalent in $\mathbb{D}$ and $\nu(\mathbb{D})$ is convex, then*

$$|\mu_n| \leqslant |\nu_1| \quad (n \geqslant 1). \tag{2.8}$$

The proof of the following lemma is similar to that of [19, Lemma 2.2].

LEMMA 5. *Suppose that the sequence $\{A_m\}_{m=2}^{\infty}$ is defined by*

$$\begin{cases} A_m = \lambda & (m=2), \\[2mm] A_m = \dfrac{\lambda}{m-1}\left(1 + \sum_{k=2}^{m-1} A_k\right) & (m \geqslant 3). \end{cases} \tag{2.9}$$

Then

$$A_m = \frac{1}{(m-1)!} \prod_{k=0}^{m-2} (\lambda + k) \quad (m \geqslant 2). \tag{2.10}$$

Proof. From (2.9), we have

$$(m-1)A_m = \lambda \left(1 + \sum_{k=2}^{m-1} A_k\right) \tag{2.11}$$

and

$$mA_{m+1} = \lambda \left(1 + \sum_{k=2}^{m} A_k\right). \tag{2.12}$$

Combining (2.11) and (2.12), we find that

$$\frac{A_{m+1}}{A_m} = \frac{\lambda + m - 1}{m} \quad (m \geqslant 2). \tag{2.13}$$

Thus,

$$
\begin{aligned}
A_m &= \frac{A_m}{A_{m-1}} \cdot \frac{A_{m-1}}{A_{m-2}} \cdots \frac{A_3}{A_2} \cdot A_2 \\
&= \frac{\lambda + m - 2}{m - 1} \cdot \frac{\lambda + m - 3}{m - 2} \cdots \frac{\lambda + 1}{2} \cdot \lambda \\
&= \frac{1}{(m-1)!} \prod_{k=0}^{m-2} (\lambda + k) \quad (m \geqslant 3).
\end{aligned}
\tag{2.14}
$$

In conjunction with (2.9), we complete the proof of Lemma 5. $\square$

3. Main results

We first discuss the absolute values of the second, third and fourth coefficients of functions in the class $\mathscr{S}_{\lambda e}^{*}$.

THEOREM 1. *Suppose that* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}_{\lambda e}^{*}$. *Then*

$$
|a_2| \leqslant \lambda,
\tag{3.1}
$$

$$
|a_3| \leqslant
\begin{cases}
\frac{\lambda}{2} & \left(0 < \lambda \leqslant \frac{2}{3}\right), \\[2mm]
\frac{3\lambda^2}{4} & \left(\frac{2}{3} < \lambda \leqslant \frac{\pi}{2}\right),
\end{cases}
\tag{3.2}
$$

and

$$
|a_4| \leqslant
\begin{cases}
\frac{\lambda}{3} & (\lambda \in (0, r_0]), \\[3mm]
\frac{\lambda(5\lambda+2)}{9} \sqrt{\frac{2(5\lambda+2)}{17\lambda^2+30\lambda+12}} & (\lambda \in (r_0, r_1]), \\[3mm]
\frac{17\lambda(25\lambda^2-16)}{252} \sqrt{\frac{25\lambda^2-16}{17\lambda^2-12}} & \left(\lambda \in \left(r_1, \sqrt{\frac{32}{43}}\,\right]\right), \\[3mm]
\frac{17\lambda^3}{36} & \left(\lambda \in \left(\sqrt{\frac{32}{43}}, \frac{\pi}{2}\right]\right),
\end{cases}
\tag{3.3}
$$

where $r_0 \approx 0.7817$ *is the unique positive root of the equation*

$$
250x^3 + 147x^2 - 150x - 92 = 0
\tag{3.4}
$$

and $r_1 \approx 0.8602$ *is the unique positive root of the equation*

$$
425\lambda^3 + 340\lambda^2 - 328\lambda - 240 = 0.
\tag{3.5}
$$

All the bounds are sharp.

Proof. Let $f \in \mathscr{S}_{\lambda e}^{*}$. Then we can write (1.5) in terms of Schwarz function as

$$
\frac{z f'(z)}{f(z)} = e^{\lambda \omega(z)}.
$$

It follows from (1.1) that

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2{}^2)z^2 + (3a_4 - 3a_2 a_3 + a_2{}^3)z^3 + \cdots. \tag{3.6}$$

Suppose that

$$\omega(z) = \sum_{n=1}^{\infty} c_n z^n.$$

From the series expansion of $e^{\lambda \omega}$ along with some calculations, we get

$$e^{\lambda \omega(z)} = 1 + \lambda c_1 z + \left(\frac{\lambda^2}{2}c_1{}^2 + \lambda c_2\right)z^2 + \left(\frac{\lambda^3}{6}c_1^3 + \lambda^2 c_1 c_2 + \lambda c_3\right)z^3 + \cdots. \tag{3.7}$$

Comparing (3.6) with (3.7), we have

$$a_2 = \lambda c_1, \tag{3.8}$$

$$a_3 = \frac{\lambda}{2}\left(c_2 + \frac{3}{2}\lambda c_1^2\right), \tag{3.9}$$

and

$$a_4 = \frac{\lambda}{3}\left(c_3 + \frac{5}{2}\lambda c_1 c_2 + \frac{17}{12}\lambda^2 c_1^3\right). \tag{3.10}$$

Since ω is a Schwarz function, we have $|c_1| \leqslant 1$. Hence, we obtain

$$|a_2| \leqslant \lambda. \tag{3.11}$$

Using a result of Carleson [1] (see also [2]), we have $|c_2| \leqslant 1 - |c_1|^2$. By virtue of (3.9), we find that

$$|a_3| \leqslant \frac{\lambda}{2}\left[1 + (\frac{3}{2}\lambda - 1)|c_1|^2\right]. \tag{3.12}$$

Then the inequality (3.2) follows from (3.12) with $|c_1| \in [0, 1]$.

Suppose that $\mu = \frac{5}{2}\lambda$, $v = \frac{17}{12}\lambda^2$ with $0 < \lambda \leqslant \frac{1}{5}$. We see that $(\mu, v) \in D_1$. From Lemma 1, we have

$$\left|c_3 + \frac{5}{2}\lambda c_1 c_2 + \frac{17}{12}\lambda^2\right| \leqslant 1. \tag{3.13}$$

By means of (3.10), we obtain

$$|a_4| \leqslant \frac{\lambda}{3} \quad \left(0 < \lambda \leqslant \frac{1}{5}\right). \tag{3.14}$$

For $\frac{1}{5} < \lambda \leqslant \frac{4}{5}$, it is clear that $\frac{1}{2} < |\mu| \leqslant 2$ and $v \geqslant -\frac{2}{3}(|\mu| + 1)$. Since

$$v \leqslant \frac{4}{27}(|\mu| + 1)^3 - (|\mu| + 1)$$

is equivalent to

$$250\lambda^3 + 147\lambda^2 - 150\lambda - 92 \geqslant 0.$$

Let

$$g_1(x) = 250x^3 + 147x^2 - 150x - 92.$$

A numerical computation shows that the unique positive root of $g_1(x) = 0$ is $r_0 \approx 0.7817$. For $\frac{1}{5} < \lambda \leqslant r_0$, we have $g_1(\lambda) \leqslant 0$. It is clear that

$$\frac{4}{27}(|\mu| + 1)^3 - (|\mu| + 1) \leqslant v \leqslant 1.$$

Thus, we see $(\mu, v) \in D_2$ for $\frac{1}{5} < \lambda \leqslant r_0$. An application of Lemma 1 leads to

$$|a_4| \leqslant \frac{\lambda}{3} \quad \left(\frac{1}{5} < \lambda \leqslant r_0\right). \tag{3.15}$$

Combing (3.14) and (3.15), we know that

$$|a_4| \leqslant \frac{\lambda}{3} \quad (0 < \lambda \leqslant r_0). \tag{3.16}$$

When $r_0 < \lambda \leqslant \frac{4}{5}$, it can be seen that $(\mu, v) \in D_3$. Therefore, the sharp bound of $|a_4|$ is given by

$$|a_4| \leqslant \frac{\lambda(5\lambda + 2)}{9}\sqrt{\frac{2(5\lambda + 2)}{17\lambda^2 + 30\lambda + 12}} \quad \left(r_0 < \lambda \leqslant \frac{4}{5}\right). \tag{3.17}$$

Now, we suppose that $\frac{4}{5} < \lambda \leqslant \frac{\pi}{2}$. It is not hard to verify that $2 \leqslant \mu < 4$. Since $v \leqslant \frac{1}{12}(\mu^2 + 8)$ is equivalent to $\lambda^2 \leqslant \frac{32}{43}$, we have

$$v > \frac{1}{12}(\mu^2 + 8)$$

for $\lambda > \sqrt{\frac{32}{43}} \approx 0.8627$. This implies that $(\mu, v) \in D_6$ for $\lambda \in \left(\sqrt{\frac{32}{43}}, \frac{\pi}{2}\right]$. Using Lemma 1, yields to

$$|a_4| \leqslant \frac{17\lambda^3}{36} \quad \left(\sqrt{\frac{32}{43}} < \lambda \leqslant \frac{\pi}{2}\right). \tag{3.18}$$

For $\frac{4}{5} < \lambda \leqslant \sqrt{\frac{32}{43}}$, we have $v \leqslant \frac{1}{12}(\mu^2 + 8)$. Then

$$v \geqslant \frac{2|\mu|(|\mu| + 1)}{\mu^2 + 2|\mu| + 4}$$

is equivalent to

$$425\lambda^3 + 340\lambda^2 - 328\lambda - 240 \geqslant 0.$$

Now, we suppose that

$$g_2(x) = 425x^3 + 340x^2 - 328x - 240. \tag{3.19}$$

The numerical computation shows that the unique positive root of $g_2(x) = 0$ is $r_1 \approx 0.8602$. For $\frac{4}{5} < \lambda \leqslant r_1$, we have $g_2(\lambda) \leqslant 0$ and hence $v \leqslant \frac{2|\mu|(|\mu|+1)}{\mu^2 + 2|\mu| + 4}$. This implies that $(\mu, v) \in D_4$, thus, we have

$$|a_4| \leqslant \frac{\lambda(5\lambda + 2)}{9}\sqrt{\frac{2(5\lambda + 2)}{17\lambda^2 + 30\lambda + 12}} \quad \left(\frac{4}{5} < \lambda \leqslant r_1\right). \tag{3.20}$$

Combing (3.17) and (3.20), we obtain

$$|a_4| \leqslant \frac{\lambda(5\lambda+2)}{9}\sqrt{\frac{2(5\lambda+2)}{17\lambda^2+30\lambda+12}} \quad (r_0 < \lambda \leqslant r_1). \tag{3.21}$$

For $\lambda \in \left(r_1, \sqrt{\frac{32}{43}}\right]$, we have $g_2(\lambda) \geqslant 0$ and hence $v \geqslant \frac{2|\mu|(|\mu|+1)}{\mu^2+2|\mu|+4}$. Obviously, $\mu = \frac{5}{2}\lambda \neq 2$. Therefore, we know that $(\mu, v) \in D_5$ and

$$|a_4| \leqslant \frac{17(25\lambda^2-16)}{252}\sqrt{\frac{25\lambda^2-16}{17\lambda^2-12}} \quad \left(r_1 < \lambda \leqslant \sqrt{\frac{32}{43}}\right). \tag{3.22}$$

By virtue of (3.16), (3.18), (3.21) and (3.22), we obtain the bound of $|a_4|$ given by (3.3). This completes the proof of Theorem 1. $\square$

From the definitions of $\mathscr{S}_{\lambda e}^*$ and $\mathscr{K}_{\lambda e}$, we know that if $f \in \mathscr{K}_{\lambda e}$, then $zf' \in \mathscr{S}_{\lambda e}^*$. We thus get the following result.

THEOREM 2. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{K}_{\lambda e}$, *then*

$$|a_2| \leqslant \frac{\lambda}{2}, \tag{3.23}$$

$$|a_3| \leqslant \begin{cases} \frac{\lambda}{6} & \left(0 < \lambda \leqslant \frac{2}{3}\right), \\ \frac{\lambda^2}{4} & \left(\frac{2}{3} < \lambda \leqslant \frac{\pi}{2}\right), \end{cases} \tag{3.24}$$

and

$$|a_4| \leqslant \begin{cases} \frac{\lambda}{12} & (\lambda \in (0, r_0]), \\ \frac{\lambda(5\lambda+2)}{36}\sqrt{\frac{2(5\lambda+2)}{17\lambda^2+30\lambda+12}} & (\lambda \in (r_0, r_1]), \\ \frac{17\lambda(25\lambda^2-16)}{1008}\sqrt{\frac{25\lambda^2-16}{17\lambda^2-12}} & \left(\lambda \in \left(r_1, \sqrt{\frac{32}{43}}\right]\right), \\ \frac{17\lambda^3}{144} & \left(\lambda \in \left(\sqrt{\frac{32}{43}}, \frac{\pi}{2}\right]\right), \end{cases} \tag{3.25}$$

where $r_0 \approx 0.7817$ *and* $r_1 \approx 0.8602$ *are the unique positive root of the equation* (3.4) *and* (3.5), *respectively. All these bounds are sharp.*

From Theorem 1, we know that $\frac{1}{2}|f''(0)| = |a_2| \leqslant \lambda$ for $f = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}_{\lambda e}^*$. Now, let

$$\mathscr{S}_{\lambda e}^*(\hat{p}) := \{f \in \mathscr{A} : f \in \mathscr{S}_{\lambda e}^*; f''(0) = \hat{p}\}, \tag{3.26}$$

where $\hat{p}$ is a given real number satisfying $-2\lambda \leqslant \hat{p} \leqslant 2\lambda$.

In what follows, we will discuss the difference of initial successive coefficients for functions in $\mathscr{S}_{\lambda e}^*(\hat{p})$.

THEOREM 3. *Let $0 \leqslant \hat{p} \leqslant 2\lambda$ and $p = \hat{p}/\lambda$. Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}_{\lambda e}^*(\hat{p})$. Then the following sharp inequalities*

$$|a_3 - a_2| \leqslant \frac{\lambda}{16} \left(p|3\lambda p - 8| + 8 - 2p^2 \right). \tag{3.27}$$

and

$$|a_4 - a_3| \leqslant \begin{cases} \Psi_1(\lambda, p) \;\; \left(0 < \lambda \leqslant \frac{3}{5}\right), \\[2mm] \Psi_2(\lambda, p) \;\; \left(\frac{3}{5} < \lambda \leqslant \frac{\pi}{2}\right), \end{cases} \tag{3.28}$$

hold, where

$$\Psi_1(\lambda, p) := \begin{cases} \frac{\lambda}{1152}\left[7\lambda^2 p^3 + (150\lambda^2 + 36\lambda - 96)p^2 + (108 - 360\lambda)p + 600\right] \\ \qquad \left(0 \leqslant p \leqslant \frac{2}{4-5\lambda}\right), \\[3mm] \frac{\lambda}{288}\left[(-17\lambda^2 + 30\lambda - 12)p^3 + (54\lambda - 36)p^2 + (48 - 120\lambda)p + 144\right] \\ \qquad \left(\frac{2}{4-5\lambda} < p \leqslant 2\right), \end{cases}$$

$$\Psi_2(\lambda, p) := \begin{cases} \frac{\lambda}{1152}\left[7\lambda^2 p^3 + (150\lambda^2 + 36\lambda - 96)p^2 + (108 - 360\lambda)p + 600\right] \\ \qquad \left(0 \leqslant p \leqslant \frac{14}{4+5\lambda}\right), \\[3mm] \frac{\lambda}{288}\left[(-17\lambda^2 - 30\lambda - 12)p^3 + (54\lambda + 36)p^2 + (48 + 120\lambda)p - 144\right] \\ \qquad \left(\frac{14}{4+5\lambda} < p \leqslant 2\right). \end{cases}$$

Proof. Let $f \in \mathscr{S}_{\lambda e}^*(\hat{p})$. In terms of Schwarz function, we can write (1.5) as

$$\frac{zf'(z)}{f(z)} = e^{\lambda \omega(z)}.$$

We define $\rho \in \mathscr{P}$ by the Schwarz function as

$$\rho(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + p_1 z + p_2 z^2 + \cdots. \tag{3.29}$$

From (3.29), we obtain

$$\begin{aligned} \omega(z) &= \frac{\rho(z) - 1}{\rho(z) + 1} \\ &= \frac{1}{2}p_1 z + \left(\frac{1}{2}p_2 - \frac{1}{4}p_1^2\right)z^2 + \left(\frac{1}{2}p_3 - \frac{1}{2}p_1 p_2 + \frac{1}{8}p_1^3\right)z^3 + \cdots. \end{aligned} \tag{3.30}$$

From the series expansion of ω along with some calculations, we have

$$\begin{aligned} e^{\lambda \omega(z)} = 1 &+ \frac{\lambda}{2}p_1 z + \left(\frac{\lambda}{2}p_2 + \frac{\lambda^2 - 2\lambda}{8}p_1^2\right)z^2 \\ &+ \left(\frac{\lambda^3 - 6\lambda^2 + 6\lambda}{48}p_1^3 + \frac{\lambda^2 - 2\lambda}{4}p_1 p_2 + \frac{\lambda}{2}p_3\right)z^3 + \cdots. \tag{3.31} \end{aligned}$$

Comparing (3.6) with (3.31), we get

$$a_2 = \frac{1}{2}\lambda p_1, \tag{3.32}$$

$$a_3 = \frac{1}{4}\lambda \left(p_2 + \frac{3\lambda - 2}{4}p_1^2 \right), \tag{3.33}$$

and

$$a_4 = \frac{1}{6}\lambda \left(p_3 + \frac{5\lambda - 4}{4}p_1 p_2 + \frac{17\lambda^2 - 30\lambda + 12}{48}p_1^3 \right). \tag{3.34}$$

Let $p = \hat{p}/\lambda$. It is obvious that $p \in [0,2]$. Since $2a_2 = f''(0) = \hat{p}$, from (3.32), we get

$$p_1 = \frac{1}{\lambda}\hat{p} = p. \tag{3.35}$$

By Lemma 2, we obtain

$$p_2 = \frac{1}{2}p^2 + \frac{1}{2}(4 - p^2)x, \tag{3.36}$$

and

$$p_3 = \frac{1}{4}p^3 + \frac{1}{2}(4 - p^2)px - \frac{1}{4}(4 - p^2)px^2 + \frac{1}{2}(4 - p^2)(1 - |x|^2)y, \tag{3.37}$$

where $x, y \in \mathbb{C}$ with $|x| \leqslant 1$, $|y| \leqslant 1$. Substituting (3.35), (3.36) and (3.37) into (3.32), (3.33) and (3.34), respectively, we obtain

$$a_2 = \frac{1}{2}\lambda p, \tag{3.38}$$

$$a_3 = \frac{3}{16}\lambda^2 p^2 + \frac{1}{8}\lambda(4 - p^2)x, \tag{3.39}$$

and

$$a_4 = \frac{17}{288}\lambda^3 p^3 + \frac{5}{48}\lambda^2(4 - p^2)px - \frac{1}{24}\lambda(4 - p^2)x^2 + \frac{1}{12}\lambda(4 - p^2)(1 - |x|^2)y \tag{3.40}$$

for some $x, y \in \mathbb{C}$ with $|x| \leqslant 1$, $|y| \leqslant 1$. For $p \in \left[0, \frac{8}{3\lambda}\right]$, we have

$$
\begin{aligned}
|a_3 - a_2| &= \left| \frac{3}{16}\lambda^2 p^2 + \frac{1}{8}\lambda(4 - p^2)x - \frac{1}{2}\lambda p \right| \\
&\leqslant \frac{1}{2}\lambda p - \frac{3}{16}\lambda^2 p^2 + \frac{1}{8}\lambda(4 - p^2) \\
&= \frac{1}{2}\lambda + \frac{1}{2}\lambda p - \frac{3\lambda + 2}{16}\lambda p^2,
\end{aligned} \tag{3.41}
$$

where equality occurs if $x = -1$. Similarly, we have

$$|a_3 - a_2| \leqslant \frac{1}{2}\lambda - \frac{1}{2}\lambda p + \frac{3\lambda - 2}{16}\lambda p^2 \tag{3.42}$$

for $p \in \left(\frac{8}{3\lambda}, 2\right]$. Hence, the inequality (3.27) follows from (3.41) and (3.42). For $p = 2$, we easily obtain

$$|a_4 - a_3| = \frac{1}{36}\lambda^2(27 - 17\lambda). \tag{3.43}$$

For $p \in [0,2)$, we have

$$|a_4 - a_3|$$

$$= \left| \frac{17}{288}\lambda^3 p^3 + \frac{5}{48}\lambda^2(4-p^2)px - \frac{1}{24}\lambda(4-p^2)px^2 + \frac{1}{12}\lambda(4-p^2)(1-|x|^2)y \right.$$

$$\left. - \frac{3}{16}\lambda^2 p^2 - \frac{1}{8}\lambda(4-p^2)x \right|$$

$$= \frac{1}{288}\left| 17\lambda^3 p^3 - 54\lambda^2 p^2 + 6\lambda(5\lambda p - 6)(4-p^2)x - 12\lambda(4-p^2)px^2 \right.$$

$$\left. + 24\lambda(4-p^2)(1-|x|^2)y \right|$$

$$\leqslant \frac{1}{12}\lambda(4-p^2)\left[\left| \frac{(17\lambda p - 54)\lambda p^2}{24(4-p^2)} + \frac{5\lambda p - 6}{4}x - \frac{p}{2}x^2 \right| + 1 - |x|^2 \right]$$

$$\leqslant \frac{1}{12}\lambda(4-p^2)Y(a,b,c),$$

$$(3.44)$$

where $Y(a,b,c)$ is given in (2.6) and

$$a = \frac{(54 - 17\lambda p)\lambda p^2}{24(4-p^2)}, \quad b = \frac{6 - 5\lambda p}{4}, \quad c = \frac{p}{2}. \qquad (3.45)$$

Since $\lambda \in (0, \frac{\pi}{2}]$, $p \in [0,2)$, we have $54 - 17\lambda p > 0$ and hence $a > 0$. Let $0 < \lambda \leqslant \frac{3}{5}$. Clearly, we have $\lambda p < \frac{6}{5}$. Then it can be verified that $|b| \leqslant 2(1-c)$ is equivalent to $0 \leqslant p \leqslant \frac{2}{4-5\lambda}$. By Lemma 3, we get

$$Y(a,b,c) = \begin{cases} \frac{7\lambda^2 p^3 + (150\lambda^2 + 36\lambda - 96)p^2 + (108 - 360\lambda)p + 600}{96(4-p^2)} & \left(0 \leqslant p \leqslant \frac{2}{4-5\lambda} \right), \\[3mm] \frac{(-17\lambda^2 + 30\lambda - 12)p^3 + (54\lambda - 36)p^2 + (48 - 120\lambda)p + 144}{24(4-p^2)} & \left(\frac{2}{4-5\lambda} < p < 2 \right). \end{cases}$$

Thus, we obtain

$$|a_4 - a_3| \leqslant \begin{cases} \frac{\lambda}{1152}\left[7\lambda^2 p^3 + (150\lambda^2 + 36\lambda - 96)p^2 + (108 - 360\lambda)p + 600 \right] \\ \qquad \left(0 \leqslant p \leqslant \frac{2}{4-5\lambda} \right), \\[3mm] \frac{\lambda}{288}\left[(-17\lambda^2 + 30\lambda - 12)p^3 + (54\lambda - 36)p^2 + (48 - 120\lambda)p + 144 \right] \\ \qquad \left(\frac{2}{4-5\lambda} < p < 2 \right). \end{cases}$$

$$(3.46)$$

Now, we suppose that $\frac{3}{5} < \lambda \leqslant \frac{\pi}{2}$. For $\lambda p \leqslant \frac{6}{5}$, we see that $|b| \leqslant 2(1-c)$ is equivalent to

$$(4 - 5\lambda)p \leqslant 2, \qquad (3.47)$$

it clearly holds for $\lambda \geqslant \frac{4}{5}$. For $\lambda \in \left(\frac{3}{5}, \frac{4}{5} \right)$, (3.47) is equivalent to

$$p \leqslant \frac{2}{4 - 5\lambda} \in (2, +\infty),$$

which always holds for $p \in [0,2)$. Therefore, we have $|b| \leqslant 2(1-c)$ provided that $p \leqslant \frac{6}{5\lambda}$. If $\lambda p > \frac{6}{5}$, a simple calculation shows that $|b| \leqslant 2(1-c)$ is equivalent to $p \leqslant \frac{14}{4+5\lambda}$. Since $\frac{14}{4+5\lambda} > \frac{6}{5\lambda}$ for $\lambda \in \left(\frac{3}{5}, \frac{\pi}{2} \right]$. Hence, we know that $|b| \leqslant 2(1-c)$ for $\frac{6}{5\lambda} < p \leqslant \frac{14}{4+5\lambda}$. From the above discussion, we obtain $|b| \leqslant 2(1-c)$ if $p \in \left[0, \frac{14}{4+5\lambda} \right]$. Also, it is clear that $|b| > 2(1-c)$ if and only if

$$ p > \max \left\{ \frac{14}{4+5\lambda}, \frac{6}{5\lambda} \right\} = \frac{14}{4+5\lambda}. $$

Now, an application of Lemma 3 leads to

$$ Y(a,b,c) = \begin{cases} \frac{7\lambda^2 p^3 + (150\lambda^2 + 36\lambda - 96)p^2 + (108 - 360\lambda)p + 600}{96(4-p^2)} & \left(0 \leqslant p \leqslant \frac{14}{4+5\lambda} \right), \\[2ex] \frac{(-17\lambda^2 - 30\lambda - 12)p^3 + (54\lambda + 36)p^2 + (120\lambda + 48)p - 144}{24(4-p^2)} & \left(\frac{14}{4+5\lambda} < p < 2 \right). \end{cases} $$

From (3.44), we deduce that

$$ |a_4 - a_3| \leqslant \begin{cases} \frac{\lambda}{1152} \left[7\lambda^2 p^3 + (150\lambda^2 + 36\lambda - 96)p^2 + (108 - 360\lambda)p + 600 \right] \\ \qquad \left(0 \leqslant p \leqslant \frac{14}{4+5\lambda} \right), \\[2ex] \frac{\lambda}{288} \left[(-17\lambda^2 - 30\lambda - 12)p^3 + (54\lambda + 36)p^2 + (120\lambda + 48)p - 144 \right] \\ \qquad \left(\frac{14}{4+5\lambda} < p < 2 \right). \end{cases} \tag{3.48} $$

Combining (3.43), (3.46) with (3.48), we obtain (3.28). This completes the proof of Theorem 3. □

Taking $\lambda = 1$ in Theorem 3, we obtain the following result.

COROLLARY 1. *Let* $0 \leqslant p \leqslant 2$ *and* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}_e^*(p)$. *Then*

$$ |a_3 - a_2| \leqslant \frac{1}{16}(-5p^2 + 8p + 8), \tag{3.49} $$

and

$$ |a_4 - a_3| \leqslant \begin{cases} \frac{1}{1152}(7p^3 + 90p^2 - 252p + 600) & \left(0 \leqslant p \leqslant \frac{14}{9} \right), \\[2ex] \frac{1}{288}(-59p^3 + 90p^2 + 168p - 144) & \left(\frac{14}{9} \leqslant p \leqslant 2 \right). \end{cases} \tag{3.50} $$

All inequalities are sharp.

If we denote by

$$ \mathscr{S}_e^*(+) = \bigcup_{0 \leqslant p \leqslant 2} \mathscr{S}_e^*(p) = \{ f : f \in \mathscr{S}_e^*; f''(0) = p \}, $$

then in view of (3.49) and (3.50), we easily obtain

$$ \sup_{f \in \mathscr{S}_e^*(+)} |a_3(f) - a_2(f)| = \frac{7}{10}, \tag{3.51} $$

and

$$\sup_{f \in \mathscr{S}_e^*(+)} |a_4(f) - a_3(f)| = \frac{25}{48}. \tag{3.52}$$

From Theorem 2, we know that for $f \in \mathscr{K}_{\lambda e}$, $\left|\frac{1}{2}f''(0)\right| = |a_2(f)| \leqslant \frac{1}{2}\lambda$. Denote by

$$\mathscr{K}_{\lambda e}(\hat{p}) = \{f \in \mathscr{K}_{\lambda e}, f''(0) = \hat{p}\}, \tag{3.53}$$

where $\hat{p}$ is a given real number with $-\lambda \leqslant \hat{p} \leqslant \lambda$.

In what follows, we will discuss the difference of initial successive coefficients for functions belonging to the class $\mathscr{K}_{\lambda e}(\hat{p})$.

THEOREM 4. *Let $0 \leqslant \hat{p} \leqslant \lambda$ and $p = \hat{p}/\lambda$. Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in the class $\mathscr{K}_{\lambda e}(\hat{p})$. Then the following sharp inequalities*

$$|a_3 - a_2| \leqslant \frac{1}{12}\lambda \left[2 + 6p - (3\lambda + 2)p^2\right], \tag{3.54}$$

and

$$|a_4 - a_3| \leqslant \begin{cases} \Theta_1(\lambda, p) \ \left(0 < \lambda < \frac{4}{5}\right), \\ \\ \Theta_2(\lambda, p) \ \left(\frac{4}{5} \leqslant \lambda \leqslant \frac{\pi}{2}\right), \end{cases} \tag{3.55}$$

hold, where

$$\Theta_1(\lambda, p) := \frac{1}{144}\lambda \left[(-17\lambda^2 + 30\lambda - 12)p^3 + (36\lambda - 24)p^2 + (12 - 30\lambda)p + 24\right],$$

$$\Theta_2(\lambda, p) := \begin{cases} \frac{1}{576}\lambda \left[7\lambda^2 p^3 + (75\lambda^2 + 24\lambda - 48)p^2 + (48 - 120\lambda)p + 96\right] \\ \left(0 \leqslant p \leqslant \frac{8}{4+5\lambda}\right), \\ \\ \frac{1}{144}\lambda \left[(-17\lambda^2 - 30\lambda - 12)p^3 + (36\lambda + 24)p^2 + (30\lambda + 12)p - 24\right] \\ \left(\frac{8}{4+5\lambda} < p \leqslant 1\right). \end{cases} \tag{3.56}$$

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{K}_{\lambda e}^*(p)$. By the Alexander relation, from (3.32), (3.33) and (3.34) we deduce that

$$a_2 = \frac{\lambda}{4}p_1, \tag{3.57}$$

$$a_3 = \frac{1}{12}\lambda \left(p_2 + \frac{3\lambda - 2}{4}p_1^2\right), \tag{3.58}$$

and

$$a_4 = \frac{1}{24}\lambda \left(p_3 + \frac{5\lambda - 4}{4}p_1 p_2 + \frac{17\lambda^2 - 30\lambda + 12}{48}p_1^3\right). \tag{3.59}$$

Let $p = \hat{p}/\lambda$, it is clear that $p \in [0, 1]$. Since $2a_2 = f''(0) = \hat{p}$, in view of (3.57), we get

$$p_1 = \frac{2}{\lambda}\hat{p} = 2p. \tag{3.60}$$

By Lemma 2, we obtain

$$p_2 = 2p^2 + 2(1-p^2)x, \tag{3.61}$$

and

$$p_3 = 2p^3 + 4(1-p^2)px - 2(1-p^2)px^2 + 2(1-p^2)(1-|x|^2)y, \tag{3.62}$$

where $x, y \in \mathbb{C}$ with $|x| \leqslant 1$, $|y| \leqslant 1$. Substituting (3.60), (3.61) and (3.62) into (3.57), (3.58) and (3.59), respectively, we have

$$a_2 = \frac{\lambda}{2}p, \tag{3.63}$$

$$a_3 = \frac{1}{4}\lambda^2 p^2 + \frac{1}{6}\lambda(1-p^2)x, \tag{3.64}$$

and

$$a_4 = \frac{17}{144}\lambda^3 p^3 + \frac{5}{24}\lambda^2(1-p^2)px - \frac{1}{12}\lambda(1-p^2)px^2 + \frac{1}{12}\lambda(1-p^2)(1-|x|^2)y \tag{3.65}$$

for some $x, y \in \mathbb{C}$ and $|x| \leqslant 1$, $|y| \leqslant 1$. Thus, we find that

$$
\begin{aligned}
|a_3 - a_2| &= \left| \frac{1}{4}\lambda^2 p^2 + \frac{1}{6}\lambda(1-p^2)x - \frac{1}{2}\lambda p \right| \\
&= \left| \frac{1}{4}\lambda p(\lambda p - 2) + \frac{1}{6}\lambda(1-p^2)x \right| \\
&\leqslant \frac{1}{6}\lambda + \frac{1}{2}\lambda p - \frac{3\lambda+2}{12}\lambda p^2,
\end{aligned}
\tag{3.66}
$$

where equality occurs if $x = -1$. The inequality (3.54) in Theorem 4 follows from (3.66). For $p = 1$, we easily obtain

$$|a_4 - a_3| = \frac{1}{144}\lambda^2(36 - 17\lambda). \tag{3.67}$$

For $p \in [0,1)$, we have

$$
\begin{aligned}
&|a_4 - a_3| \\
&= \left| \frac{17}{144}\lambda^3 p^3 + \frac{5}{24}\lambda^2(1-p^2)px - \frac{1}{12}\lambda(1-p^2)px^2 + \frac{1}{12}\lambda(1-p^2)(1-|x|^2)y \right. \\
&\quad \left. -\frac{1}{4}\lambda^2 p^2 - \frac{1}{6}\lambda(1-p^2)x \right| \\
&= \frac{1}{144}\lambda \left| (17\lambda p - 36)\lambda p^2 + 6(5\lambda p - 4)(1-p^2)x - 12(1-p^2)px^2 \right. \\
&\quad \left. + 12(1-p^2)(1-|x|^2)y \right| \\
&\leqslant \frac{1}{12}\lambda(1-p^2)\left[\left| \frac{(17\lambda p - 36)\lambda p^2}{12(1-p^2)} + \frac{5\lambda p - 4}{2}x - px^2 \right| + 1 - |x|^2 \right] \\
&\leqslant \frac{1}{12}\lambda(1-p^2)Y(a,b,c),
\end{aligned}
$$

where $Y(a,b,c)$ is given in (2.6) and

$$a = \frac{(36 - 17\lambda p)\lambda p^2}{12(1 - p^2)}, \quad b = \frac{4 - 5\lambda p}{2}, \quad c = p.$$

For $\lambda \in (0, \frac{\pi}{2}]$, $p \in [0,1)$, it can be seen that $a > 0$. Since $|b| \leqslant 2(1-c)$ is equivalent to $b^2 \leqslant 4(1-c)^2$, we see that $|b| \leqslant 2(1-c)$ if and only if

$$(4 - 5\lambda)\left(1 - \frac{5\lambda + 4}{8}p\right) \leqslant 0. \tag{3.68}$$

Clearly, (3.68) holds only if $\lambda \geqslant \frac{4}{5}$. This means that $|b| > 2(1-c)$ for all $\lambda \in \left(0, \frac{4}{5}\right)$. Let $\lambda \in \left(0, \frac{4}{5}\right)$, by using Lemma 3, we see that

$$Y(a,b,c) = \frac{(-17\lambda^2 + 30\lambda - 12)p^3 + (36\lambda - 24)p^2 + (12 - 30\lambda)p + 24}{12(1 - p^2)}.$$

This induces that

$$|a_4 - a_3| \leqslant \frac{1}{144}\lambda\left[(-17\lambda^2 + 30\lambda - 12)p^3 + (36\lambda - 24)p^2 + (12 - 30\lambda)p + 24\right]. \tag{3.69}$$

Now, we suppose that $\frac{4}{5} \leqslant \lambda \leqslant \frac{\pi}{2}$. From (3.68), we obtain $|b| \leqslant 2(1-c)$ if and only if $p \in \left[0, \frac{8}{4+5\lambda}\right]$. By noting that $\frac{8}{4+5\lambda} \geqslant \frac{4}{5\lambda}$ for $\lambda \in \left[\frac{4}{5}, \frac{\pi}{2}\right]$, an application of Lemma 3 yields

$$Y(a,b,c) = \begin{cases} \frac{7\lambda^2 p^3 + (75\lambda^2 + 24\lambda - 48)p^2 + (48 - 120\lambda)p + 96}{48(1 - p^2)} & \left(0 \leqslant p \leqslant \frac{8}{4+5\lambda}\right), \\ \frac{(-17\lambda^2 - 30\lambda - 12)p^3 + (36\lambda + 24)p^2 + (30\lambda + 12)p - 24}{12(1 - p^2)} & \left(\frac{8}{4+5\lambda} < p < 1\right). \end{cases}$$

Thus, we deduce that

$$|a_4 - a_3| \leqslant \begin{cases} \frac{1}{576}\lambda\left[7\lambda^2 p^3 + (75\lambda^2 + 24\lambda - 48)p^2 + (48 - 120\lambda)p + 96\right] \\ \left(0 \leqslant p \leqslant \frac{8}{4+5\lambda}\right), \\ \frac{1}{144}\lambda\left[(-17\lambda^2 - 30\lambda - 12)p^3 + (36\lambda + 24)p^2 + (30\lambda + 12)p - 24\right] \\ \left(\frac{8}{4+5\lambda} < p < 1\right). \end{cases} \tag{3.70}$$

Combining (3.67), (3.69) with (3.70), we obtain the inequality (3.55) in Theorem 4. The proof is thus completed. $\quad\square$

By choosing $\lambda = 1$ in Theorem 4, we obtain the following result.

COROLLARY 2. *Let* $0 \leqslant p \leqslant 1$. *Suppose that* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in the class* $\mathscr{K}_e(p)$. *Then the following sharp inequalities*

$$|a_3 - a_2| \leqslant \frac{1}{12}(-5p^2 + 6p + 2), \tag{3.71}$$

and

$$|a_4 - a_3| \leqslant \begin{cases} \frac{1}{576}\left(7p^3 + 51p^2 - 72p + 96\right) & \left(0 \leqslant p \leqslant \frac{8}{9}\right), \\ \frac{1}{144}\left(-59p^3 + 60p^2 + 42p - 24\right) & \left(\frac{8}{9} < p \leqslant 1\right), \end{cases} \tag{3.72}$$

hold.

Now, we denote by

$$\mathscr{K}_e^*(+) = \bigcup_{0 \leqslant p \leqslant 1} \mathscr{K}_e(p) = \left\{ f : f \in \mathscr{K}_e; f''(0) = p \right\}. \tag{3.73}$$

By virtue of (3.71) and (3.72), we easily find that

$$\sup_{f \in \mathscr{K}_e^*(+)} |a_3(f) - a_2(f)| = \frac{19}{60}, \tag{3.74}$$

and

$$\sup_{f \in \mathscr{K}_e^*(+)} |a_4(f) - a_3(f)| = \frac{1}{6}. \tag{3.75}$$

Finally, we will give the upper bounds of $|a_n|\,(n \geqslant 2)$ for functions in the class $\mathscr{S}_{\lambda e}^*$ and $\mathscr{K}_{\lambda e}$ for $\lambda \in (0,1]$. However, they are not always sharp.

THEOREM 5. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}_{\lambda e}^* (0 < \lambda \leqslant 1)$, *then*

$$|a_n| \leqslant \frac{1}{(n-1)!} \prod_{k=0}^{n-2} (\lambda + k) \quad (n \geqslant 2). \tag{3.76}$$

Proof. Let

$$\psi(z) = \frac{z f'(z)}{f(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

Since $f \in \mathscr{S}_{\lambda e}^*$, we know that

$$\psi(z) \prec \chi(z) = e^{\lambda z} = 1 + \lambda z + \frac{\lambda^2}{2!} z^2 + \cdots.$$

Note that $\chi(z)$ is univalent and convex in $\mathbb{D}$ for $0 < \lambda \leqslant 1$, by Lemma 4, we obtain

$$|c_n| \leqslant \lambda \quad (n \geqslant 1).$$

In view of $z f'(z) = \psi(z) f(z)$, by comparing the coefficients of z^n on both sides, it follows that

$$a_2 = c_1$$

and

$$(n-1)a_n = c_{n-1} + \sum_{k=2}^{n-1} c_{n-k} a_k \quad (n \geqslant 3).$$

Thus, we have

$$|a_2| = |c_1| \leqslant \lambda,$$

and

$$|a_n| \leqslant \frac{\lambda}{n-1}\left(1 + \sum_{k=2}^{n-1}|a_k|\right) \quad (n \geqslant 3). \tag{3.77}$$

Now, we define the sequence $\{A_m\}_{m=2}^{\infty}$ as follows:

$$\begin{cases} A_m = \lambda & (m = 2), \\ A_m = \dfrac{\lambda}{m-1}\left(1 + \displaystyle\sum_{k=2}^{m-1} A_k\right) & (m \geqslant 3). \end{cases} \tag{3.78}$$

In order to prove that

$$|a_m| \leqslant A_m \quad (m \geqslant 2), \tag{3.79}$$

we use the principle of mathematical induction. It is easy to verify that

$$|a_2| \leqslant A_2 = \lambda. \tag{3.80}$$

Thus, assuming that

$$|a_l| \leqslant A_l \quad (l = 2, 3, \ldots, m), \tag{3.81}$$

we find from (3.77) and (3.81) that

$$\begin{aligned} |a_{m+1}| &\leqslant \frac{\lambda}{m}\left(1 + \sum_{k=2}^{m}|a_k|\right) \\ &\leqslant \frac{\lambda}{m}\left(1 + \sum_{k=2}^{m} A_k\right) = A_{m+1}. \end{aligned} \tag{3.82}$$

Therefore, by the principle of mathematical induction, we have

$$|a_m| \leqslant A_m \quad (m \geqslant 2). \tag{3.83}$$

By means of Lemma 5 and (3.78), we see that

$$A_m = \frac{1}{(m-1)!}\prod_{k=0}^{m-2}(\lambda + k) \quad (m \geqslant 2). \tag{3.84}$$

Combining (3.83) with (3.84), we readily get the coefficient estimates (3.76) asserted by Theorem 5. $\square$

According to the relationship between the classes $\mathscr{S}_{\lambda e}^{*}$ and $\mathscr{K}_{\lambda e}$, we easily obtain the follow result.

THEOREM 6. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{K}_{\lambda e}\,(0 < \lambda \leqslant 1)$, *then*

$$|a_n| \leqslant \frac{1}{n!}\prod_{k=0}^{n-2}(\lambda + k) \quad (n \geqslant 2).$$

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Lei Shi
School of Mathematics and Statistics
Anyang Normal University
Anyang 455002, Henan, P. R. China
e-mail: shimath@163.com

Zhi-Gang Wang
School of Mathematics and Computing Science
Hunan First Normal University
Changsha 410205, Hunan, P. R. China
e-mail: wangmath@163.com

Ren-Li Su
School of Mathematics and Statistics
Changsha University of Science and Technology
Hunan Provincial Key Laboratory of Mathematical Modeling and
Analysis in Engineering
Changsha 410114, Hunan, P. R. China
e-mail: 1579206189@qq.com

Muhammad Arif
Department of Mathematics
Abdul Wali Khan University
Mardan 23200, Pakistan
e-mail: marimaths @ awkum.edu.pk

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www.ele-math.com
jmi@ele-math.com