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ON A MORE ACCURATE REVERSE HILBERT–TYPE INEQUALITY IN THE WHOLE PLANE

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Abstract. In the present paper, using weight coefficients and applying Hermite-Hadamard's inequality, we derive a new, more accurate reverse Hilbert-type inequality in the whole plane with multi-parameters involving the cosine and natural logarithm functions. The corresponding constant factor is proved to be the best possible. We additionally consider some equivalent forms and a few particular inequalities. As an application, the obtained results are compared with some previously known results and we show that these new results are more accurate than the earlier ones.

1. Introduction

Let us consider $p>1, \frac{1}{p}+\frac{1}{q}=1,\ a_m, b_n \geqslant 0, a=\{a_m\}_{m=1}^{\infty}\in l^p,\ b=\{b_n\}_{n=1}^{\infty}\in l^q,$

$$||a||_p=\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}>0,\ \ ||b||_q>0.$$

We have the following well known classical Hardy-Hilbert inequality:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\frac{\pi}{p})}||a||_p||b||_q, \tag{1}$$

where the constant factor

$$\frac{\pi}{\sin(\pi/p)}$$

is the best possible (cf. [1]). Following the assumptions of (1), the following Hilbert-type inequality (cf. [2], Theorem 342) holds true as well:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(\frac{m}{n})a_m b_n}{m-n}<\left[\frac{\pi}{\sin(\frac{\pi}{p})}\right]^2||a||_p||b||_q, \tag{2}$$

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where the constant factor

$$\left[\frac{\pi}{\sin(\pi/p)}\right]^2$$

is the best possible.

In 2009, by introducing parameters $\lambda_1, \lambda_2 \in (0,1]$ $(\lambda_1+\lambda_2=\lambda)$, Yang [3] provided the following extension of (2) with its reverse (for $0<p<1, \frac{1}{p}+\frac{1}{q}=1$):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(\frac{m}{n})a_m b_n}{m^\lambda - n^\lambda} < \left[\frac{\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}\right]^2 \left\{\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1}a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1}b_n^q\right\}^{\frac{1}{q}}, \tag{3}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(\frac{m}{n})a_m b_n}{m^\lambda - n^\lambda} > \left[\frac{\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}\right]^2 \times\left\{\sum_{m=1}^{\infty}(1-\theta_\lambda(\lambda_1,m)) m^{p(1-\lambda_1)-1}a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1}b_n^q\right\}^{\frac{1}{q}}, \tag{4}$$

where

$$\left[\frac{\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}\right]^2$$

is the best possible and

$$\theta_\lambda(\lambda_1,m) := \left[\frac{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}{\pi}\right]^2 \int_m^\infty \frac{\ln u}{u^\lambda - 1}u^{\lambda_1-1}du = O\left(\frac{1}{m^{\lambda_1/2}}\right) \in (0,1).$$

In 2011, Yang [4] established the following more accurate extension of (1): If $0<\lambda_1,\lambda_2\leqslant 1$, $\lambda_1+\lambda_2=\lambda$, $a_m, b_n \geqslant 0$,

$$||a||_{p,\varphi} = \{\sum_{m=1}^{\infty}(m-\alpha)^{p(1-\lambda_1)-1}a_m^p\}^{\frac{1}{p}} \in (0,\infty),$$

$$||b||_{q,\psi} = \{\sum_{n=1}^{\infty}(n-\alpha)^{q(1-\lambda_2)-1}b_n^q\}^{\frac{1}{q}} \in (0,\infty),$$

then we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n-2\alpha)^\lambda} < B(\lambda_1,\lambda_2)||a||_{p,\varphi}||b||_{q,\psi} \quad \left(0\leqslant\alpha\leqslant\frac{1}{2}\right), \tag{5}$$

where the constant factor $B(\lambda_1,\lambda_2)$ is the best possible and $B(u,v)$ stands for the beta function defined by (cf. [5])

$$B(u,v) := \int_0^\infty \frac{1}{(1+t)^{u+v}}t^{u-1}dt \quad (u,v>0). \tag{6}$$

For $\lambda = 1,\ \lambda_1 = \frac{1}{q},\ \lambda_2 = \frac{1}{p}, \alpha = 0,$ inequality (5) reduces to (1). Some further results related to (1)–(5) are provided in [6]–[38].

In the present paper, using weight coefficients and applying Hermite-Hadamard's inequality, we derive the following more accurate extension of (4) with parameters in the whole plane:

If $\xi, \eta \in [0,\ \frac{1}{2}],\ 0 < \lambda_1,\ \lambda_2 < 1,\ \lambda_1 + \lambda_2 = \lambda \leqslant 1,\ a_m,\ b_n \geqslant 0,$

$$\sum_{|m|=1}^{\infty} |m-\xi|^{p(1-\lambda_1)-1} a_m^p \in (0,\infty), \quad \sum_{|n|=1}^{\infty} |n-\eta|^{q(1-\lambda_2)-1} b_n^q \in (0,\infty),$$

$$\begin{aligned}\widetilde{\theta}(\lambda_2, m) &:= \frac{1}{\left[\frac{\pi}{\lambda \sin(\pi\lambda_1/\lambda)}\right]^2} \int_0^{\left(\frac{1+\eta}{|m-\xi|}\right)^{\lambda}} \frac{u^{(\lambda_2/\lambda)-1} \ln u}{u-1} du \\ &= O\left(\frac{1}{|m-\xi|^{\lambda_2/2}}\right) \in (0,1), |m| \in \mathbf{N},\end{aligned}$$

then we have

$$\begin{aligned}&\sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \frac{\ln|\frac{m-\xi}{n-\eta}|}{|m-\xi|^{\lambda} - |n-\eta|^{\lambda}} a_m b_n \\ &> 2\left[\frac{\pi}{\lambda \sin(\pi\lambda_1/\lambda)}\right]^2 \left[\sum_{|m|=1}^{\infty} (1-\widetilde{\theta}(\lambda_2, m))|m-\xi|^{p(1-\lambda_1)-1} a_m^p\right]^{\frac{1}{p}} \\ &\times \left[\sum_{|n|=1}^{\infty} |n-\eta|^{q(1-\lambda_2)-1} b_n^q\right]^{\frac{1}{q}}. \qquad (7)\end{aligned}$$

Furthermore, we provide a generalization of (7) with multi-parameters and a best possible constant factor. Certain equivalent forms and a few particular inequalities are also considered.

2. Some lemmas

In what follows, we shall assume that $\mathbf{N} = \{1, 2, \cdots\},\ 0 < p < 1\ (q < 0),\ \frac{1}{p} + \frac{1}{q} = 1,$ $\alpha, \beta \in (0, \pi),\ \xi, \eta \in [0, \frac{1}{2}],\ 0 < \lambda_1, \lambda_2 < 1, \lambda_1 + \lambda_2 = \lambda \leqslant 1.$ For $|x|, |y| > \frac{1}{2}$ we set

$$k(x,y) := \frac{\ln\left[\frac{|x-\xi|+(x-\xi)\cos\alpha}{|y-\eta|+(y-\eta)\cos\beta}\right]}{[|x-\xi|+(x-\xi)\cos\alpha]^{\lambda} - [|y-\eta|+(y-\eta)\cos\beta]^{\lambda}}. \qquad (8)$$

In particular, for $\alpha = \beta = \frac{\pi}{2}$ we set

$$h(x,y) := \frac{\ln|\frac{x-\xi}{y-\eta}|}{|x-\xi|^{\lambda} - |y-\eta|^{\lambda}}, \ |x|, |y| > \frac{1}{2}. \qquad (9)$$

DEFINITION 1. Define the following weight coefficients:

$$\omega(\lambda_2,m) := \sum_{|n|=1}^{\infty} k(m,n)\frac{[|m-\xi|+(m-\xi)\cos\alpha]^{\lambda_1}}{[|n-\eta|+(n-\eta)\cos\beta]^{1-\lambda_2}}, \quad |m|\in\mathbf{N}, \tag{10}$$

$$\varpi(\lambda_1,n) := \sum_{|m|=1}^{\infty} k(m,n)\frac{[|n-\eta|+(n-\eta)\cos\beta]^{\lambda_2}}{[|m-\xi|+(m-\xi)\cos\alpha]^{1-\lambda_1}}, \quad |n|\in\mathbf{N}, \tag{11}$$

where

$$\sum_{|j|=1}^{\infty}\cdots = \sum_{j=-1}^{-\infty}\cdots + \sum_{j=1}^{\infty}\cdots \quad (j=m,n).$$

LEMMA 1. *For*

$$k_\beta(\lambda_1) := 2\left[\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}\right]^2\csc^2\beta,$$

we have

$$k_\beta(\lambda_1)(1-\theta(\lambda_2,m)) < \omega(\lambda_2,m) < k_\beta(\lambda_1), \quad |m|\in\mathbf{N}, \tag{12}$$

where $\theta(\lambda_2,m)$ *is defined by*

$$\begin{aligned}\theta(\lambda_2,m) &:= \frac{1}{[\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}]^2}\int_0^{[\frac{(1+\eta)(1+\cos\beta)}{|m-\xi|+(m-\xi)\cos\alpha}]^\lambda}\frac{u^{(\lambda_2/\lambda)-1}\ln u}{u-1}du\\ &= O\left(\frac{1}{[|m-\xi|+(m-\xi)\cos\alpha]^{\lambda_2/2}}\right)\in(0,1), \quad |m|\in\mathbf{N}.\end{aligned} \tag{13}$$

Proof. For $|x|>\frac{1}{2}$, we set

$$k^{(1)}(x,y) := \frac{\ln\left[\frac{|x-\xi|+(x-\xi)\cos\alpha}{(y-\eta)(\cos\beta-1)}\right]}{[|x-\xi|+(x-\xi)\cos\alpha]^\lambda-[(y-\eta)(\cos\beta-1)]^\lambda}, \quad y<-\frac{1}{2},$$

$$k^{(2)}(x,y) := \frac{\ln\left[\frac{|x-\xi|+(x-\xi)\cos\alpha}{(y-\eta)(\cos\beta+1)}\right]}{[|x-\xi|+(x-\xi)\cos\alpha]^\lambda-[(y-\eta)(\cos\beta+1)]^\lambda}, \quad y>\frac{1}{2},$$

and then for $y>\frac{1}{2}$,

$$k^{(1)}(x,-y) = \frac{\ln\left[\frac{|x-\xi|+(x-\xi)\cos\alpha}{(y+\eta)(1-\cos\beta)}\right]}{[|x-\xi|+(x-\xi)\cos\alpha]^\lambda-[(y+\eta)(1-\cos\beta)]^\lambda}.$$

We obtain that

$$\begin{aligned}\omega(\lambda_2,m) &= \sum_{n=-1}^{-\infty}k^{(1)}(m,n)\frac{[|m-\xi|+(m-\xi)\cos\alpha]^{\lambda_1}}{[(n-\eta)(\cos\beta-1)]^{1-\lambda_2}}\\ &\quad+\sum_{n=1}^{\infty}k^{(2)}(m,n)\frac{[|m-\xi|+(m-\xi)\cos\alpha]^{\lambda_1}}{[(n-\eta)(1+\cos\beta)]^{1-\lambda_2}}\end{aligned}$$

$$= \frac{[|m-\xi|+(m-\xi)\cos\alpha]^{\lambda_1}}{(1-\cos\beta)^{1-\lambda_2}}\sum_{n=1}^{\infty}\frac{k^{(1)}(m,-n)}{(n+\eta)^{1-\lambda_2}} + \frac{[|m-\xi|+(m-\xi)\cos\alpha]^{\lambda_1}}{(1+\cos\beta)^{1-\lambda_2}}\sum_{n=1}^{\infty}\frac{k^{(2)}(m,n)}{(n-\eta)^{1-\lambda_2}}. \tag{14}$$

It is evident that for fixed $m \in \mathbf{N},\ \lambda_2 < 1\ \ (0<\lambda\leqslant 1),$ both

$$\frac{k^{(1)}(m,-y)}{(y+\eta)^{1-\lambda_2}} \text{ and } \frac{k^{(2)}(m,y)}{(y-\eta)^{1-\lambda_2}}$$

are strictly decreasing and strictly convex with respect to $y \in (\frac{1}{2},\infty),$ satisfying (cf. [4])

$$\frac{d}{dy}\frac{k^{(i)}(m,(-1)^i y)}{[y+(-1)^i\eta]^{1-\lambda_2}} < 0, \qquad \frac{d^2}{dy^2}\frac{k^{(i)}(m,(-1)^i y)}{[y+(-1)^i\eta]^{1-\lambda_2}} > 0\ \ (i=1,2).$$

By Hermite-Hadamard's inequality (cf. [39]), we derive that

$$\begin{aligned}\omega(\lambda_2,m) &< \frac{[|m-\xi|+(m-\xi)\cos\alpha]^{\lambda_1}}{(1-\cos\beta)^{1-\lambda_2}}\int_{\frac{1}{2}}^{\infty}\frac{k^{(1)}(m,-y)}{(y+\eta)^{1-\lambda_2}}dy \\ &\quad+\frac{[|m-\xi|+(m-\xi)\cos\alpha]^{\lambda_1}}{(1+\cos\beta)^{1-\lambda_2}}\int_{\frac{1}{2}}^{\infty}\frac{k^{(2)}(m,y)}{(y-\eta)^{1-\lambda_2}}dy \\ &< \frac{[|m-\xi|+(m-\xi)\cos\alpha]^{\lambda_1}}{(1-\cos\beta)^{1-\lambda_2}}\int_{-\eta}^{\infty}\frac{k^{(1)}(m,-y)}{(y+\eta)^{1-\lambda_2}}dy \\ &\quad+\frac{[|m-\xi|+(m-\xi)\cos\alpha]^{\lambda_1}}{(1+\cos\beta)^{1-\lambda_2}}\int_{\eta}^{\infty}\frac{k^{(2)}(m,y)}{(y-\eta)^{1-\lambda_2}}dy.\end{aligned}$$

Setting

$$u = \left[\frac{(y+\eta)(1-\cos\beta)}{|m-\xi|+(m-\xi)\cos\alpha}\right]^{\lambda}\ \left(\text{resp. } \left[\frac{(y-\eta)(1+\cos\beta)}{|m-\xi|+(m-\xi)\cos\alpha}\right]^{\lambda}\right)$$

in the above first (resp. second) integral, by simplification, we deduce that

$$\begin{aligned}\omega(\lambda_2,m) &< \left(\frac{1}{1-\cos\beta}+\frac{1}{1+\cos\beta}\right)\frac{1}{\lambda^2}\int_0^{\infty}\frac{u^{(\lambda_2/\lambda)-1}\ln u}{u-1}du \\ &= 2\left[\frac{\pi}{\lambda\sin(\pi\lambda_2/\lambda)}\right]^2\csc^2\beta \\ &= 2\left[\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}\right]^2\csc^2\beta = k_\beta(\lambda_1).\end{aligned}$$

By (14), since both

$$\frac{k^{(1)}(m,-y)}{(y+\eta)^{1-\lambda_2}} \text{ and } \frac{k^{(2)}(m,y)}{(y-\eta)^{1-\lambda_2}}$$

are strictly decreasing, we also obtain that

$$\begin{aligned}
\omega(\lambda_2,m) &> \frac{[|m-\xi|+(m-\xi)\cos\alpha]^{\lambda_1}}{(1-\cos\beta)^{1-\lambda_2}}\int_1^\infty \frac{k^{(1)}(m,-y)}{(y+\eta)^{1-\lambda_2}}dy \\
&\quad + \frac{[|m-\xi|+(m-\xi)\cos\alpha]^{\lambda_1}}{(1+\cos\beta)^{1-\lambda_2}}\int_1^\infty \frac{k^{(2)}(m,y)}{(y-\eta)^{1-\lambda_2}}dy \\
&\geqslant \frac{1}{1-\cos\beta}\int_{[\frac{(1+\eta)(1+\cos\beta)}{|m-\xi|+(m-\xi)\cos\alpha}]^\lambda}^\infty \frac{u^{(\lambda_2/\lambda)-1}\ln u}{u-1}du \\
&\quad + \frac{1}{1+\cos\beta}\int_{[\frac{(1+\eta)(1+\cos\beta)}{|m-\xi|+(m-\xi)\cos\alpha}]}^\infty \frac{u^{(\lambda_2/\lambda)-1}\ln u}{u-1}du \\
&= k_\beta(\lambda_2)(1-\theta(\lambda_2,m)) > 0,
\end{aligned}$$

where $\theta(\lambda_2,m)$ is as defined by (13).

Since

$$\frac{u^{(\lambda_2/2\lambda)-1}\ln u}{u-1} \to 0 \;\; (u \to 0^+),$$

and

$$\frac{u^{(\lambda_2/2\lambda)-1}\ln u}{u-1} \to 1 \;\; (u \to 1),$$

there exists a constant $M > 0$, such that

$$0 < \frac{u^{(\lambda_2/2\lambda)-1}\ln u}{u-1} \leqslant M \quad \left(u \in \left(0, \left[\frac{(1+\eta)(1+\cos\beta)}{|m-\xi|+(m-\xi)\cos\alpha}\right]^\lambda\right]\right).$$

Then we derive that

$$\begin{aligned}
0 < \theta(\lambda_2,m) &= \frac{1}{[\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}]^2}\int_0^{[\frac{(1+\eta)(1+\cos\beta)}{|m-\xi|+(m-\xi)\cos\alpha}]^\lambda} \frac{u^{(\lambda_2/2\lambda)-1}\ln u}{u-1}u^{\frac{\lambda_2}{2\lambda}-1}du \\
&\leqslant \frac{M}{[\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}]^2}\int_0^{[\frac{(1+\eta)(1+\cos\beta)}{|m-\xi|+(m-\xi)\cos\alpha}]^\lambda} u^{(\lambda_2/2\lambda)-1}du \\
&= \frac{2\lambda M}{\lambda_2[\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}]^2}\left[\frac{(1+\eta)(1+\cos\beta)}{|m-\xi|+(m-\xi)\cos\alpha}\right]^{\lambda_2/2}.
\end{aligned}$$

Hence, we obtain (12) and (13).

This completes the proof of lemma 1 (A. Raigorodskii). □

Similarly, we also obtain the following:

LEMMA 2. *For*

$$k_\alpha(\lambda_1) = 2\left[\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}\right]^2 \csc^2\alpha,$$

we have

$$k_\alpha(\lambda_1)(1-\vartheta(\lambda_1,n)) < \varpi(\lambda_1,n) < k_\alpha(\lambda_1), \quad |n| \in \mathbf{N}, \tag{15}$$

where

$$\vartheta(\lambda_1,n) := \frac{1}{[\frac{\pi}{\lambda \sin(\pi\lambda_1/\lambda)}]^2} \int_0^{[\frac{(1+\xi)(1+\cos\alpha)}{|n-\eta|+(n-\eta)\cos\beta}]^\lambda} \frac{u^{(\lambda_1/\lambda)-1}\ln u}{u-1} du$$
$$= O\left(\frac{1}{[|n-\eta|+(n-\eta)\cos\beta]^{\lambda_1/2}}\right) \in (0,1), \quad |n| \in \mathbf{N}. \tag{16}$$

LEMMA 3. *If* $\zeta \in [0,\frac{1}{2}]$, $\theta \in (0,\pi)$, *then for* $\rho > 0$, *we have*

$$H_\rho(\zeta,\theta) := \sum_{|n|=1}^{\infty} \frac{1}{[|n-\zeta|+(n-\zeta)\cos\theta]^{1+\rho}}$$
$$= \frac{1+o(1)}{\rho}\left[\frac{1}{(1+\cos\theta)^{1+\rho}} + \frac{1}{(1-\cos\theta)^{1+\rho}}\right] \quad (\rho \to 0^+). \tag{17}$$

Proof. We find

$$H_\rho(\zeta,\theta) = \sum_{n=-1}^{-\infty} \frac{1}{[(n-\zeta)(\cos\theta-1)]^{1+\rho}} + \sum_{n=1}^{\infty} \frac{1}{[(n-\zeta)(\cos\theta+1)]^{1+\rho}}$$
$$= \frac{1}{(1-\cos\theta)^{1+\rho}} \sum_{n=1}^{\infty} \frac{1}{(n+\zeta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}} \sum_{n=1}^{\infty} \frac{1}{(n-\zeta)^{1+\rho}}.$$

For $a = \frac{1}{(1-\zeta)^{1+\rho}}$, we obtain

$$H_\rho(\zeta,\theta) \leqslant \left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right]\left[a + \sum_{n=2}^{\infty} \frac{1}{(n-\zeta)^{1+\rho}}\right]$$
$$< \left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right]\left[a + \int_1^\infty \frac{dy}{(y-\zeta)^{1+\rho}}\right]$$
$$= \frac{1}{\rho}\left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right]$$
$$\times \left\{1 + \left[a\rho + \frac{1}{(1-\zeta)^\rho} - 1\right]\right\},$$

and

$$H_\rho(\zeta,\theta) \geqslant \left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right] \sum_{n=1}^{\infty} \frac{1}{(n+\zeta)^{1+\rho}}$$
$$> \left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right] \int_1^\infty \frac{dy}{(y+\zeta)^{1+\rho}}$$
$$= \frac{1+[(1+\zeta)^{-\rho}-1]}{\rho}\left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right].$$

Hence, we get (17).

This completes the proof of lemma 3. □

3. Main results and some particular inequalities

THEOREM 1. *If* $a_m, b_n \geqslant 0 \ \ (|m|, |n| \in \mathbf{N})$, *for which the following inequalities hold:*

$$0 < \sum_{|m|=1}^{\infty} [|m-\xi| + (m-\xi)\cos\alpha]^{p(1-\lambda_1)-1} a_m^p < \infty,$$

$$0 < \sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\lambda_2)-1} b_n^q < \infty,$$

setting

$$K(\lambda_1) := k_\beta^{\frac{1}{p}}(\lambda_1) k_\alpha^{\frac{1}{q}}(\lambda_1) = 2\left[\frac{\pi}{\lambda \sin(\pi\lambda_1/\lambda)}\right]^2 \csc^{\frac{2}{p}}\beta \csc^{\frac{2}{q}}\alpha, \tag{18}$$

we have the following equivalent inequalities:

$$\begin{aligned} I &:= \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} k(m,n) a_m b_n \\ &> K(\lambda_1) \left\{ \sum_{|m|=1}^{\infty} (1-\theta(\lambda_2,m))[|m-\xi| + (m-\xi)\cos\alpha]^{p(1-\lambda_1)-1} a_m^p \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\lambda_2)-1} b_n^q \right\}^{\frac{1}{q}}, \end{aligned} \tag{19}$$

$$\begin{aligned} J_1 &:= \left\{ \sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{p\lambda_2-1} \left(\sum_{|m|=1}^{\infty} k(m,n) a_m \right)^p \right\}^{\frac{1}{p}} \\ &> K(\lambda_1) \left\{ \sum_{|m|=1}^{\infty} (1-\theta(\lambda_2,m))[|m-\xi| + (m-\xi)\cos\alpha]^{p(1-\lambda_1)-1} a_m^p \right\}^{\frac{1}{p}}. \end{aligned} \tag{20}$$

$$\begin{aligned} J_2 &:= \left\{ \sum_{|m|=1}^{\infty} \frac{[|m-\xi| + (m-\xi)\cos\alpha]^{q\lambda_1-1}}{(1-\theta(\lambda_2,m))^{1-q}} \left(\sum_{|n|=1}^{\infty} k(m,n) b_n \right)^q \right\}^{\frac{1}{q}} \\ &> K(\lambda_1) \left\{ \sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\lambda_2)-1} b_n^q \right\}^{\frac{1}{q}}. \end{aligned} \tag{21}$$

Proof. By the reverse Hölder inequality (cf. [39]) and (11), we get

$$\begin{aligned} \left(\sum_{|m|=1}^{\infty} k(m,n) a_m \right)^p &= \left\{ \sum_{|m|=1}^{\infty} k(m,n) \frac{[|m-\xi| + (m-\xi)\cos\alpha]^{(1-\lambda_1)/q}}{[|n-\eta| + (n-\eta)\cos\beta]^{(1-\lambda_2)/p}} a_m \right. \\ &\quad \left. \times \frac{[|n-\eta| + (n-\eta)\cos\beta]^{(1-\lambda_2)/p}}{[|m-\xi| + (m-\xi)\cos\alpha]^{(1-\lambda_1)/q}} \right\}^p \end{aligned}$$

$$\geqslant \sum_{|m|=1}^{\infty} k(m,n) \frac{[|m-\xi|+(m-\xi)\cos\alpha]^{(1-\lambda_1)p/q}}{[|n-\eta|+(n-\eta)\cos\beta]^{1-\lambda_2}} a_m^p$$
$$\times \left\{ \sum_{|m|=1}^{\infty} k(m,n) \frac{[|n-\eta|+(n-\eta)\cos\beta]^{(1-\lambda_2)q/p}}{[|m-\xi|+(m-\xi)\cos\alpha]^{1-\lambda_1}} \right\}^{p-1}$$
$$= \frac{(\varpi(\lambda_1,n))^{p-1}}{[|n-\eta|+(n-\eta)\cos\beta]^{p\lambda_2-1}}$$
$$\times \sum_{|m|=1}^{\infty} k(m,n) \frac{[|m-\xi|+(m-\xi)\cos\alpha]^{(1-\lambda_1)p/q}}{[|n-\eta|+(n-\eta)\cos\beta]^{1-\lambda_2}} a_m^p.$$

By (15), we obtain

$$J_1 > k_\alpha^{\frac{1}{q}}(\lambda_1) \left\{ \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} k(m,n) \frac{[|m-\xi|+(m-\xi)\cos\alpha]^{(1-\lambda_1)p/q}}{[|n-\eta|+(n-\eta)\cos\beta]^{1-\lambda_2}} a_m^p \right\}^{\frac{1}{p}}$$
$$= k_\alpha^{\frac{1}{q}}(\lambda_1) \left\{ \sum_{|m|=1}^{\infty} \sum_{|n|=1}^{\infty} k(m,n) \frac{[|m-\xi|+(m-\xi)\cos\alpha]^{(1-\lambda_1)p/q}}{[|n-\eta|+(n-\eta)\cos\beta]^{1-\lambda_2}} a_m^p \right\}^{\frac{1}{p}}$$
$$= k_\alpha^{\frac{1}{q}}(\lambda_1) \left\{ \sum_{|m|=1}^{\infty} \omega(\lambda_2,m)[|m-\xi|+(m-\xi)\cos\alpha]^{p(1-\lambda_1)-1} a_m^p \right\}^{\frac{1}{p}}. \quad (22)$$

Then, by (12) for $0 < p < 1$, we derive (20).

By the reverse Hölder inequality (cf. [39]), we get

$$I = \sum_{|n|=1}^{\infty} \left\{ [|n-\eta|+(n-\eta)\cos\beta]^{\lambda_2-\frac{1}{p}} \sum_{|m|=1}^{\infty} k(m,n)a_m \right\}$$
$$\times [|n-\eta|+(n-\eta)\cos\beta]^{\frac{1}{p}-\lambda_2} b_n$$
$$\geqslant J_1 \left\{ \sum_{|n|=1}^{\infty} [|n-\eta|+(n-\eta)\cos\beta]^{q(1-\lambda_2)-1} b_n^q \right\}^{\frac{1}{q}}. \quad (23)$$

Then by (20), we deduce (19).

On the other hand, assuming that (19) is valid and setting

$$b_n := [|n-\eta|+(n-\eta)\cos\beta]^{p\lambda_2-1} \left(\sum_{|m|=1}^{\infty} k(m,n)a_m \right)^{p-1}, \quad |n| \in \mathbf{N},$$

it follows that

$$J_1 = \left\{ \sum_{|n|=1}^{\infty} [|n-\eta|+(n-\eta)\cos\beta]^{q(1-\lambda_2)-1} b_n^q \right\}^{\frac{1}{p}}.$$

By (22), we obtain that $J_1 > 0$. If $J_1 = \infty$, then (20) is evidently valid; if $J_1 < \infty$, then by (19), we have

$$\begin{aligned}
\infty &> \sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\lambda_2)-1} b_n^q = J_1^p = I \\
&> K(\lambda_1)\left\{\sum_{|m|=1}^{\infty} (1-\theta(\lambda_2,m))[|m-\xi| + (m-\xi)\cos\alpha]^{p(1-\lambda_1)-1} a_m^p\right\}^{\frac{1}{p}} \\
&\quad \times \left\{\sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\lambda_2)-1} b_n^q\right\}^{\frac{1}{q}} > 0, \\
J_1 &= \left\{\sum_{|n|=1}^{\infty} [|n-\eta| + (n-\eta)\cos\beta]^{q(1-\lambda_2)-1} b_n^q\right\}^{\frac{1}{p}} \\
&> K(\lambda_1)\left\{\sum_{|m|=1}^{\infty} (1-\theta(\lambda_2,m))[|m-\xi| + (m-\xi)\cos\alpha]^{p(1-\lambda_1)-1} a_m^p\right\}^{\frac{1}{p}},
\end{aligned}$$

namely, (20) follows which is equivalent to (19).

Assuming that (21) is valid, by the reverse Hölder inequality (cf. [39]), we obtain that

$$\begin{aligned}
I &= \sum_{|m|=1}^{\infty} \left\{(1-\theta(\lambda_2,m))^{\frac{1}{p}} [|m-\xi| + (m-\xi)\cos\alpha]^{\frac{1}{q}-\lambda_1} a_m\right\} \\
&\quad \times \left\{\frac{[|m-\xi| + (m-\xi)\cos\alpha]^{\frac{-1}{q}+\lambda_1}}{(1-\theta(\lambda_2,m))^{\frac{-1}{p}}} \sum_{|n|=1}^{\infty} k(m,n) b_n\right\} \\
&\geqslant \left\{\sum_{|m|=1}^{\infty} (1-\theta(\lambda_2,m))[|m-\xi| + (m-\xi)\cos\alpha]^{p(1-\lambda_1)-1} a_m^p\right\}^{\frac{1}{p}} J_2. \qquad (24)
\end{aligned}$$

Then by (21), we deduce (19).

On the other hand, assuming that (19) is valid and setting

$$a_m := \frac{[|m-\xi| + (m-\xi)\cos\alpha]^{q\lambda_1-1}}{(1-\theta(\lambda_2,m))^{1-q}} \left(\sum_{|n|=1}^{\infty} k(m,n) b_n\right)^{q-1}, \quad |m| \in \mathbf{N},$$

it follows that

$$J_2 = \left\{\sum_{|m|=1}^{\infty} (1-\theta(\lambda_2,m))[|m-\xi| + (m-\xi)\cos\alpha]^{p(1-\lambda_1)-1} a_m^p\right\}^{\frac{1}{q}}.$$

If $J_2 = 0$, then it is impossible for (21) to be satisfied, namely $J_2 > 0$; if $J_2 = \infty$, then (21) is evidently valid. Suppose that $0 < J_2 < \infty$. By (19), we have

$$\begin{aligned}
\infty &> \sum_{|m|=1}^{\infty} (1-\theta(\lambda_2,m))[|m-\xi|+(m-\xi)\cos\alpha]^{p(1-\lambda_1)-1} a_m^p = J_2^q = I \\
&> K(\lambda_1) \left\{ \sum_{|m|=1}^{\infty} (1-\theta(\lambda_2,m))[|m-\xi|+(m-\xi)\cos\alpha]^{p(1-\lambda_1)-1} a_m^p \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \sum_{|n|=1}^{\infty} [|n-\eta|+(n-\eta)\cos\beta]^{q(1-\lambda_2)-1} b_n^q \right\}^{\frac{1}{q}} > 0,
\end{aligned}$$

$$\begin{aligned}
J_2 &= \left\{ \sum_{|m|=1}^{\infty} (1-\theta(\lambda_2,m))[|m-\xi|+(m-\xi)\cos\alpha]^{p(1-\lambda_1)-1} a_m^p \right\}^{\frac{1}{q}} \\
&> K(\lambda_1) \left\{ \sum_{|n|=1}^{\infty} [|n-\eta|+(n-\eta)\cos\beta]^{q(1-\lambda_2)-1} b_n^q \right\}^{\frac{1}{q}},
\end{aligned}$$

namely, (21) follows, which is equivalent to (19).

Hence, (19), (20) and (21) are equivalent.

This completes the proof of the theorem. □

THEOREM 2. *Subject to the assumptions of Theorem 1, the constant factor $K(\lambda_1)$ in (19), (20) and (21) is the best possible.*

Proof. For any $\varepsilon \in (0, |q|(1-\lambda_2))$, we set

$$\widetilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q}, \ \ \widetilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q} \ (\in (0,1)),$$

and

$$\begin{aligned}
\widetilde{a}_m &:= [|m-\xi|+(m-\xi)\cos\alpha]^{(\lambda_1-\frac{\varepsilon}{p})-1} \\
&= [|m-\xi|+(m-\xi)\cos\alpha]^{\widetilde{\lambda}_1-\varepsilon-1} \ (|m| \in \mathbf{N}), \\
\widetilde{b}_n &:= [|n-\eta|+(n-\eta)\cos\beta]^{(\lambda_2-\frac{\varepsilon}{q})-1} \\
&= [|n-\eta|+(n-\eta)\cos\beta]^{\widetilde{\lambda}_2-1} \ (|n| \in \mathbf{N}).
\end{aligned}$$

Then by (17) and (12), we derive that

$$\begin{aligned}
\widetilde{I}_1 &:= \left\{ \sum_{|m|=1}^{\infty} (1-\theta(\lambda_2,m))[|m-\xi|+(m-\xi)\cos\alpha]^{p(1-\lambda_1)-1} \widetilde{a}_m^p \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \sum_{|n|=1}^{\infty} [|n-\eta|+(n-\eta)\cos\beta]^{q(1-\lambda_2)-1} \widetilde{b}_n^q \right\}^{\frac{1}{q}}
\end{aligned}$$

$$= \left\{ \sum_{|m|=1}^{\infty} \frac{1}{[|m-\xi|+(m-\xi)\cos\alpha]^{1+\varepsilon}} \right.$$

$$\left. - \sum_{|m|=1}^{\infty} O\Big(\frac{1}{[|m-\xi|+(m-\xi)\cos\alpha]^{1+\frac{\lambda_2}{2}+\varepsilon}}\Big) \right\}^{\frac{1}{p}}$$

$$\times \left\{ \sum_{|n|=1}^{\infty} \frac{1}{[|n-\eta|+(n-\eta)\cos\beta]^{1+\varepsilon}} \right\}^{\frac{1}{q}}$$

$$= \frac{1}{\varepsilon}\left[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}}+\frac{1}{(1-\cos\alpha)^{1+\varepsilon}}-\varepsilon O(1)\right]^{\frac{1}{p}}$$

$$\times \left[\frac{1}{(1+\cos\beta)^{1+\varepsilon}}+\frac{1}{(1-\cos\beta)^{1+\varepsilon}}\right]^{\frac{1}{q}}$$

$$\times (1+o_1(1)-\varepsilon O(1))^{\frac{1}{p}}(1+o_2(1))^{\frac{1}{q}},$$

and

$$\widetilde{I} := \sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty} k(m,n)\widetilde{a}_m\widetilde{b}_n$$

$$= \sum_{|m|=1}^{\infty}\sum_{|m|=1}^{\infty} k(m,n)\frac{[|m-\xi|+(m-\xi)\cos\alpha]^{\widetilde{\lambda}_1-\varepsilon-1}}{[|n-\eta|+(n-\eta)\cos\beta]^{1-\widetilde{\lambda}_2}}$$

$$= \sum_{|m|=1}^{\infty} \frac{\omega(\widetilde{\lambda}_2,m)}{[|m-\xi|+(m-\xi)\cos\alpha]^{\varepsilon+1}}$$

$$\leqslant k_\beta(\widetilde{\lambda}_1)\sum_{|m|=1}^{\infty} \frac{1}{[|m-\xi|+(m-\xi)\cos\alpha]^{\varepsilon+1}}$$

$$= \frac{k_\beta(\widetilde{\lambda}_1)}{\varepsilon}\left\{\Big[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}}+\frac{1}{(1-\cos\alpha)^{1+\varepsilon}}\Big](1+o_1(1))\right\}.$$

If there exists a constant $k \geqslant K(\lambda_1)$, such that (19) is satisfied when replacing $K(\lambda_1)$ by k, then in particular we have $\varepsilon\widetilde{I} > k\widetilde{I}_1$, namely,

$$k_\beta(\widetilde{\lambda}_1)\left\{\Big[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}}+\frac{1}{(1-\cos\alpha)^{1+\varepsilon}}\Big](1+o_1(1))\right\}$$

$$> k\left[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}}+\frac{1}{(1-\cos\alpha)^{1+\varepsilon}}-\varepsilon O(1)\right]^{\frac{1}{p}}(1+o_1(1))^{\frac{1}{p}}$$

$$\times\left[\frac{1}{(1+\cos\beta)^{1+\varepsilon}}+\frac{1}{(1-\cos\beta)^{1+\varepsilon}}\right]^{\frac{1}{q}}(1+o_2(1))^{\frac{1}{q}}.$$

It follows that

$$4\Big[\frac{\pi}{\lambda\sin(\pi\lambda_1/\lambda)}\Big]^2\csc^2\beta\csc^2\alpha \geqslant 2k\csc^{\frac{2}{p}}\alpha\csc^{\frac{2}{q}}\beta\ (\varepsilon\to 0^+),$$

that is

$$K(\lambda_1) = 2\Big[\frac{\pi}{\lambda \sin(\pi\lambda_1/\lambda)}\Big]^2 \csc^{\frac{2}{p}}\beta \csc^{\frac{2}{q}}\alpha \geqslant k.$$

Hence, $k = K(\lambda_1)$ is the best possible constant factor of (19).

The constant factor $K(\lambda_1)$ in (20) (resp. (21)) is still the best possible. Otherwise, we would reach a contradiction by (23) (resp. (24)) that the constant factor in (19) is not the best possible.

This completes the proof of the theorem. □

REMARK 1. (i) For $\alpha = \beta = \frac{\pi}{2}$, the inequality (19) reduces to (7); inequalities (20) and (21) reduce to the equivalent forms of (7) as follows:

$$\left[\sum_{|n|=1}^{\infty} |n-\eta|^{p\lambda_2-1}\left(\sum_{|m|=1}^{\infty} h(m,n)a_m\right)^p\right]^{\frac{1}{p}}$$

$$> 2\left[\frac{\pi}{\lambda \sin(\pi\lambda_1/\lambda)}\right]^2 \left[\sum_{|m|=1}^{\infty} (1-\widetilde{\theta}(\lambda_2,m))|m-\xi|^{p(1-\lambda_1)-1} a_m^p\right]^{\frac{1}{p}}, \tag{25}$$

$$\left\{\sum_{|m|=1}^{\infty} \frac{|m-\xi|^{q\lambda_1-1}}{(1-\widetilde{\theta}(\lambda_2,m))^{1-q}}\left(\sum_{|n|=1}^{\infty} h(m,n)b_n\right)^q\right\}^{\frac{1}{q}}$$

$$> 2\left[\frac{\pi}{\lambda \sin(\pi\lambda_1/\lambda)}\right]^2 \left\{\sum_{|n|=1}^{\infty} |n-\eta|^{q(1-\lambda_2)-1} b_n^q\right\}^{\frac{1}{q}}. \tag{26}$$

Hence, (19) provides an extension of (7).

(ii) For $\xi = \eta = 0$,

$$\widehat{\theta}(\lambda_2,m) := \frac{1}{[\frac{\pi}{\lambda \sin(\pi\lambda_1/\lambda)}]^2} \int_0^{(\frac{1+\cos\beta}{|m|+m\cos\alpha})^{\lambda}} \frac{u^{(\lambda_2/\lambda)-1}\ln u}{u-1} du$$

$$= O\left(\frac{1}{(|m|+m\cos\alpha)^{\lambda_2/2}}\right) \in (0,1), \quad |m| \in \mathbf{N}.$$

the inequalities (19), (20) and (21) reduce to the following equivalent inequalities:

$$\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty} \frac{\ln[\frac{|m|+m\cos\alpha}{|n|+n\cos\beta}]}{(|m|+m\cos\alpha)^{\lambda} - (|n|+n\cos\beta)^{\lambda}} a_m b_n s$$

$$> K(\lambda_1)\left[\sum_{|m|=1}^{\infty} (1-\widehat{\theta}(\lambda_2,m))(|m|+m\cos\alpha)^{p(1-\lambda_1)-1} a_m^p\right]^{\frac{1}{p}}$$

$$\times \left[\sum_{|n|=1}^{\infty} (|n|+n\cos\beta)^{q(1-\lambda_2)-1} b_n^q\right]^{\frac{1}{q}}, \tag{27}$$

$$\left\{\sum_{|n|=1}^{\infty}[|n|+n\cos\beta]^{p\lambda_2-1}\left[\sum_{|m|=1}^{\infty}\frac{\ln[\frac{|m|+m\cos\alpha}{|n|+n\cos\beta}]}{(|m|+m\cos\alpha)^{\lambda}-(|n|+n\cos\beta)^{\lambda}}a_m\right]^p\right\}^{\frac{1}{p}}$$

$$> K(\lambda_1)\left\{\sum_{|m|=1}^{\infty}(1-\widehat{\theta}(\lambda_2,m))(|m|+m\cos\alpha)^{p(1-\lambda_1)-1}a_m^p\right\}^{\frac{1}{p}}. \tag{28}$$

$$\left\{\sum_{|m|=1}^{\infty}\frac{(|m|+m\cos\alpha)^{q\lambda_1-1}}{(1-\widehat{\theta}(\lambda_2,m))^{1-q}}\left[\sum_{|n|=1}^{\infty}\frac{\ln[\frac{|m|+m\cos\alpha}{|n|+n\cos\beta}]}{(|m|+m\cos\alpha)^{\lambda}-(|n|+n\cos\beta)^{\lambda}}b_n\right]^q\right\}^{\frac{1}{q}}$$

$$> K(\lambda_1)\left\{\sum_{|n|=1}^{\infty}(|n|+n\cos\beta)^{q(1-\lambda_2)-1}b_n^q\right\}^{\frac{1}{q}}. \tag{29}$$

Hence, inequality (19) constitutes a more accurate extension of (27).

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REFERENCES

[1] G. H. HARDY, *Note on a theorem of Hilbert concerning series of positive terms*, Proceedings London Math. Soc., vol. 23, no. 2, Records of Proc. xlv–xlvi, 1925.

[2] G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, *Inequalities*, Cambridge University Press, Cambridge, 1934.

[3] B. C. YANG, *The Norm of Operator and Hilbert-type Inequalities*, Science Press, Beijing, China, 2009.

[4] B. C. YANG, *Discrete Hilbert-type Inequalities*, Bentham Science Publishers Ltd., The United Arab Emirates, 2011.

[5] Z. Q. WANG AND D. R. GUO, *Introduction to Special Functions*, Science Press, Beijing, China, 1979.

[6] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK, *Inequalities Involving Functions and their Integrals and Derivatives*, Kluwer Acaremic Publishers, Boston, 1991.

[7] M. Z. GAO AND B. C. YANG, *On extened Hilbert's inequality*, Proceedings of the American Math. Society, 1998, 126 (3): 751–759.

[8] Y. HONG, *All-side generalization about Hardy-Hilbert integral inequalities*, Acta Mathematica Sinica, 2001, 44 (4): 619–625.

[9] Y. HONG, B. HE, B. YANG, *Necessary and sufficient conditions for the validity of Hilbert type integral inequalities with a class of quasi-homogeneous kernels and its application in operator theory*, J. Math. Inequal. 12 (2018), 777–788.

[10] K. W. ZHANG, *A bilinear inequality*, Journal of Mathematical Analysis and Applications, 2002, 271: 188–296.

[11] A. BENYI AND C. OH, *Best constant for certain multilinear integral operator*, Journal of Inequalities and Applications, vol. 2006, Article ID 28582, 2006.

[12] J. C. KUANG AND L. DEBNATH, *On Hilbert's type integral inequalities on the weighted Orlicz spaces*, Pacific J. Appl. Math., 2006, 1 (1): 95–104.

[13] Y. J. LI AND B. HE, *On inequalities of Hilbert's type*, Bulletin of the Australian Mathematical Society, 2007, 76 (1): 1–13.

[14] L. E. AZAR, *On some extensions of Hardy-Hilbert's inequality and applications*, Journal of Inequalities and Applications, vol. 2008, Article ID 546829, 2008.

[15] W. Y. ZHONG, *The Hilbert-type integral inequality with a homogeneous kernel of Lambda-degree*, Journal of Inequalities and Applications, vol. 2008, Article ID 917392, 2008.

[16] J. J. JIN AND L. DEBNATH, *On a Hilbert-type linear series operator and its applications*, Journal of Mathematical Analysis and Applications, 2010, 371 (2): 691–404.

[17] Q. L. HUANG, *On a multiple Hilbert-type integral operator and applications*, Journal of Inequalities and Applications, vol. 2010, Article ID 309319, 2010.

[18] B. C. YANG AND M. KRNIĆ, *On the norm of a multi-dimensional Hilbert-type operator*, Sarajevo Journal of Mathematics, 2013, 7 (20): 223–243.

[19] M. KRNIĆ AND P. VUKOVIĆ, *On a multidimensional version of the Hilbert-type inequality*, Analysis Mathematica, 2012, 38: 291–303.

[20] G. V. MILOVANOVIĆ AND M. TH. RASSIAS (eds.), *Analytic Number Theory, Approximation Theory and Special Functions*, Springer, 2014.

[21] M. TH. RASSIAS AND B. C. YANG, *On half-discrete Hilbert's inequality*, Applied Mathematics and Computation, 2013, 220: 75–93.

[22] M. TH. RASSIAS AND B. C. YANG, *A multidimensional half-discrete Hilbert-type inequality and the Riemann zeta function*, Applied Mathematics and Computation, 2013, 225: 263–277.

[23] M. TH. RASSIAS AND B. C. YANG, *On a multidimensional half-discrete Hilbert-type inequality related to the hyperbolic cotangent function*, Applied Mathematics and Computation, 2014, 242: 800–813.

[24] M. TH. RASSIAS AND B. C. YANG, *On a multidimensional Hilbert-type integral inequality associated to the gamma function*, Applied Mathematics and Computation, 2014, 249: 408–418.

[25] M. TH. RASSIAS AND B. C. YANG, *A Hilbert-type integral inequality in the whole plane related to the hyper geometric function and the beta function*, Journal of Mathematical Analysis and Applications, 2015, 428 (2): 1286–1308.

[26] V. ADIYASUREN, TS. BATBOLD AND M. KRNIĆ, *Half-discrete Hilbert-type inequalities with mean operators, the best constants, and applications*, Appl. Math. Comput., 2014, 231: 148–159.

[27] Q. L. HUANG, *A new extension of Hardy-Hilbert-type inequality*, Journal of Inequalities and Applications (2015), 2015: 397.

[28] B. HE, *A multiple Hilbert-type discrete inequality with a new kernel and best possible constant factor*, Journal of Mathematical Analysis and Applications, 2015, 431: 990–902.

[29] Y. P. SHI AND B. C. YANG, *A new Hardy-Hilbert-type inequality with multiparameters and a best possible constant factor*, Journal of Inequalities and Applications (2015), 2015: 380.

[30] M. TH. RASSIAS, B. C. YANG, *On a Hardy-Hilbert-type inequality with a general homogeneous kernel*, Int. J. Nonlinear Anal. Appl. 2016, 7 (1): 249–269.

[31] V. ADIYASUREN, TS. BATBOLD, AND M. KRNIĆ, *Multiple Hilbert-type inequalities involving some differential operators*, Banach J. Math. Anal., 2016, 10 (2), 320–337.

[32] M. TH. RASSIAS, B. C. YANG, *A more accurate half-discrete Hardy-Hilbert-type inequality with the best possible constant factor related to the extended Riemann-Zeta function*, Int. J. Nonlinear Anal. Appl., 2016, 7 (2): 1–27.

[33] M. TH. RASSIAS AND B. C. YANG, *Equivalent conditions of a Hardy-type integral inequality related to the extended Riemann zeta function*, Adv. Oper. Theory, 2017, 2 (3): 237–256.

[34] M. TH. RASSIAS AND B. C. YANG, *Equivalent properties of a Hilbert-type integral inequality with the best constant factor related the Hurwitz zeta function*, Ann. Funct. Anal., 2018, 9 (2): 282–295.

[35] M. TH. RASSIAS, B. C. YANG AND A. RAIGORODSKII, *Two Kinds of the Reverse Hardy-Type Integral Inequalities with the equivalent forms related to the extended Riemann zeta function*, Appl. Anal. Discrete Math., 2018, 12: 273–296.

[36] M. TH. RASSIAS AND B. C. YANG, *On an equivalent property of a reverse Hilbert-type integral inequality related to the extended Hurwitz-zeta function*, Journal of Mathematical Inequalities, 2019, 13(2): 315–334.
[37] M. TH. RASSIAS AND B. C. YANG, *A reverse Mulholland-type inequality in the whole plane with multi-parameters*, Appl. Anal. Discrete Math., 2019, 13: 290–308.
[38] M. TH. RASSIAS, B. YANG, *On a Hilbert-Type Integral InequalityRelated to the Extended Hurwitz Zeta Function in the Whole Plane*, Acta Appl. Math. 160 (2019), 67–80.
[39] J. KUANG, *Applied inequalities*, Shangdong Science and Technology Press, Jinan, China, 2004.

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