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INEQUALITIES AND CONVEXITY PROPERTIES FOR THE WEIGHTED EXPONENTIAL–BETA FUNCTION

FARZAD KHOSROWSHAHI AND SILVESTRU SEVER DRAGOMIR

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Abstract. In this paper we establish several analytic inequalities and convexity properties for the Weighted Exponential-Beta function

$$F(\alpha,\beta;\gamma) := \int_0^1 \exp\left[\gamma x^{\alpha}(1-x)^{\beta}\right] dx,$$

where α, β and γ are positive numbers.

1. Introduction

Construction industry is notoriously recognized for high level of liquidation. To this end, timely availability of the necessary capital is critical for the success of the project and possibly the organization, as cash-flow mishaps account for much of the failures of contractors. The graphical representation of the incurred cumulative expenditure of construction projects typically assumes the familiar S-curve, which inherits the characteristics of an exponential behavior. The history of systematic and scientific approach to construction cash-flow forecasting dates back to 1970s. The subsequent generations of models can be categorized into nomothetic vs idiographic (see [7]), stochastic vs deterministic, elemental vs mathematical and parametric. The elemental approach places the emphasis on the exact details of quantity, rates and timing of expenditure of all construction elements and activities. The time and cost overhead of this approach, in conjunction with lack of evidence on the accuracy led to a surge of alternative approaches in the 1980s. In particular, easy, fast, cheap and pragmatic solutions gained popularity. This trend paved the way for the development of mathematical models.

Extensive analysis of project expenditure patterns has revealed that the main features of the shape Y_C of the project periodic expenditure pattern are defined in terms of a number of variables, which can be simulated using the following expression (see, [8]):

$$Y_C := \exp\left[bx^{a}(1-x)^{d}\right] - 1.$$

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Here

$$x_p := R = \frac{a}{a+d} \text{ and } y_p := Q = \exp\left[bR^a(1-R)^d\right] - 1,$$

$Q,\ R$, represent the positions of the project expenditure peak on both the cost and time access, a and b are parameterized in terms of x_p and y_p as follows

$$a = \frac{x_p d}{1-x_p},\ b = \frac{\ln(1+y_p)}{x_p^a(1-x_p)^d}$$

and the parameter d is calculated through numerical method that is derived to rapidly converge towards a solution within desired error tolerance.

A relationship is established between the properties of the project and the physical shape of the project expenditure pattern. These are then related and reflected on the mathematical expression through its parameters.

Motivated by the above considerations, in this paper we consider the three-parameter family of functions introduced in [6]

$$f_{\alpha,\beta,\gamma}(x) := \exp\left[\gamma x^{\alpha}(1-x)^{\beta}\right],\ x \in [0,1],\ \alpha\ ,\beta,\ \gamma \geqslant 0$$

and the "*Weighted Exponential-Beta*" (WEB) function defined by the integral

$$F(\alpha,\beta;\gamma) := \int_0^1 f_{\alpha,\beta,\gamma}(x)\,dx = \int_0^1 \exp\left[\gamma x^{\alpha}(1-x)^{\beta}\right] dx,\ \alpha,\ \beta,\ \gamma \geqslant 0.$$

In the same paper [6] we obtained the following representation for the generating function $f_{\alpha,\beta,\gamma}$:

THEOREM 1. *Let* $\alpha,\ \beta,\ \gamma > 0$, *then we have function series expansion*

$$f_{\alpha,\beta,\gamma}(x) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!}\gamma^k x^{\alpha k}(1-x)^{\beta k} \tag{1.1}$$

uniformly on the interval $[0,1]$.

As an important consequence, we also have the following series expansion for the WEB-function:

COROLLARY 1. *We have the Beta-Taylor series expansion*

$$F(\alpha,\beta;\gamma) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!}\gamma^k B(\alpha k+1, \beta k+1) \tag{1.2}$$

for all $\alpha,\ \beta,\ \gamma > 0$.

We recall that the *Beta function*, also called the *Euler integral of the first kind*, is a special function defined by

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt,\ \alpha > 0,\ \beta > 0. \tag{1.3}$$

For some various inequalities for the Beta function see the survey paper [4].

In this paper we establish several analytic inequalities and convexity properties for the generating function $f_{\alpha,\beta,\gamma}$ and the Weighted Exponential-Beta function $F(\alpha,\beta;\gamma)$ where $\alpha,\ \beta$ and γ are positive numbers.

2. Some analytic inequalities for $f_{\alpha,\beta,\gamma}$

We start with the following fact:

THEOREM 2. *Let* $\alpha,\ \beta,\ \gamma>0$. *For any* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *we have*

$$0\leqslant f_{\alpha,\beta,\gamma}(x)-1\leqslant\left[\exp\left(\gamma x^{\alpha p}\right)-1\right]^{1/p}\left[\exp\left(\gamma(1-x)^{q\beta}\right)-1\right]^{1/q} \tag{2.1}$$

for all $x\in[0,1]$.

In particular, we have

$$\left[f_{\alpha,\beta,\gamma}(x)-1\right]^{2}\leqslant\left[\exp\left(\gamma x^{2\alpha}\right)-1\right]\left[\exp\left(\gamma(1-x)^{2\beta}\right)-1\right] \tag{2.2}$$

for all $x\in[0,1]$.

Proof. If we make use of Hölder's discrete weighted inequality

$$0\leqslant\sum_{k=1}^{n}m_k a_k b_k\leqslant\left(\sum_{k=1}^{n}m_k a_k^{p}\right)^{1/p}\left(\sum_{k=1}^{n}m_k b_k^{q}\right)^{1/q},$$

where $m_k,\ a_k,\ b_k\geqslant 0,\ k\in\{1,\dots,n\}$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we can write for $m_k=\frac{1}{k!}\gamma^k,\ a_k=x^{\alpha k}$ and $b_k=(1-x)^{\beta k}$ the following inequality

$$\begin{aligned}0&\leqslant\sum_{k=1}^{n}\frac{1}{k!}\gamma^k x^{\alpha k}(1-x)^{\beta k}\leqslant\left(\sum_{k=1}^{n}\frac{1}{k!}\gamma^k x^{\alpha pk}\right)^{1/p}\left(\sum_{k=1}^{n}\frac{1}{k!}\gamma^k(1-x)^{q\beta k}\right)^{1/q}\\&=\left(\sum_{k=1}^{n}\frac{1}{k!}\gamma^k\left(x^{\alpha p}\right)^k\right)^{1/p}\left(\sum_{k=1}^{n}\frac{1}{k!}\gamma^k\left[(1-x)^{q\beta}\right]^k\right)^{1/q}\end{aligned} \tag{2.3}$$

for all $n>1$ and $x\in[0,1]$.

Since the series

$$\sum_{k=1}^{\infty}\frac{1}{k!}\gamma^k x^{\alpha k}(1-x)^{\beta k},\qquad\sum_{k=1}^{\infty}\frac{1}{k!}\gamma^k\left(x^{\alpha p}\right)^k$$

and

$$\sum_{k=1}^{\infty}\frac{1}{k!}\gamma^k\left[(1-x)^{q\beta}\right]^k$$

are convergent and

$$\sum_{k=1}^{\infty} \frac{1}{k!} \gamma^k x^{\alpha k} (1-x)^{\beta k} = \exp\left[\gamma x^{\alpha} (1-x)^{\beta}\right] - 1 = f_{\alpha,\beta,\gamma}(x) - 1,$$

$$\sum_{k=1}^{\infty} \frac{1}{k!} \gamma^k (x^{\alpha p})^k = \exp(\gamma x^{\alpha p}) - 1$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k!} \gamma^k \left[(1-x)^{q\beta}\right]^k = \exp\left[\gamma (1-x)^{q\beta}\right] - 1,$$

then by taking the limit over $n \to \infty$ in (2.3) and utilising the representation (1.1) we get the desired result (2.1) □

The following result also provides some lower bounds for $f_{\alpha,\beta,\gamma}(x)$.

THEOREM 3. *Let* $\alpha,\ \beta,\ \gamma > 0$. *Then for all* $x \in [0,1]$ *we have*

$$\begin{aligned} &\frac{1}{e^{\gamma}-1} [\exp(\gamma x^{\alpha}) - 1] \left[\exp\left(\gamma (1-x)^{\beta}\right) - 1\right] \\ &\leqslant f_{\alpha,\beta,\gamma}(x) - 1 \\ &\leqslant \frac{1}{e^{\gamma}-1} [\exp(\gamma x^{\alpha}) - 1] \left[\exp\left(\gamma (1-x)^{\beta}\right) - 1\right] + \frac{1}{4} (e^{\gamma} - 1) x^{\alpha} (1-x)^{\beta}. \end{aligned} \tag{2.4}$$

Proof. We use the weighted *Chebyshev's inequality* for sequences $a_k,\ b_k,\ k \in \{1,\ldots,n\}$ that have the same monotonicity

$$\sum_{k=1}^{n} m_k a_k \sum_{k=1}^{n} m_k b_k \leqslant \sum_{k=1}^{n} m_k \sum_{k=1}^{n} m_k a_k b_k, \tag{2.5}$$

where $m_k \geqslant 0,\ k \in \{1,\ldots,n\}$.

Consider the sequences $a_k := x^{\alpha k},\ b_k := (1-x)^{\beta k},\ k \in \{1,\ldots,n\}$, for $x \in [0,1]$. We observe that both sequences are monotonic nonincreasing and by applying Chebyshev's inequality for the positive weights $m_k := \frac{1}{k!}\gamma^k$ we get

$$\sum_{k=1}^{n} \frac{1}{k!} \gamma^k x^{\alpha k} \sum_{k=1}^{n} \frac{1}{k!} \gamma^k (1-x)^{\beta k} \leqslant \sum_{k=1}^{n} \frac{1}{k!} \gamma^k \sum_{k=1}^{n} \frac{1}{k!} \gamma^k x^{\alpha k} (1-x)^{\beta k}, \tag{2.6}$$

for all $x \in [0,1]$ and $n \geqslant 1$.

Since the series $\sum_{k=1}^{\infty} \frac{1}{k!} \gamma^k x^{\alpha k}$, $\sum_{k=1}^{\infty} \frac{1}{k!} \gamma^k (1-x)^{\beta k}$ and $\sum_{k=1}^{\infty} \frac{1}{k!} \gamma^k$ are convergent and

$$\sum_{k=1}^{\infty} \frac{1}{k!} \gamma^k x^{\alpha k} = \sum_{k=1}^{\infty} \frac{1}{k!} \gamma^k (x^{\alpha})^k = \exp(\gamma x^{\alpha}) - 1,$$

$$\sum_{k=1}^{\infty} \frac{1}{k!} \gamma^k (1-x)^{\beta k} = \sum_{k=1}^{\infty} \frac{1}{k!} \gamma^k \left((1-x)^{\beta}\right)^k = \exp\left(\gamma (1-x)^{\beta}\right) - 1$$

and $\sum_{k=1}^{\infty}\frac{1}{k!}\gamma^k = e^\gamma - 1$, then by taking the limit over $n\to\infty$ in (2.6), we get the first inequality in (2.4).

Further, we use the weighted Grüss inequality for the bounded sequences $a \leqslant a_k \leqslant A, b \leqslant b_k \leqslant B,\ k\in\{1,\dots,n\}$ and nonnegative weights $m_k \geqslant 0,\ k\in\{1,\dots,n\}$, see for instance [2],

$$\begin{aligned}&\left|\sum_{k=1}^{n} m_k \sum_{k=1}^{n} m_k a_k b_k - \sum_{k=1}^{n} m_k a_k \sum_{k=1}^{n} m_k b_k\right| \\ &\leqslant \frac{1}{4}\left(\sum_{k=1}^{n} m_k\right)^2 (A-a)(B-b).\end{aligned} \tag{2.7}$$

Now, if we consider the sequences $a_k := x^{\alpha k},\ b_k := (1-x)^{\beta k}\ k\in\{1,\dots,n\}$, for $x\in[0,1]$, then we observe that $0\leqslant a_k \leqslant x^\alpha$ and $0\leqslant b_k \leqslant (1-x)^\beta$ for all positive integer k. So, by utilising the inequality (2.7) for $a=b=0$, $A=x^\alpha,\ B=(1-x)^\beta$ and $m_k := \frac{1}{k!}\gamma^k$ we get

$$\begin{aligned}&\left|\sum_{k=1}^{n}\frac{1}{k!}\gamma^k \sum_{k=1}^{n}\frac{1}{k!}\gamma^k x^{\alpha k}(1-x)^{\beta k} - \sum_{k=1}^{n}\frac{1}{k!}\gamma^k x^{\alpha k}\sum_{k=1}^{n}\frac{1}{k!}\gamma^k (1-x)^{\beta k}\right| \\ &\leqslant \frac{1}{4}\left(\sum_{k=1}^{n}\frac{1}{k!}\gamma^k\right)^2 x^\alpha (1-x)^\beta,\end{aligned} \tag{2.8}$$

that holds for all $x\in[0,1]$ and $n\geqslant 1$.

Since all the series involved in (2.8) are convergent, then by taking the limit over $n\to\infty$ in this inequality, we get the second part of (2.4). □

THEOREM 4. *Let $\alpha,\ \beta > 0$. Then for all $x\in(0,1)$ we have*

$$(e^\gamma - 1)\left[x^\alpha (1-x)^\beta\right]^{\frac{\gamma e^\gamma}{e^\gamma - 1}} \leqslant f_{\alpha,\beta,\gamma}(x) - 1 \tag{2.9}$$

and

$$\begin{aligned}0 &\leqslant \ln\left(\frac{f_{\alpha,\beta,\gamma}(x)-1}{e^\gamma - 1}\right) - \ln\left\{\left[x^\alpha (1-x)^\beta\right]^{\frac{\gamma e^\gamma}{e^\gamma-1}}\right\} \\ &\leqslant \frac{1}{(e^\gamma-1)^2}\left\{\exp\left[\gamma x^{-\alpha}(1-x)^{-\beta}\right]-1\right\}\left\{\exp\left[\gamma x^{\alpha}(1-x)^{\beta}\right]-1\right\} - 1.\end{aligned} \tag{2.10}$$

Proof. Since ln is a concave function, then by *Jensen's discrete inequality* for concave functions g, namely

$$g\left(\frac{\sum_{k=1}^{n} p_k x_k}{\sum_{k=1}^{n} p_k}\right) \geqslant \frac{\sum_{k=1}^{n} p_k g(x_k)}{\sum_{k=1}^{n} p_k}$$

where $p_k > 0,\ k \in \{1,\dots,n\}$, we have for $x_k = x^{\alpha k}(1-x)^{\beta k}$ and $p_k = \frac{1}{k!}\gamma^k,\ k \in \{1,\dots,n\}$ that

$$\ln\left(\frac{\sum_{k=1}^n \frac{1}{k!}\gamma^k x^{\alpha k}(1-x)^{\beta k}}{\sum_{k=1}^n \gamma^k \frac{1}{k!}}\right) \geqslant \frac{\sum_{k=1}^n \frac{1}{k!}\gamma^k \ln\left[x^{\alpha k}(1-x)^{\beta k}\right]}{\sum_{k=1}^n \frac{1}{k!}\gamma^k} \tag{2.11}$$
$$= \frac{\sum_{k=1}^n \frac{k}{k!}\gamma^k \ln\left[x^{\alpha}(1-x)^{\beta}\right]}{\sum_{k=1}^n \frac{1}{k!}\gamma^k}$$
$$= \ln\left[x^{\alpha}(1-x)^{\beta}\right]\frac{\sum_{k=1}^n \frac{k}{k!}\gamma^k}{\sum_{k=1}^n \frac{1}{k!}\gamma^k}$$
$$= \ln\left[x^{\alpha}(1-x)^{\beta}\right]\frac{\sum_{k=1}^n \frac{1}{(k-1)!}\gamma^k}{\sum_{k=1}^n \frac{1}{k!}\gamma^k},$$

for all $x \in (0,1)$ and $n \geqslant 1$.

Since the series $\sum_{k=1}^\infty \frac{1}{(k-1)!}\gamma^k = \gamma\sum_{k=1}^\infty \frac{1}{(k-1)!}\gamma^{k-1} = \gamma e^\gamma$ and $\sum_{k=1}^\infty \frac{1}{k!}\gamma^k = e^\gamma - 1$, then by taking the limit over $n \to \infty$ in (2.11) and using representation (1.1) we get

$$\ln\left(\frac{f_{\alpha,\beta,\gamma}(x)-1}{e^\gamma - 1}\right) \geqslant \frac{\gamma e^\gamma}{e^\gamma - 1}\ln\left[x^{\alpha}(1-x)^{\beta}\right]$$
$$= \ln\left\{\left[x^{\alpha}(1-x)^{\beta}\right]^{\frac{\gamma e^\gamma}{e^\gamma - 1}}\right\}$$

that is equivalent to the first inequality in (2.9).

Further, we use Dragomir-Ionescu's reverse of Jensen's inequality [5] for concave functions

$$0 \leqslant g\left(\frac{\sum_{k=1}^n p_k x_k}{\sum_{k=1}^n p_k}\right) - \frac{\sum_{k=1}^n p_k g(x_k)}{\sum_{k=1}^n p_k} \tag{2.12}$$
$$\leqslant \frac{1}{\sum_{k=1}^n p_k}\sum_{k=1}^n p_k g'(x_k)\frac{1}{\sum_{k=1}^n p_k}\sum_{k=1}^n p_k x_k - \frac{1}{\sum_{k=1}^n p_k}\sum_{k=1}^n p_k x_k g'(x_k),$$

which gives for $g(x) = \ln x,\ x_k = x^{\alpha k}(1-x)^{\beta k}$ and $p_k = \frac{1}{k!}\gamma^k,\ k \in \{1,\dots,n\}$ that

$$0 \leqslant \ln\left(\frac{\sum_{k=1}^n \frac{1}{k!}\gamma^k x^{\alpha k}(1-x)^{\beta k}}{\sum_{k=1}^n \frac{1}{k!}\gamma^k}\right) - \frac{\sum_{k=1}^n \frac{1}{k!}\gamma^k \ln\left[x^{\alpha k}(1-x)^{\beta k}\right]}{\sum_{k=1}^n \frac{1}{k!}\gamma^k} \tag{2.13}$$
$$\leqslant \frac{1}{\sum_{k=1}^n \frac{1}{k!}\gamma^k}\sum_{k=1}^n \frac{1}{k!}\gamma^k x^{-\alpha k}(1-x)^{-\beta k}\frac{1}{\sum_{k=1}^n \frac{1}{k!}\gamma^k}\sum_{k=1}^n \frac{1}{k!}\gamma^k x^{\alpha k}(1-x)^{\beta k} - 1$$

for $x \in (0,1)$ and $n \geqslant 1$.

Since the series $\sum_{k=1}^{\infty}\frac{1}{k!}\gamma^k x^{-\alpha k}(1-x)^{-\beta k}$ is convergent and

$$\sum_{k=1}^{\infty}\frac{1}{k!}\gamma^k x^{-\alpha k}(1-x)^{-\beta k}=\sum_{k=1}^{\infty}\frac{1}{k!}\gamma^k\left[x^{-\alpha}(1-x)^{-\beta}\right]^k$$
$$=\exp\left[\gamma x^{-\alpha}(1-x)^{-\beta}\right]-1$$

then by letting $n\to\infty$ in (2.13) and using representation (1.1), we get

$$0\leqslant\ln\left(\frac{f_{\alpha,\beta,\gamma}(x)-1}{e^{\gamma}-1}\right)-\ln\left\{\left[x^{\alpha}(1-x)^{\beta}\right]^{\frac{\gamma e^{\gamma}}{e^{\gamma}-1}}\right\}$$
$$\leqslant\frac{1}{(e^{\gamma}-1)^2}\left\{\exp\left[\gamma x^{-\alpha}(1-x)^{-\beta}\right]-1\right\}\left\{\exp\left[\gamma x^{\alpha}(1-x)^{\beta}\right]-1\right\}-1,$$

which is equivalent to (2.10). □

REMARK 1. As a simple consequence of the inequality (2.10) we note that

$$(e^{\gamma}-1)^2\leqslant\left[f_{-\alpha,-\beta,\gamma}(x)-1\right]\left[f_{\alpha,\beta,\gamma}(x)-1\right]\tag{2.14}$$

for all positive $\alpha,\ \beta,\ \gamma>0$ and $x\in(0,1)$.

In 1984, S. S. Dragomir obtained in [1] the following Cauchy-Bunyakovsky-Schwarz related weighted inequality, see also [3, Theorem 2.20]

$$\sum_{k=1}^{n}m_k a_k^2\sum_{k=1}^{n}m_k b_k^2\geqslant\frac{\sum_{k=1}^{n}m_k a_k\sum_{k=1}^{n}m_k b_k\sum_{k=1}^{n}m_k a_k b_k}{\sum_{k=1}^{n}m_k}\tag{2.15}$$

where $a_k,\ b_k$ are real numbers and $m_k\geqslant 0$ for $k\in\{1,\dots,n\}$ and $\sum_{k=1}^{n}m_k>0$.

THEOREM 5. *Let* $\alpha,\ \beta,\ \gamma>0$. *Then for all* $x\in(0,1)$ *we have*

$$0\leqslant f_{\alpha,\beta,\gamma}(x)-1\leqslant(e^{\gamma}-1)\frac{\left[\exp\left(\gamma x^{2\alpha}\right)-1\right]\left[\exp\left(\gamma(1-x)^{2\beta}\right)-1\right]}{\left[\exp\left(\gamma x^{\alpha}\right)-1\right]\left[\exp\left(\gamma(1-x)^{\beta}\right)-1\right]}.\tag{2.16}$$

Proof. By taking $a_k:=x^{\alpha k},\ b_k:=(1-x)^{\beta k}$ and $m_k:=\frac{1}{k!}\gamma^k$ in (2.15) we get

$$\sum_{k=1}^{n}\frac{1}{k!}\gamma^k x^{2\alpha k}\sum_{k=1}^{n}\frac{1}{k!}\gamma^k(1-x)^{2\beta k}\tag{2.17}$$
$$\geqslant\frac{\sum_{k=1}^{n}\frac{1}{k!}\gamma^k x^{\alpha k}\sum_{k=1}^{n}\frac{1}{k!}\gamma^k(1-x)^{\beta k}\sum_{k=1}^{n}\frac{1}{k!}\gamma^k x^{\alpha k}(1-x)^{\beta k}}{\sum_{k=1}^{n}\frac{1}{k!}\gamma^k}.$$

Since all the series involved in (2.17) are convergent, then by taking the limit over $n\to\infty$ in this inequality and using representation (1.1), we get the desired result (2.16). □

3. Convexity of $f_{\alpha,\beta,\gamma}$

Now, recall the well known inequality between the *weighted arithmetic mean* and *weighted geometric mean*

$$a^{1-t}b^{t} \leqslant (1-t)a+tb \tag{3.1}$$

that holds for all $a,\ b>0$ and $t\in[0,1]$. This inequality is also known in the literature as *Young's inequality*.

We have the following global convexity result for the function $f_{\alpha,\beta,\gamma}(x)$ as a mapping of the positive parameters (α,β,γ). More precisely, we have

THEOREM 6. *For any* $(\alpha_1,\beta_1,\gamma_1),\ (\alpha_2,\beta_2,\gamma_2)\in(0,\infty)\times(0,\infty)\times(0,\infty)$ *and* $t\in[0,1]$ *we have*

$$\begin{aligned} &f_{(1-t)\alpha_1+t\alpha_2,(1-t)\beta_1+t\beta_2,(1-t)\gamma_1+t\gamma_2}(x) \\ &\leqslant (1-t)^2 f_{\alpha_1,\beta_1,\gamma_1}(x)+(1-t)t f_{\alpha_2,\beta_2,\gamma_1}(x) \\ &\quad +t(1-t) f_{\alpha_1,\beta_1,\gamma_2}(x)+t^2 f_{\alpha_2,\beta_2,\gamma_2}(x) \end{aligned} \tag{3.2}$$

for all $x\in(0,1)$.

Proof. Fix $x\in(0,1)$. Let $(\alpha_1,\beta_1,\gamma_1),\ (\alpha_2,\beta_2,\gamma_2)\in(0,\infty)\times(0,\infty)\times(0,\infty)$ and $t\in[0,1]$. Then

$$\begin{aligned} &(1-t)(\alpha_1,\beta_1,\gamma_1)+t(\alpha_2,\beta_2,\gamma_2) \\ &=((1-t)\alpha_1+t\alpha_2,(1-t)\beta_1+t\beta_2,(1-t)\gamma_1+t\gamma_2)\in(0,\infty)\times(0,\infty)\times(0,\infty) \end{aligned}$$

and

$$\begin{aligned} &f_{(1-t)\alpha_1+t\alpha_2,(1-t)\beta_1+t\beta_2,(1-t)\gamma_1+t\gamma_2}(x)-1 \\ &=\sum_{k=1}^{\infty}\frac{1}{k!}((1-t)\gamma_1+t\gamma_2)^k x^{[(1-t)\alpha_1+t\alpha_2]k}(1-x)^{[(1-t)\beta_1+t\beta_2]k} \\ &=\sum_{k=1}^{\infty}\frac{1}{k!}((1-t)\gamma_1+t\gamma_2)^k x^{(1-t)\alpha_1 k+t\alpha_2 k}(1-x)^{k(1-t)\beta_1+t\beta_2 k} \\ &=\sum_{k=1}^{\infty}\frac{1}{k!}((1-t)\gamma_1+t\gamma_2)^k x^{(1-t)\alpha_1 k}x^{t\alpha_2 k}(1-x)^{k(1-t)\beta_1}(1-x)^{t\beta_2 k} \\ &=\sum_{k=1}^{\infty}\frac{1}{k!}((1-t)\gamma_1+t\gamma_2)^k x^{(1-t)\alpha_1 k}(1-x)^{k(1-t)\beta_1}x^{t\alpha_2 k}(1-x)^{t\beta_2 k} \\ &=\sum_{k=1}^{\infty}\frac{1}{k!}((1-t)\gamma_1+t\gamma_2)^k\left[x^{\alpha_1 k}(1-x)^{k\beta_1}\right]^{(1-t)}\left[x^{\alpha_2 k}(1-x)^{\beta_2 k}\right]^{t} =: A. \end{aligned}$$

By the convexity of the power function, we have

$$((1-t)\gamma_1+t\gamma_2)^k \leqslant (1-t)\gamma_1^k+t\gamma_2^k$$

for all $k \geqslant 1$ and $t \in [0,1]$.

Therefore

$$A \leqslant \sum_{k=1}^{\infty} \frac{1}{k!} \left[(1-t)\gamma_1^k + t\gamma_2^k \right] \left[x^{\alpha_1 k} (1-x)^{k\beta_1} \right]^{(1-t)} \left[x^{\alpha_2 k} (1-x)^{\beta_2 k} \right]^t$$
$$= (1-t) \sum_{k=1}^{\infty} \frac{1}{k!} \gamma_1^k \left[x^{\alpha_1 k} (1-x)^{k\beta_1} \right]^{(1-t)} \left[x^{\alpha_2 k} (1-x)^{\beta_2 k} \right]^t$$
$$+ t \sum_{k=1}^{\infty} \frac{1}{k!} \gamma_2^k \left[x^{\alpha_1 k} (1-x)^{k\beta_1} \right]^{(1-t)} \left[x^{\alpha_2 k} (1-x)^{\beta_2 k} \right]^t.$$

By Young's inequality (3.1) we have

$$\left[x^{\alpha_1 k} (1-x)^{k\beta_1} \right]^{(1-t)} \left[x^{\alpha_2 k} (1-x)^{\beta_2 k} \right]^t$$
$$\leqslant (1-t) x^{\alpha_1 k} (1-x)^{k\beta_1} + t x^{\alpha_2 k} (1-x)^{\beta_2 k}$$
$$= (1-t) \left[x^{\alpha_1} (1-x)^{\beta_1} \right]^k + t \left[x^{\alpha_2} (1-x)^{\beta_2} \right]^k$$

for all $k \geqslant 1$, and by taking the sum in this inequality we get

$$A \leqslant (1-t) \sum_{k=1}^{\infty} \frac{1}{k!} \gamma_1^k \left\{ (1-t) \left[x^{\alpha_1} (1-x)^{\beta_1} \right]^k + t \left[x^{\alpha_2} (1-x)^{\beta_2} \right]^k \right\}$$
$$+ t \sum_{k=1}^{\infty} \frac{1}{k!} \gamma_2^k \left\{ (1-t) \left[x^{\alpha_1} (1-x)^{\beta_1} \right]^k + t \left[x^{\alpha_2} (1-x)^{\beta_2} \right]^k \right\}$$
$$= (1-t)^2 \sum_{k=1}^{\infty} \frac{1}{k!} \gamma_1^k \left[x^{\alpha_1} (1-x)^{\beta_1} \right]^k + (1-t) t \sum_{k=1}^{\infty} \frac{1}{k!} \gamma_1^k \left[x^{\alpha_2} (1-x)^{\beta_2} \right]^k$$
$$+ t(1-t) \sum_{k=1}^{\infty} \frac{1}{k!} \gamma_2^k \left[x^{\alpha_1} (1-x)^{\beta_1} \right]^k + t^2 \sum_{k=1}^{\infty} \frac{1}{k!} \gamma_2^k \left[x^{\alpha_2} (1-x)^{\beta_2} \right]^k$$
$$= (1-t)^2 \left[f_{\alpha_1,\beta_1,\gamma_1}(x) - 1 \right] + (1-t) t \left[f_{\alpha_2,\beta_2,\gamma_1}(x) - 1 \right]$$
$$+ t(1-t) \left[f_{\alpha_1,\beta_1,\gamma_2}(x) - 1 \right] + t^2 \left[f_{\alpha_2,\beta_2,\gamma_2}(x) - 1 \right],$$

which implies that

$$f_{(1-t)\alpha_1 + t\alpha_2, (1-t)\beta_1 + t\beta_2, (1-t)\gamma_1 + t\gamma_2}(x) - 1$$
$$\leqslant (1-t)^2 \left[f_{\alpha_1,\beta_1,\gamma_1}(x) - 1 \right] + (1-t) t \left[f_{\alpha_2,\beta_2,\gamma_1}(x) - 1 \right]$$
$$+ t(1-t) \left[f_{\alpha_1,\beta_1,\gamma_2}(x) - 1 \right] + t^2 \left[f_{\alpha_2,\beta_2,\gamma_2}(x) - 1 \right]$$
$$= (1-t)^2 f_{\alpha_1,\beta_1,\gamma_1}(x) + (1-t) t f_{\alpha_2,\beta_2,\gamma_1}(x)$$
$$+ t(1-t) f_{\alpha_1,\beta_1,\gamma_2}(x) + t^2 f_{\alpha_2,\beta_2,\gamma_2}(x)$$
$$- (1-t)^2 - (1-t)t - t(1-t) - t^2$$
$$= (1-t)^2 f_{\alpha_1,\beta_1,\gamma_1}(x) + (1-t) t f_{\alpha_2,\beta_2,\gamma_1}(x)$$
$$+ t(1-t) f_{\alpha_1,\beta_1,\gamma_2}(x) + t^2 f_{\alpha_2,\beta_2,\gamma_2}(x) - 1$$

and the claim is thus proved. □

COROLLARY 2. *The function*

$$(0,\infty)\times(0,\infty)\ni(\alpha,\beta)\mapsto f_{\alpha,\beta,\gamma}(x)\in[1,\infty)$$

is globally convex on $(0,\infty)\times(0,\infty)$ *for any* $x\in(0,1)$ *and* $\gamma>0$.

Proof. Fix $x\in(0,1)$ and $\gamma>0$. Let (α_1,β_1), $(\alpha_2,\beta_2)\in(0,\infty)\times(0,\infty)$ and $t\in[0,1]$. Then by (3.2) for $\gamma_1=\gamma_2=\gamma$ we get

$$\begin{aligned}
&f_{(1-t)\alpha_1+t\alpha_2,(1-t)\beta_1+t\beta_2,\gamma}(x)\\
&=f_{(1-t)\alpha_1+t\alpha_2,(1-t)\beta_1+t\beta_2,(1-t)\gamma+t\gamma}(x)\\
&\leqslant(1-t)^2 f_{\alpha_1,\beta_1,\gamma}(x)+(1-t)tf_{\alpha_2,\beta_2,\gamma}(x)\\
&\quad+t(1-t)f_{\alpha_1,\beta_1,\gamma}(x)+t^2 f_{\alpha_2,\beta_2,\gamma}(x)\\
&=\left[(1-t)^2+t(1-t)\right]f_{\alpha_1,\beta_1,\gamma}(x)+\left[(1-t)t+t^2\right]f_{\alpha_2,\beta_2,\gamma}(x)\\
&=(1-t)f_{\alpha_1,\beta_1,\gamma}(x)+tf_{\alpha_2,\beta_2,\gamma}(x),
\end{aligned}$$

which proves the global convexity for the variables $(\alpha,\beta)\in(0,\infty)\times(0,\infty)$. □

We also have:

COROLLARY 3. *The function*

$$(0,\infty)\ni\gamma\mapsto f_{\alpha,\beta,\gamma}(x)\in[1,\infty)$$

is convex on $(0,\infty)$ *for any* $x\in(0,1)$ *and* $\alpha,\ \beta>0$.

Proof. Fix $x\in(0,1)$ and $\alpha,\ \beta>0$. Let $\gamma_1,\ \gamma_2>0$ and $t\in[0,1]$. Then by (3.2) for $\alpha_1=\alpha_2=\alpha$ and $\beta_1=\beta_2=\beta$ we get

$$\begin{aligned}
&f_{\alpha,\beta,(1-t)\gamma_1+t\gamma_2}(x)\\
&=f_{(1-t)\alpha+t\alpha,(1-t)\beta+t\beta,(1-t)\gamma_1+t\gamma_2}(x)\\
&\leqslant(1-t)^2 f_{\alpha,\beta,\gamma_1}(x)+(1-t)tf_{\alpha,\beta,\gamma_1}(x)+t(1-t)f_{\alpha,\beta,\gamma_2}(x)+t^2 f_{\alpha,\beta,\gamma_2}(x)\\
&=\left[(1-t)^2+(1-t)t\right]f_{\alpha,\beta,\gamma_1}(x)+\left[t(1-t)+t^2\right]f_{\alpha,\beta,\gamma_2}(x)\\
&=(1-t)f_{\alpha,\beta,\gamma_1}(x)+tf_{\alpha,\beta,\gamma_2}(x),
\end{aligned}$$

which proves the desired convexity. □

4. Inequalities and convexity properties for $F(\alpha,\beta;\gamma)$

We have:

THEOREM 7. *Let* $\alpha,\ \beta,\ \gamma>0$. *For any* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *we have*

$$0\leqslant F(\alpha,\beta;\gamma)-1\leqslant[F(p\alpha,0;\gamma)-1]^{1/p}[F(0,q\beta;\gamma)-1]^{1/q}. \tag{4.1}$$

In particular, we have

$$[F(\alpha,\beta;\gamma)-1]^2\leqslant[F(2\alpha,0;\gamma)-1][F(0,2\beta;\gamma)-1]. \tag{4.2}$$

Proof. Using Hölder's integral inequality

$$\int_0^1 f(x)g(x)\,dx\leqslant\left(\int_0^1 f^p(x)\,dx\right)^{1/p}\left(\int_0^1 g^q(x)\,dx\right)^{1/q}$$

for any $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and (2.1), we have

$$\begin{aligned}
0\leqslant F(\alpha,\beta;\gamma)-1&\leqslant\int_0^1[\exp(\gamma x^{\alpha p})-1]^{1/p}\left[\exp\left(\gamma(1-x)^{q\beta}\right)-1\right]^{1/q}dx\\
&\leqslant\left[\int_0^1\left([\exp(\gamma x^{\alpha p})-1]^{1/p}\right)^p dx\right]^{1/p}\left[\int_0^1\left(\left[\exp\left(\gamma(1-x)^{q\beta}\right)-1\right]^{1/q}\right)^q dx\right]^{1/q}\\
&=\left[\int_0^1[\exp(\gamma x^{\alpha p})-1]\,dx\right]^{1/p}\left[\int_0^1\left[\exp\left(\gamma(1-x)^{q\beta}\right)-1\right]dx\right]^{1/q}\\
&=[F(p\alpha,0;\gamma)-1]^{1/p}[F(0,q\beta;\gamma)-1]^{1/q},
\end{aligned}$$

which proves (4.1). □

From a different view point we also have:

THEOREM 8. *Let* $\alpha,\ \beta,\ \gamma>0$. *Then*

$$\begin{aligned}
&0\leqslant F(\alpha,\beta;\gamma)-1\\
&\quad\leqslant\frac{1}{e^\gamma-1}[F(\alpha,0;\gamma)-1][F(0,\beta;\gamma)]+\frac{1}{4}(e^\gamma-1)B(\alpha+1,\beta+1).
\end{aligned} \tag{4.3}$$

Proof. If we take the integral in the second inequality of (2.4) we get

$$\begin{aligned}
0&\leqslant F(\alpha,\beta;\gamma)-1\\
&\leqslant\frac{1}{e^\gamma-1}\int_0^1[\exp(\gamma x^\alpha)-1]\left[\exp\left(\gamma(1-x)^\beta\right)-1\right]dx\\
&\quad+\frac{1}{4}(e^\gamma-1)\int_0^1 x^\alpha(1-x)^\beta\\
&=\frac{1}{e^\gamma-1}\int_0^1[\exp(\gamma x^\alpha)-1]\left[\exp\left(\gamma(1-x)^\beta\right)-1\right]dx\\
&\quad+\frac{1}{4}(e^\gamma-1)B(\alpha+1,\beta+1).
\end{aligned} \tag{4.4}$$

We use the Chebyshev's inequality for functions of opposite monotonicities $f,\ g: [0,1] \to \mathbb{R}$

$$\int_0^1 f(x)g(x)\,dx \leqslant \int_0^1 f(x)\,dx \int_0^1 g(x)\,dx$$

for the increasing function $f(x) = \exp(\gamma x^{\alpha}) - 1$ and decreasing function $g(x) = \exp(\gamma(1-x)^{\beta}) - 1$ to get

$$\begin{aligned}
&\int_0^1 [\exp(\gamma x^{\alpha}) - 1]\left[\exp\left(\gamma(1-x)^{\beta}\right) - 1\right] dx \\
&\leqslant \int_0^1 [\exp(\gamma x^{\alpha}) - 1]\,dx \int_0^1 \left[\exp\left(\gamma(1-x)^{\beta}\right) - 1\right] dx \\
&= [F(\alpha,0;\gamma) - 1]\,[F(0,\beta;\gamma)].
\end{aligned}$$

By utilizing (4.4) we obtain the desired result (4.3). □

We have:

THEOREM 9. *For any* $(\alpha_1,\beta_1,\gamma_1),\ (\alpha_2,\beta_2,\gamma_2) \in (0,\infty)\times(0,\infty)\times(0,\infty)$ *and* $t \in [0,1]$ *we have*

$$\begin{aligned}
&F((1-t)(\alpha_1,\beta_1,\gamma_1) + t(\alpha_2,\beta_2,\gamma_2)) \\
&\leqslant (1-t)^2 F(\alpha_1,\beta_1,\gamma_1) + (1-t)tF(\alpha_2,\beta_2,\gamma_1) \\
&\quad + t(1-t)F(\alpha_1,\beta_1,\gamma_2) + t^2 F(\alpha_2,\beta_2,\gamma_2).
\end{aligned} \tag{4.5}$$

Proof. Let $(\alpha_1,\beta_1,\gamma_1),\ (\alpha_2,\beta_2,\gamma_2) \in (0,\infty)\times(0,\infty)\times(0,\infty)$ and $t \in [0,1]$. From (3.2) we have by integration over x on $[0,1]$ that

$$\begin{aligned}
&F((1-t)(\alpha_1,\beta_1,\gamma_1) + t(\alpha_2,\beta_2,\gamma_2)) \\
&= F((1-t)\alpha_1 + t\alpha_2, (1-t)\beta_1 + t\beta_2; (1-t)\gamma_1 + t\gamma_2) \\
&= \int_0^1 f_{(1-t)\alpha_1+t\alpha_2,(1-t)\beta_1+t\beta_2,(1-t)\gamma_1+t\gamma_2}(x)\,dx \\
&\leqslant (1-t)^2 \int_0^1 f_{\alpha_1,\beta_1,\gamma_1}(x)\,dx + (1-t)t\int_0^1 f_{\alpha_2,\beta_2,\gamma_1}(x)\,dx \\
&\quad + t(1-t)\int_0^1 f_{\alpha_1,\beta_1,\gamma_2}(x)\,dx + t^2\int_0^1 f_{\alpha_2,\beta_2,\gamma_2}(x)\,dx \\
&= (1-t)^2 F(\alpha_1,\beta_1,\gamma_1) + (1-t)tF(\alpha_2,\beta_2,\gamma_1) \\
&\quad + t(1-t)F(\alpha_1,\beta_1,\gamma_2) + t^2 F(\alpha_2,\beta_2,\gamma_2),
\end{aligned}$$

which proves (4.5). □

COROLLARY 4. *The function* $F(\cdot,\cdot;\gamma)$ *is globally convex on* $(0,\infty)\times(0,\infty)$ *for any* $\gamma > 0$. *Also, the function* $F(\alpha,\beta;\cdot)$ *is convex on* $(0,\infty)$ *for any* $\alpha,\ \beta > 0$.

Finally we have the following logarithmic convexity property:

THEOREM 10. *For each* $\gamma>0$, *the function* $F(\cdot,\cdot;\gamma)$ *is globally logarithmically convex on* $(0,\infty)\times(0,\infty)$.

Proof. Fix $\gamma>0$. Let (α_1,β_1), $(\alpha_2,\beta_2)\in(0,\infty)\times(0,\infty)$ and $t\in[0,1]$. Then by the representation (1.2) we have

$$\begin{aligned}
&F((1-t)\alpha_1+t\alpha_2,(1-t)\beta_1+t\beta_2;\gamma)-1\\
&=\sum_{k=1}^{\infty}\frac{1}{k!}\gamma^k B([(1-t)\alpha_1+t\alpha_2]k+1,[(1-t)\beta_1+t\beta_2]k+1)\\
&=\sum_{k=1}^{\infty}\frac{1}{k!}\gamma^k B[(1-t)(\alpha_1k+1)+t(\alpha_2k+1),(1-t)(\beta_1k+1)+t(\beta_2k+1)]\\
&=\sum_{k=1}^{\infty}\frac{1}{k!}\gamma^k B[(1-t)(\alpha_1k+1,\beta_1k+1)+t(\alpha_2k+1,\beta_2k+1)]=:T.
\end{aligned}$$

By the global logarithmic convexity of the beta function that was proved in [4], we have

$$\begin{aligned}
&B[(1-t)(\alpha_1k+1,\beta_1k+1)+t(\alpha_2k+1,\beta_2k+1)]\\
&\leqslant[B(\alpha_1k+1,\beta_1k+1)]^{1-t}[B(\alpha_2k+1,\beta_2k+1)]^t
\end{aligned}$$

for (α_1,β_1), $(\alpha_2,\beta_2)\in(0,\infty)\times(0,\infty)$ and $t\in[0,1]$.

This implies that

$$\begin{aligned}
T&\leqslant\sum_{k=1}^{\infty}\frac{1}{k!}\gamma^k[B(\alpha_1k+1,\beta_1k+1)]^{1-t}[B(\alpha_2k+1,\beta_2k+1)]^t\\
&\leqslant\left[\sum_{k=1}^{\infty}\frac{1}{k!}\gamma^k\left([B(\alpha_1k+1,\beta_1k+1)]^{1-t}\right)^{\frac{1}{1-t}}\right]^{1-t}\left[\sum_{k=1}^{\infty}\frac{1}{k!}\gamma^k\left([B(\alpha_2k+1,\beta_2k+1)]^t\right)^{\frac{1}{t}}\right]^t\\
&=\left[\sum_{k=1}^{\infty}\frac{1}{k!}\gamma^k[B(\alpha_1k+1,\beta_1k+1)]\right]^{1-t}\left[\sum_{k=1}^{\infty}\frac{1}{k!}\gamma^k[B(\alpha_2k+1,\beta_2k+1)]\right]^t,
\end{aligned}$$

where for the last inequality we used the weighted Hölder's inequality with $p=\frac{1}{1-t}$ and $q=\frac{1}{t}$ for which we have $\frac{1}{p}+\frac{1}{q}=1$ with $p,\ q>1$.

Therefore, we have

$$\begin{aligned}
&F((1-t)\alpha_1+t\alpha_2,(1-t)\beta_1+t\beta_2;\gamma)-1\\
&\leqslant[F(\alpha_1,\beta_1;\gamma)-1]^{1-t}[F(\alpha_2,\beta_2;\gamma)-1]^t
\end{aligned}\tag{4.6}$$

for (α_1,β_1), $(\alpha_2,\beta_2)\in(0,\infty)\times(0,\infty)$ and $t\in[0,1]$.

Now, by utilising (4.6) and Hölder's discrete inequality we have

$$\begin{aligned}
&F((1-t)\alpha_1+t\alpha_2,(1-t)\beta_1+t\beta_2;\gamma)\\
&\leqslant[F(\alpha_1,\beta_1;\gamma)-1]^{1-t}[F(\alpha_2,\beta_2;\gamma)-1]^t+1
\end{aligned}$$

$$
\begin{aligned}
&= [F(\alpha_1,\beta_1;\gamma)-1]^{1-t}[F(\alpha_2,\beta_2;\gamma)-1]^t + 1^{1-t}1^t \\
&\leqslant \left[\left([F(\alpha_1,\beta_1;\gamma)-1]^{1-t}\right)^{\frac{1}{1-t}}+1\right]^{1-t}\left[\left([F(\alpha_2,\beta_2;\gamma)-1]^t\right)^{\frac{1}{t}}+1\right]^t \\
&= [F(\alpha_1,\beta_1;\gamma)]^{1-t}[F(\alpha_2,\beta_2;\gamma)]^t
\end{aligned}
$$

for (α_1,β_1), $(\alpha_2,\beta_2)\in(0,\infty)\times(0,\infty)$ and $t\in[0,1]$, which proves the logarithmically convexity of $F(\cdot,\cdot;\gamma)$ on $(0,\infty)\times(0,\infty)$. $\square$

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Farzad Khosrowshahi
College of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia
e-mail: farzad.khosrow@vu.edu.au

Silvestru Sever Dragomir
College of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia
and
DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand
Private Bag 3, Johannesburg 2050, South Africa
e-mail: sever.dragomir@vu.edu.au