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STABILITIES OF THE GENERALIZED MIXED WIDTH AND DUAL MIXED RADIAL INEQUALITIES

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Abstract. In this note, stability properties of the generalized mixed width inequality and the generalized dual mixed radial inequality are obtained in the Hausdorff distance, L_2-metric and the dual L_2-metric, respectively.

1. Introduction

There are many important geometric inequalities, for example, the classical isoperimetric inequality, the reverse isoperimetric inequality, the Aleksandrov-Fenchel inequality, etc.. It is well know that stabilities of these geometric inequalities have been extensively investigated, see [2, 3, 4, 5, 6, 9]. Roughly speaking, these investigations focus on the geometric implications if the inequalities are in a certain sense close equalities. For more information of the stability problem one may consult [2, 5]. An inequality in convex geometry can be written

$$\Phi(K) \geqslant 0, \tag{1.1}$$

where $\Phi : \mathscr{C}^n \to \mathbb{R}$ is a real valued function and (1.1) is supposed to hold for all $K \in \mathscr{C}^n$. Let $\mathscr{C}^n_\Phi$ denote those elements $K \in \mathscr{C}^n$ for which the equality sign in (1.1) holds, i.e., $\Phi(K) = 0$ for all $K \in \mathscr{C}^n_\Phi$.

We are interested in the stability problem associated with geometric inequalities of type (1.1). That means, we ask if K must be close to a member of $\mathscr{C}^n_\Phi$ whenever $\Phi(K)$ is close to zero. In order to give a precise formulation of this problem, it is necessary for us to be given a measurement function $g : \mathscr{C}^n \times \mathscr{C}^n \to \mathbb{R}$ that describes in some sense the deviation between two convex bodies. Function g should satisfy following conditions:

(i) $g(K,L) \geqslant 0$ for all $K,L \in \mathscr{C}^n$;

(ii) $g(K,L) = 0$ if and only if $K = L$.

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If Φ, $\mathscr{C}^n_\Phi$ and g are given, the stability problem associated with the geometric inequality (1.1) can now be formulated as follows:

Find positive constants c,α with property that whenever

$$\Phi(K) \leqslant \varepsilon$$

(for some $\varepsilon \geqslant 0$), then there exists an $L \in \mathscr{C}^n_\Phi$ such that

$$g(K,L) \leqslant c\varepsilon^\alpha.$$

Or equivalently, find positive constants c,α with property that for each $K \in \mathscr{C}^n$, there exists an $L \in \mathscr{C}^n_\Phi$ (L may depend on K) such that

$$\Phi(K) \geqslant cg(K,L)^\alpha.$$

A bounded convex subset in the Euclidean space $\mathbb{R}^n$ is said to be an n-dimensional convex body if it is closed and has interior points. Let $\mathscr{C}^n$ denote the class of all n-dimensional convex bodies. If $n=2$, a 2-dimensional convex body is usually called a convex domain. Let K be a convex domain with area $A(K)$ and width function $\omega(K,\theta)$. In 1969, Chernoff [1] proved an inequality that says

$$A(K) \leqslant \frac{1}{2}\int_0^{\frac{\pi}{2}} \omega(K,\theta)\omega\Big(K,\theta+\frac{\pi}{2}\Big)d\theta,$$

where the equality holds if and only if K is a disc. In 2010, Ou-Pan in [8] introduced the k-order width function $\omega_k(K,\theta)$ by

$$\omega_k(K,\theta) = H(K,\theta)+H\Big(K,\theta+\frac{2\pi}{k}\Big)+\cdots+H\Big(K,\theta+\frac{2(k-1)\pi}{k}\Big),\ k\geqslant 2,$$

where $H(K,\theta)$ is the support function of K. By the k-order width function $\omega_k(K,\theta)$, Ou-Pan got the following Chernoff-Ou-Pan inequality

$$A(K) \leqslant \frac{1}{k}\int_0^{\frac{\pi}{k}} \omega_k(K,\theta)\omega_k\Big(K,\theta+\frac{\pi}{k}\Big)d\theta, \tag{1.2}$$

and the equality holds if and only if K is a disc.

Moreover, for two convex domains K and L, Mao-Yang [7] got the generalized mixed width inequality that states

$$\sqrt{A(K)A(L)} \leqslant \frac{1}{2k^2}\int_0^{2\pi} \omega_k(K,\theta)\omega_k(L,\theta)d\theta, \tag{1.3}$$

and the equality holds if and only if K and L are discs.

Let P be a planar star body with area $A(P)$ and radial function $\rho(P,\theta)$, Zhang-Yang in [12] introduced the k-order radial function $\rho_k(P,\theta)$ by

$$\rho_k(P,\theta) = \rho(P,\theta)+\rho\Big(P,\theta+\frac{2\pi}{k}\Big)+\cdots+\rho\Big(P,\theta+\frac{2(k-1)\pi}{k}\Big),\ k\geqslant 2.$$

By the k-order radial function $\rho_k(P,\theta)$, Zhang-Yang obtained the following dual Chernoff-Ou-Pan inequality

$$A(P) \geqslant \frac{1}{k}\int_0^{\frac{\pi}{k}} \rho_k(P,\theta)\rho_k\Big(P,\theta+\frac{\pi}{k}\Big)d\theta, \tag{1.4}$$

where the equality holds if and only if the radial function $\rho(P,\theta)$ of P is of the form

$$\rho(P,\theta) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty}(c_{2nk}\cos 2nk\theta + d_{2nk}\sin 2nk\theta).$$

Moreover, for two planar star bodies P and Q, Mao-Yang [7] got the generalized dual mixed radial inequality that states

$$\sqrt{A(P)A(Q)} \geqslant \frac{1}{2k^2}\int_0^{2\pi} \rho_k(P,\theta)\rho_k(Q,\theta)d\theta, \tag{1.5}$$

and the equality holds if and only if the radial functions of P and Q have the same form

$$\rho(\theta) = \frac{1}{2}c_0 + \sum_{n=1}^{\infty}(c_{nk}\cos nk\theta + d_{nk}\sin nk\theta).$$

In this note we will focus our attention on stabilities of the generalized mixed width inequality (1.3) in the Hausdorff distance, the L_2-metric and the generalized dual mixed radial inequality (1.5) in the dual L_2-metric, respectively.

2. Preliminaries

In this section, we will first recall some basic facts about plane convex geometry which will be used later on. Let $K \in \mathscr{C}^2$ be a convex domain and assume that the origin O of $\mathbb{R}^2$ lies in the interior of K, and let $\mathbf{u}$ be a unit vector in $\mathbb{R}^2$ and $l(\mathbf{u})$ denote the supporting line of K that is perpendicular to $\mathbf{u}$ and on the same side of the origin. The oriented distance from O to $l(\mathbf{u})$, denoted by $H(K,\mathbf{u})$, is called the Minkowski support function of K. Since $\mathbf{u}$ is usually determined by the oriented angle, say θ, from the positive x-axis to $\mathbf{u}$, one also writes $H(K,\theta)$ instead of $H(K,\mathbf{u})$. It is clear that $H(K,\theta)$ is a continuous 2π-periodic function. A more comprehensive introduction to the theory of convex bodies can be found in [11].

Let $p(K)$ denote the perimeter and $A(K)$ the area of K, one can find (see [5]),

$$p(K) = \int_0^{2\pi} H(K,\theta)d\theta, \tag{2.1}$$

and if H is sufficiently smooth, then

$$A(K) = \frac{1}{2}\int_0^{2\pi} \left(H^2(K,\theta) - H'^2(K,\theta)\right)d\theta, \tag{2.2}$$

where $'$ denotes the derivative with respect to θ.

The Steiner disc of K, denoted by $S(K)$, is the disc with radius $p(K)/2\pi$ and center at the Steiner point which can be defined in terms of the Minkowski support function

$$\vec{s}(K)=\frac{1}{\pi}\int_0^{2\pi}\mathbf{u}(\theta)H(K,\theta)d\theta. \tag{2.3}$$

The Steiner disc of K will play a role in our stability statement in Section 3 below. For more information on the Steiner point of a convex body one may consult [10].

The width of K in a direction $\mathbf{u}(\theta)=(\cos\theta,\sin\theta)$, denoted by $\omega(K,\theta)$, is defined to be the distance between two tangents to a perpendicular to $\mathbf{u}(\theta)$. It is clear that

$$\omega(K,\theta)=H(K,\theta)+H(K,\theta+\pi).$$

The convex domain K is said to be of constant width if its width in any direction is a positive constant ω_0, i.e., $\omega(K,\theta)=H(K,\theta)+H(K,\theta+\pi)=\omega_0$ for any $\theta\in[0,2\pi]$. For a convex domain K, Ou-Pan [8] introduced the k-order width function $\omega_k(K,\theta)$ by

$$\omega_k(K,\theta)=H(K,\theta)+H\Big(K,\theta+\frac{2\pi}{k}\Big)+\cdots+H\Big(K,\theta+\frac{2(k-1)\pi}{k}\Big),\ k\geqslant 2.$$

We wish to express $p(K),\vec{s}(K),A(K)$ in terms of the Fourier coefficients of $H(K,\theta)$. Since the support function of a given convex domain K is always continuous, bounded and 2π-periodic, it has a Fourier series of the form

$$H(K,\theta)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta),$$

where

$$\begin{aligned}a_0&=\frac{1}{\pi}\int_0^{2\pi}H(K,\theta)d\theta,\\ a_n&=\frac{1}{\pi}\int_0^{2\pi}H(K,\theta)\cos n\theta d\theta,\ b_n=\frac{1}{\pi}\int_0^{2\pi}H(K,\theta)\sin n\theta d\theta,\ n\in\mathbb{Z}^+.\end{aligned}$$

Differentiation of this with respect to θ gives us

$$H'(K,\theta)=\sum_{n=1}^{\infty}n(-a_n\sin n\theta+b_n\cos n\theta).$$

From (2.1) and (2.3), it follows immediately that

$$p(K)=\pi a_0, \tag{2.4}$$

$$\vec{s}(K)=(a_1,b_1). \tag{2.5}$$

By the Parseval equality, one can get

$$\int_0^{2\pi}H^2(K,\theta)d\theta=\frac{1}{2}\pi a_0^2+\pi\sum_{n=1}^{\infty}(a_n^2+b_n^2),$$

$$\int_0^{2\pi} H'^2(K,\theta)d\theta = \pi\sum_{n=1}^{\infty} n^2(a_n^2+b_n^2),$$

which together with (2.2) gives us

$$A(K) = \frac{1}{4}\pi a_0^2 - \frac{1}{2}\pi\sum_{n=2}^{\infty}(n^2-1)(a_n^2+b_n^2). \tag{2.6}$$

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$, a compact subset P of $\mathbb{R}^n$ be star-shaped with respect to the origin, for $\mathbf{u}\in S^{n-1}$, its radial function $\rho(P,\mathbf{u})$ is defined by

$$\rho(P,\mathbf{u}) = \max\{\lambda>0 : \lambda\mathbf{u}\in P\}.$$

Since $\mathbf{u}$ is usually determined by the oriented angle, say θ, from the positive x-axis to $\mathbf{u}$, one also writes $\rho(P,\theta)$ instead of $\rho(P,\mathbf{u})$. It is clear that $\rho(P,\theta)$ is a continuous 2π-periodic function. For a planar star body P, Zhang-Yang [12] introduced the k-order radial function $\rho_k(P,\theta)$ by

$$\rho_k(P,\theta) = \rho(P,\theta)+\rho\left(P,\theta+\frac{2\pi}{k}\right)+\cdots+\rho\left(P,\theta+\frac{2(k-1)\pi}{k}\right),\ k\geqslant 2,\ \theta\in[0,2\pi].$$

For a planar star body P, its area $A(P)$ can be expressed by

$$A(P) = \frac{1}{2}\int_0^{2\pi}\rho^2(P,\theta)d\theta. \tag{2.7}$$

Since $\rho(P,\theta)$ is continuous, bounded and 2π-periodic, it has a Fourier series of the form

$$\rho(P,\theta) = \frac{1}{2}c_0+\sum_{n=1}^{\infty}(c_n\cos n\theta + d_n\sin n\theta), \tag{2.8}$$

where

$$c_0 = \frac{1}{\pi}\int_0^{2\pi}\rho(P,\theta)d\theta,$$
$$c_n = \frac{1}{\pi}\int_0^{2\pi}\rho(P,\theta)\cos n\theta d\theta,\ d_n = \frac{1}{\pi}\int_0^{2\pi}\rho(P,\theta)\sin n\theta d\theta,\ n\in\mathbb{Z}^+.$$

We will express $A(P)$ in terms of the Fourier coefficients of $\rho(P,\theta)$. By the Parseval equality and (2.8), we get

$$A(P) = \frac{1}{4}\pi c_0^2+\frac{1}{2}\pi\sum_{n=1}^{\infty}(c_n^2+d_n^2).$$

For the star body P, if its radial function

$$\rho(P,\theta) = \frac{1}{2}c_0+\sum_{n=1}^{\infty}(c_n\cos n\theta + d_n\sin n\theta)$$

satisfies

$$\frac{1}{2}c_0+\sum_{l=1}^{\infty}(c_{kl}\cos kl\theta+d_{kl}\sin kl\theta)>0,$$

then we can define a star body $\widetilde{P}$ associated with P by

$$\rho(\widetilde{P},\theta)=\frac{1}{2}c_0+\sum_{l=1}^{\infty}(c_{kl}\cos kl\theta+d_{kl}\sin kl\theta). \tag{2.9}$$

The star body $\widetilde{P}$ associated with P will play a role in our stability statement in Section 4 below.

3. Stability properties of the generalized mixed width inequality

Let K and L be two convex domains with support functions $H(K,\theta)$ and $H(L,\theta)$ respectively, the most frequently used function to measure the deviation between K and L is the Hausdorff distance

$$h(K,L)=\max_{\theta}|H(K,\theta)-H(L,\theta)|.$$

Another such measure with respect to stability problem is the L_2-metric, which is defined by

$$h_2(K,L)=\left(\int_0^{2\pi}|H(K,\theta)-H(L,\theta)|^2d\theta\right)^{1/2}.$$

It is obvious that $h(K,L)=0$ (or $h_2(K,L)=0$) if and only if $K=L$.

We consider now stabilities of the generalized mixed width inequality (1.3) with respect to the deviation measures h_2 and h.

THEOREM 3.1. *Let K,L be two convex domains with areas $A(K)$ and $A(L)$ respectively. For $k\in\mathbb{Z}^+$ and $k\geqslant 2$, then*

$$\frac{1}{2k^2}\int_0^{2\pi}\omega_k(K,\theta)\omega_k(L,\theta)d\theta-\sqrt{A(K)A(L)}\geqslant h_2(K,S(K))h_2(L,S(L)), \tag{3.1}$$

and the equality (3.1) holds if and only if K and L are discs, where $S(K)$ and $S(L)$ are the Steiner discs of K and L, respectively.

Proof. Since the support function $H(K,\theta)$ of the convex domain K has the following Fourier series

$$H(K,\theta)=\frac{1}{2\pi}p(K)+\sum_{n=1}^{\infty}(a_n^K\cos n\theta+b_n^K n\sin n\theta),$$

where

$$a_n^K=\frac{1}{\pi}\int_0^{2\pi}H(K,\theta)\cos n\theta d\theta,\ b_n^K=\frac{1}{\pi}\int_0^{2\pi}H(K,\theta)\sin n\theta d\theta,\ n\in\mathbb{Z}^+,$$

and the support function of $S(K)$ is

$$H(S(K),\theta) = \frac{1}{2\pi}p(K) + a_1^K\cos\theta + b_1^K\sin\theta.$$

By using Parseval's equality one can obtain

$$h_2(K,S(K)) = \left(\int_0^{2\pi}|H(K,\theta)-H(S(K),\theta)|^2 d\theta\right)^{1/2} = \sqrt{\pi\sum_{n=2}^{\infty}({a_n^K}^2+{b_n^K}^2)}. \quad (3.2)$$

Similarly, one has

$$h_2(L,S(L)) = \left(\int_0^{2\pi}|H(L,\theta)-H(S(L),\theta)|^2 d\theta\right)^{1/2} = \sqrt{\pi\sum_{m=2}^{\infty}({a_m^L}^2+{b_m^L}^2)}. \quad (3.3)$$

By the proof of Theorem 3.1 in [7], it follows that

$$\begin{aligned}
&\frac{1}{k^2}\int_0^{2\pi}\omega_k(K,\theta)\omega_k(L,\theta)d\theta - \frac{a_0^L}{a_0^K}A(K) - \frac{a_0^K}{a_0^L}A(L)\\
&= \frac{\pi}{2}\sum_{n=2}^{\infty}(n^2-1)\left[\frac{a_0^L}{a_0^K}({a_n^K}^2+{b_n^K}^2) + \frac{a_0^K}{a_0^L}({a_n^L}^2+{b_n^L}^2)\right] + \pi\sum_{l=1}^{\infty}(a_{kl}^K a_{kl}^L + b_{kl}^K b_{kl}^L).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\frac{1}{k^2}\int_0^{2\pi}\omega_k(K,\theta)\omega_k(L,\theta)d\theta - \frac{a_0^L}{a_0^K}A(K) - \frac{a_0^K}{a_0^L}A(L)\\
&\geqslant \frac{\pi}{2}\sum_{n=2}^{\infty}(n^2-2)\left[\frac{a_0^L}{a_0^K}({a_n^K}^2+{b_n^K}^2) + \frac{a_0^K}{a_0^L}({a_n^L}^2+{b_n^L}^2)\right]\\
&\quad + \frac{\pi}{2}\sum_{n=2}^{\infty}\left[\left(\frac{a_0^L}{a_0^K}{a_n^K}^2 + \frac{a_0^K}{a_0^L}{a_n^L}^2\right) + \left(\frac{a_0^L}{a_0^K}{b_n^K}^2 + \frac{a_0^K}{a_0^L}{b_n^L}^2\right)\right] - \frac{\pi}{2}\sum_{l=1}^{\infty}\left(|2a_{kl}^K a_{kl}^L| + |2b_{kl}^K b_{kl}^L|\right)\\
&\geqslant \frac{\pi}{2}\sum_{n=2}^{\infty}(n^2-2)\left[\frac{a_0^L}{a_0^K}({a_n^K}^2+{b_n^K}^2) + \frac{a_0^K}{a_0^L}({a_n^L}^2+{b_n^L}^2)\right],
\end{aligned} \quad (3.4)$$

which together with (3.2) and (3.3) implies that

$$\begin{aligned}
&\frac{1}{k^2}\int_0^{2\pi}\omega_k(K,\theta)\omega_k(L,\theta)d\theta - \frac{a_0^L}{a_0^K}A(K) - \frac{a_0^K}{a_0^L}A(L)\\
&\geqslant \pi\left[\frac{a_0^L}{a_0^K}\sum_{n=2}^{\infty}({a_n^K}^2+{b_n^K}^2) + \frac{a_0^K}{a_0^L}\sum_{m=2}^{\infty}({a_m^L}^2+{b_m^L}^2)\right]\\
&\geqslant 2\sqrt{\pi\sum_{n=2}^{\infty}({a_n^K}^2+{b_n^K}^2)}\sqrt{\pi\sum_{m=2}^{\infty}({a_m^L}^2+{b_m^L}^2)}\\
&= 2h_2(K,S(K))h_2(L,S(L)).
\end{aligned}$$

Furthermore, one can get

$$\begin{aligned}\frac{1}{k^2}\int_0^{2\pi}\omega_k(K,\theta)\omega_k(L,\theta)d\theta &\geqslant \frac{a_0^L}{a_0^K}A(K)+\frac{a_0^K}{a_0^L}A(L)+2h_2(K,S(K))h_2(L,S(L))\\ &\geqslant 2\sqrt{A(K)A(L)}+2h_2(K,S(K))h_2(L,S(L)).\end{aligned}$$

Hence,

$$\frac{1}{2k^2}\int_0^{2\pi}\omega_k(K,\theta)\omega_k(L,\theta)d\theta-\sqrt{A(K)A(L)}\geqslant h_2(K,S(K))h_2(L,S(L)),$$

and the equality holds if and only if K and L are discs. □

THEOREM 3.2. *Under the same assumptions of Theorem 3.1 , one gets*

$$\frac{1}{2k^2}\int_0^{2\pi}\omega_k(K,\theta)\omega_k(L,\theta)d\theta-\sqrt{A(K)A(L)}\geqslant\frac{2\pi}{5-\sqrt{2}\pi\cot(\sqrt{2}\pi)}h(K,S(K))h(L,S(L)), \tag{3.5}$$

and the equality holds if K and L are discs.

Proof. Since it is easily see that

$$|a_n\cos n\theta+b_n\sin n\theta|\leqslant\sqrt{a_n^2+b_n^2},$$

one can get

$$\begin{aligned}&|H(K,\theta)-H(S(K),\theta)|\\ &=\left|\frac{1}{2}a_0^K+\sum_{n=1}^{\infty}(a_n^K\cos n\theta+b_n^K\sin n\theta)-\left(\frac{1}{2}a_0^K+a_1^K\cos\theta+b_1^K\sin\theta\right)\right|\\ &\leqslant\sum_{n=2}^{\infty}\left|a_n^K\cos n\theta+b_n^K\sin n\theta\right|\leqslant\sum_{n=2}^{\infty}\sqrt{{a_n^K}^2+{b_n^K}^2}.\end{aligned}$$

Similarly,

$$|H(L,\theta)-H(S(L),\theta)|\leqslant\sum_{m=2}^{\infty}\sqrt{{a_m^L}^2+{b_m^L}^2}.$$

It follows from the Hölder inequality that

$$\begin{aligned}h(K,S(K))&\leqslant\sum_{n=2}^{\infty}\sqrt{{a_n^K}^2+{b_n^K}^2}\\ &\leqslant\left(\sum_{n=2}^{\infty}\frac{1}{n^2-2}\right)^{1/2}\left[\sum_{n=2}^{\infty}(n^2-2)({a_n^K}^2+{b_n^K}^2)\right]^{1/2}\end{aligned} \tag{3.6}$$

and

$$h(L,S(L))\leqslant\left(\sum_{m=2}^{\infty}\frac{1}{m^2-2}\right)^{1/2}\left[\sum_{m=2}^{\infty}(m^2-2)({a_m^L}^2+{b_m^L}^2)\right]^{1/2}. \tag{3.7}$$

Recall that if q is not an integer by Fourier series calculation,

$$\pi \cot q\pi = \frac{1}{q} - 2q \sum_{n=1}^{\infty} \frac{1}{n^2 - q^2}.$$

Furthermore, one can calculate

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 2} = \frac{5 - \sqrt{2}\pi \cot(\sqrt{2}\pi)}{4}.$$

It follows from (3.4), (3.6) and (3.7) that

$$\begin{aligned}
&\frac{1}{k^2}\int_0^{2\pi} \omega_k(K,\theta)\omega_k(L,\theta)d\theta - \frac{a_0^L}{a_0^K}A(K) - \frac{a_0^K}{a_0^L}A(L) \\
&\geqslant \frac{\pi}{2}\left[\frac{a_0^L}{a_0^K}\sum_{n=2}^{\infty}(n^2-2)(a_n^{K2}+b_n^{K2}) + \frac{a_0^K}{a_0^L}\sum_{m=2}^{\infty}(m^2-2)(a_m^{L\,2}+b_m^{L\,2})\right] \\
&\geqslant \pi\sqrt{\sum_{n=2}^{\infty}(n^2-2)(a_n^{K2}+b_n^{K2})}\sqrt{\sum_{m=2}^{\infty}(m^2-2)(a_m^{L\,2}+b_m^{L\,2})} \\
&\geqslant \frac{4\pi}{5-\sqrt{2}\pi\cot(\sqrt{2}\pi)}h(K,S(K))h(L,S(L)).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\frac{1}{k^2}\int_0^{2\pi} \omega_k(K,\theta)\omega_k(L,\theta)d\theta \\
&\geqslant \frac{a_0^L}{a_0^K}A(K) + \frac{a_0^K}{a_0^L}A(L) + \frac{4\pi}{5-\sqrt{2}\pi\cot(\sqrt{2}\pi)}h(K,S(K))h(L,S(L)) \\
&\geqslant 2\sqrt{A(K)A(L)} + \frac{4\pi}{5-\sqrt{2}\pi\cot(\sqrt{2}\pi)}h(K,S(K))h(L,S(L)).
\end{aligned}$$

Hence,

$$\frac{1}{2k^2}\int_0^{2\pi} \omega_k(K,\theta)\omega_k(L,\theta)d\theta - \sqrt{A(K)A(L)} \geqslant \frac{2\pi}{5-\sqrt{2}\pi\cot(\sqrt{2}\pi)}h(K,S(K))h(L,S(L)),$$

and the equality holds if K and L are discs. □

REMARK. (i) Theorems 3.1 and 3.2 can be looked upon as strengthened forms of the generalized mixed width inequality (1.3).

(ii) Observe that although (3.1) cannot be improved for all $K, L \in \mathscr{C}^2$, it is possible to prove stronger inequalities for particular kinds of convex domains. For example, if K, L are of constant width, the Fourier expression of the support functions of K and L

have the properties that $a_{2n}^K = a_{2m}^L = b_{2n}^K = b_{2m}^L = 0$ for all $n, m \in \mathbb{Z}^+$. Checking the proof of (3.1),

$$h_2(K, S(K)) = \sqrt{\pi \sum_{n=1}^{\infty} ({a_{2n+1}^K}^2 + {b_{2n+1}^K}^2)},$$

$$h_2(L, S(L)) = \sqrt{\pi \sum_{m=1}^{\infty} ({a_{2m+1}^L}^2 + {b_{2m+1}^L}^2)}.$$

It follows from (3.4) that

$$\begin{aligned}
&\frac{1}{k^2}\int_0^{2\pi} \omega_k(K,\theta)\omega_k(L,\theta)d\theta - \frac{a_0^L}{a_0^K}A(K) - \frac{a_0^K}{a_0^L}A(L) \\
&\geqslant \frac{7\pi}{2}\left[\frac{a_0^L}{a_0^K}\sum_{n=1}^{\infty}({a_{2n+1}^K}^2 + {b_{2n+1}^K}^2) + \frac{a_0^K}{a_0^L}\sum_{m=1}^{\infty}({a_{2m+1}^L}^2 + {b_{2m+1}^L}^2)\right] \\
&\geqslant 7\sqrt{\pi\sum_{n=1}^{\infty}({a_{2n+1}^K}^2 + {b_{2n+1}^K}^2)}\sqrt{\pi\sum_{m=1}^{\infty}({a_{2m+1}^L}^2 + {b_{2m+1}^L}^2)} \\
&= 7h_2(K,S(K))h_2(L,S(L)).
\end{aligned}$$

Hence,

$$\frac{1}{2k^2}\int_0^{2\pi} \omega_k(K,\theta)\omega_k(L,\theta)d\theta - \sqrt{A(K)A(L)} \geqslant \frac{7}{2}h_2(K,S(K))h_2(L,S(L)),$$

and the equality holds if and only if K and L are discs.

Similarly, (3.5) can also be strengthened in this case. Since

$$\begin{aligned}
&|H(K,\theta) - H(S(K),\theta)| \\
&\leqslant \sum_{n=1}^{\infty}\sqrt{{a_{2n+1}^K}^2 + {b_{2n+1}^K}^2} \\
&\leqslant \left(\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2-2}\right)^{1/2}\left[\sum_{n=1}^{\infty}((2n+1)^2-2)({a_{2n+1}^K}^2 + {b_{2n+1}^K}^2)\right]^{1/2} \\
&< \left(\sum_{n=1}^{\infty}\frac{1}{4n^2+4n-3}\right)^{1/2}\left[\sum_{n=1}^{\infty}((2n+1)^2-2)({a_{2n+1}^K}^2 + {b_{2n+1}^K}^2)\right]^{1/2} \\
&= \sqrt{\frac{1}{3}}\left[\sum_{n=1}^{\infty}((2n+1)^2-2)({a_{2n+1}^K}^2 + {b_{2n+1}^K}^2)\right]^{1/2}.
\end{aligned}$$

Similarly, we can get

$$|H_L(\theta) - H_{S(L)}(\theta)| < \sqrt{\frac{1}{3}}\left[\sum_{m=1}^{\infty}((2m+1)^2-2)({a_{2m+1}^L}^2 + {b_{2m+1}^L}^2)\right]^{1/2}.$$

It follows from (3.4) that

$$\begin{aligned}
&\frac{1}{k^2}\int_0^{2\pi}\omega_k(K,\theta)\omega_k(L,\theta)d\theta-\frac{a_0^L}{a_0^K}A(K)-\frac{a_0^K}{a_0^L}A(L)\\
&\geqslant \frac{\pi}{2}\left[\sum_{n=1}^{\infty}\left((2n+1)^2-2\right)\frac{a_0^L}{a_0^K}({a_{2n+1}^K}^2+{b_{2n+1}^K}^2)+\sum_{m=1}^{\infty}\left((2m+1)^2-2\right)\frac{a_0^K}{a_0^L}({a_{2m+1}^L}^2+{b_{2m+1}^L}^2)\right]\\
&\geqslant \pi\sqrt{\sum_{n=1}^{\infty}((2n+1)^2-2)({a_{2n+1}^K}^2+{b_{2n+1}^K}^2)}\sqrt{\sum_{m=1}^{\infty}((2m+1)^2-2)({a_{2m+1}^L}^2+{b_{2m+1}^L}^2)}\\
&> 3\pi h(K,S(K))h(L,S(L)).
\end{aligned}$$

Hence,

$$\frac{1}{2k^2}\int_0^{2\pi}\omega_k(K,\theta)\omega_k(L,\theta)d\theta-\sqrt{A(K)A(L)}>\frac{3\pi}{2}h(K,S(K))h(L,S(L)).$$

4. Stability property of the generalized dual mixed radial inequality

Let P and Q be two planar star bodies with radial functions $\rho(P,\theta)$ and $\rho(Q,\theta)$ respectively, similar to L_2-metric, the dual L_2-metric between P and Q can be defined by

$$\delta_2(P,Q)=\left(\int_0^{2\pi}|\rho(P,\theta)-\rho(Q,\theta)|^2d\theta\right)^{1/2}.$$

It is obvious that $\delta_2(P,Q)=0$ if and only if $P=Q$.

We consider now the stability of the generalized dual mixed radial inequality (1.5) with respect to the deviation measure δ_2.

THEOREM 4.1. *Let P,Q be two planar star bodies with areas $A(P)$ and $A(Q)$ respectively. If $\widetilde{P},\widetilde{Q}$ are star bodies associated with P and Q, respectively. For $k\in\mathbb{Z}^+$ and $k\geqslant 2$, then*

$$A(P)A(Q)-\left(\frac{1}{2k^2}\int_0^{2\pi}\rho_k(P,\theta)\rho_k(Q,\theta)d\theta\right)^2\geqslant\frac{1}{4}\delta_2(P,\widetilde{P})^2\delta_2(Q,\widetilde{Q})^2, \qquad (4.1)$$

and the equality holds if and only if P and Q are discs.

Proof. We assume that $\rho(P,\theta)$ and $\rho(Q,\theta)$ have the following Fourier series

$$\rho(P,\theta)=\frac{1}{2}c_0^P+\sum_{n=1}^{\infty}(c_n^P\cos n\theta+d_n^P\sin n\theta),$$
$$\rho(Q,\theta)=\frac{1}{2}c_0^Q+\sum_{n=1}^{\infty}(c_n^Q\cos n\theta+d_n^Q\sin n\theta),$$

then areas $A(P)$ and $A(Q)$ follow from (2.7) and (2.8) that

$$A(P)=\frac{1}{4}\pi c_0^{P2}+\frac{1}{2}\pi\sum_{n=1}^{\infty}(c_n^{P2}+d_n^{P2}),\quad A(Q)=\frac{1}{4}\pi c_0^{Q2}+\frac{1}{2}\pi\sum_{n=1}^{\infty}(c_n^{Q2}+d_n^{Q2}).$$

Similarly, by the proof of Theorem 3.1 in [7], it follows that

$$\frac{1}{2k^2}\int_0^{2\pi}\rho_k(P,\theta)\rho_k(Q,\theta)d\theta=\frac{1}{4}\pi c_0^P c_0^Q+\frac{1}{2}\pi\sum_{l=1}^{\infty}(c_{kl}^P c_{kl}^Q+d_{kl}^P d_{kl}^Q).$$

By using Parseval's equality one can obtain

$$\delta_2(P,\widetilde{P})^2=\int_0^{2\pi}\left|\rho(P,\theta)-\rho(\widetilde{P},\theta)\right|^2 d\theta=\pi\sum_{n\neq kl}^{\infty}(c_n^{P2}+d_n^{P2}),$$

$$\delta_2(Q,\widetilde{Q})^2=\int_0^{2\pi}\left|\rho(Q,\theta)-\rho(\widetilde{Q},\theta)\right|^2 d\theta=\pi\sum_{n\neq kl}^{\infty}(c_n^{Q2}+d_n^{Q2}).$$

Therefore,

$$\begin{aligned}
&A(P)A(Q)-\left(\frac{1}{2k^2}\int_0^{2\pi}\rho_k(P,\theta)\rho_k(Q,\theta)d\theta\right)^2\\
&=\frac{1}{8}\pi^2 c_0^{P2}\sum_{n=1}^{\infty}(c_n^{Q2}+d_n^{Q2})+\frac{1}{8}\pi^2 c_0^{Q2}\sum_{n=1}^{\infty}(c_n^{P2}+d_n^{P2})\\
&+\frac{1}{4}\pi^2\sum_{n=1}^{\infty}(c_n^{P2}+d_n^{P2})\sum_{n=1}^{\infty}(c_n^{Q2}+d_n^{Q2})\\
&-\frac{1}{4}\pi^2 c_0^P c_0^Q\sum_{l=1}^{\infty}(c_{kl}^P c_{kl}^Q+d_{kl}^P d_{kl}^Q)-\frac{1}{4}\pi^2\left[\sum_{l=1}^{\infty}(c_{kl}^P c_{kl}^Q+d_{kl}^P d_{kl}^Q)\right]^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&A(P)A(Q)-\left(\frac{1}{2k^2}\int_0^{2\pi}\rho_k(P,\theta)\rho_k(Q,\theta)d\theta\right)^2\\
&\geqslant\frac{1}{8}\pi^2 c_0^{P2}\sum_{n\neq kl}^{\infty}(c_n^{Q2}+d_n^{Q2})+\frac{1}{8}\pi^2 c_0^{Q2}\sum_{n\neq kl}^{\infty}(c_n^{P2}+d_n^{P2})\\
&+\frac{1}{4}\pi^2\sum_{n\neq kl}^{\infty}(c_n^{P2}+d_n^{P2})\sum_{n\neq kl}^{\infty}(c_n^{Q2}+d_n^{Q2})\\
&\geqslant\frac{1}{4}\pi^2\sum_{n\neq kl}^{\infty}(c_n^{P2}+d_n^{P2})\sum_{n\neq kl}^{\infty}(c_n^{Q2}+d_n^{Q2})\\
&=\frac{1}{4}\delta_2(P,\widetilde{P})^2\delta_2(Q,\widetilde{Q})^2.
\end{aligned}$$

Hence,

$$A(P)A(Q)-\left(\frac{1}{2k^2}\int_0^{2\pi}\rho_k(P,\theta)\rho_k(Q,\theta)d\theta\right)^2\geqslant\frac{1}{4}\delta_2(P,\widetilde{P})^2\delta_2(Q,\widetilde{Q})^2,$$

and the equality holds if and only if P and Q are discs. □

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