AN EIGENVALUE INEQUALITY FOR POSITIVE SEMIDEFINITE $k \times k$ BLOCK MATRICES

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Abstract. In this paper, we give some generalized results on matrix eigenvalue majorization inequality for positive semidefinite block matrices under a condition, which is a natural extended result given by Lin [4].

1. Introduction

First, we recall the definition of majorization. Given a real vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we rearrange its components as $x[1] \geq x[2] \geq \ldots \geq x[n]$. For $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, if

$$\sum_{i=1}^k x[i] \leq \sum_{i=1}^k y[i], \quad k = 1, 2, \ldots, n,$$

then we say that $x$ is weakly majorized by $y$ and denote $x \prec_w y$. If $x \prec_w y$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ hold, then we say that $x$ is majorized by $y$ and denote $x \prec y$.

As usual, the set of $m \times n$ complex matrices is denoted by $M_{m,n}$. For $A \in M_{n,n}$, we use $s_i(A)$ to present the singular values of $A$ with $s_1(A) \geq \ldots \geq s_n(A)$. Let $s(A) = (s_1(A), \ldots, s_n(A))$. If $A \in M_{n,n}$ is Hermitian, then all eigenvalues of $A$ are real and ordered as $\lambda_1(A) \geq \ldots \geq \lambda_n(A)$ and set $\lambda (A) = (\lambda_1(A), \ldots, \lambda_n(A))$. Note that $s_i(A) = \lambda_i(|A|)$, where $|A|$ is the modulus of $A$, i.e. $|A| = (A^*A)^{1/2}$ and $A^*$ is the conjugate transpose of $A$. $A \geq 0$ means that $A$ is positive semidefinite. In this paper, we use $A \oplus B$ to present the block matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

The study of eigenvalues is of central importance in matrix analysis. In 1923, Schur [1] showed that the diagonal entries of a Hermitian matrix are majorized by its eigenvalues, i.e.

$$\text{diag} (H) \prec \lambda (H).$$

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Let $H = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$ be a partitioned Hermitian matrix, where $A, B \in M_{n,n}$. Ky Fan extended Schur’s result to block Hermitian matrices, i.e.

$$\lambda (A \oplus B) \prec \lambda (H).$$

Lin and Wolkowicz [4] gave a reverse majorization result of above:

$$\lambda (H) \prec \lambda ((A + B) \oplus 0)$$

holds under the conditions that $C$ is Hermitian and $H$ is a positive semidefinite matrix. In 2012, Turkmen, Paksoy and Zhang [7] proved (1), where $C$ is skew-Hermitian and $H$ is a positive semidefinite matrix. Zhang [8] showed that

$$\lambda (H) \prec \frac{1}{2} \lambda ([A + B + \sqrt{-1}(zC^* - z^*C)] \oplus 0) + \frac{1}{2} \lambda ([A + B + \sqrt{-1}(z^*C - zC^*)] \oplus 0),$$

where $|z| = 1$. One may see [9] and its references for more results on majorization inequalities.

Motivated by the above, we generalize (1) to following:

**Theorem 1.** Let $A_{ij} \in M_{n,n}$, $i, j = 1, 2, \ldots, s$ ($s \geq 2$). Let $(i \neq j)$ be skew-Hermitian matrices. Let $H = [A_{ij}] \in M_{sn,sn}$ be positive semidefinite matrix. Then

$$\lambda (H) \prec \lambda ((\sum_{i=1}^{s} A_{ii}) \oplus 0).$$

2. Proofs of the main results and corollaries

Before we prove the main results, we first recall some well known results on majorization:

**Lemma 2.** [10] Let $A, B \in M_{n,n}$ be Hermitian matrices. Then we have

$$\lambda (A + B) \prec \lambda (A) + \lambda (B).$$

**Lemma 3.** (Lemma 1.3 of [4]) Let $A \in M_{m,n}$ and $m \geq n$. Then

$$\lambda (AA^*) = \lambda ((A^*A) \oplus 0).$$

**Lemma 4.** (Theorem 2.3.3 of [3]) Suppose $f(t)$ is a monotonically increasing and convex function, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$. Then $x \prec_w y$ implies

$$(f(x_1), \ldots, f(x_n)) \prec_w (f(y_1), \ldots, f(y_n)).$$

Let $\text{Span}\{P_1, P_2, \ldots, P_s\} = \{k_1, \ldots, k_s \in R| k_1P_1 + k_2P_2 + \ldots + k_sP_s\}$. We assume that $P_i^*P_j = -P_j^*P_i (i \neq j)$. Now we use mathematical induction to deduce the following lemma.
LEMMA 5. Let $P_1, P_2, \ldots, P_s \in M_{m,n}$ ($s \geq 2$) satisfying $P_i^* P_j = -P_j^* P_i (i \neq j)$. Then there exist $2^{s-1}$ matrices $V_1, \ldots, V_{2^{s-1}} \in \text{Span}\{P_1, P_2, \ldots, P_s\}$ such that

$$\sum_{i=1}^{2^{s-1}} V_i V_i^* = 2^{s-1} (\sum_{i=1}^{s} P_i P_i^*)$$

and for all $j = 1, \ldots, 2^{s-1}$

$$V_j^* V_j = \sum_{i=1}^{s} P_i^* P_i.$$

Proof. When $s = 2$, let $V_1 = P_1 + P_2$, $V_2 = P_1 - P_2$, we obtain

$$V_1 V_1^* + V_2 V_2^* = 2(P_1 P_1^* + P_2^* P_2)$$

and

$$V_j^* V_j = \sum_{i=1}^{2} P_i^* P_i$$

for $j = 1, 2$. Then the inequality holds.

Suppose that the Lemma holds for $s = t$, that is, there exist $2^{t-1}$ matrices $U_1, \ldots, U_{2^{t-1}} \in \text{Span}\{P_1, P_2, \ldots, P_t\}$ satisfying

$$\sum_{i=1}^{2^{t-1}} U_i U_i^* = 2^{t-1} (\sum_{i=1}^{t} P_i P_i^*) \quad (2)$$

and

$$U_j^* U_j = \sum_{i=1}^{t} P_i^* P_i \quad (3)$$

for $j = 1, \ldots, 2^{t-1}$.

Then for $s = t + 1$, set $B_i = U_i + P_{t+1}$, $C_i = U_i - P_{t+1}$, $1 \leq i \leq 2^{t-1}$,

$$\sum_{i=1}^{2^{t-1}} B_i B_i^* + \sum_{i=1}^{2^{t-1}} C_i C_i^* = 2^{2^{t-1}} (\sum_{i=1}^{t} U_i U_i^*) + 2^{t} P_{t+1}^* P_{t+1}^*$$

$$= 2^{t} (\sum_{i=1}^{t} P_i P_i^*) + 2^{t} P_{t+1}^* P_{t+1}^*$$

$$= 2^{t} (\sum_{i=1}^{t+1} P_i P_i^*).$$

The equality (4) follows from $P_i^* P_j = -P_j^* P_i$ and equality (2).

Let $V_i = B_i$, $V_{2^{t-1}+i} = C_i$, ($1 \leq i \leq 2^{t-1}$). Then $V_1, \ldots, V_{2^t} \in \text{Span}\{P_1, P_2, \ldots, P_t, P_{t+1}\}$ and

$$\sum_{i=1}^{2^t} V_i V_i^* = 2^{t} (\sum_{i=1}^{t+1} P_i P_i^*).$$
By \( P_i^*P_j = -P_j^*P_i \), we have
\[
V_j^*V_j = (U_j + P_{t+1})^*(U_j + P_{t+1}) = U_j^*U_j + P_{t+1}^*P_{t+1}
\]
\[
= \sum_{i=1}^t P_i^*P_i + P_{t+1}^*P_{t+1} = \sum_{i=1}^{t+1} P_i^*P_i
\]
for \( 1 \leq j \leq 2^{t-1} \) and
\[
V_j^*V_j = (U_j - P_{t+1})^*(U_j - P_{t+1}) = U_j^*U_j + P_{t+1}^*P_{t+1}
\]
\[
= \sum_{i=1}^t P_i^*P_i + P_{t+1}^*P_{t+1} = \sum_{i=1}^{t+1} P_i^*P_i
\]
for \( 2^{t-1} + 1 \leq j \leq 2^t \).

That is, the equality
\[
V_j^*V_j = \sum_{i=1}^{t+1} P_i^*P_i
\]
holds for \( 1 \leq j \leq 2^t \). Thus we have finished the proof. \( \square \)

*Proof of Theorem 1.* Since \( H \) is positive semidefinite, it follows that there exists a matrix \( P \in M_{sn,sn} \) such that \( H = P^*P \). Let \( P = [P_1, P_2, \ldots, P_s] \), where \( P_1, P_2, \ldots, P_s \in M_{sn,n} \). An easy computation shows that
\[
A_{ij} = P_i^*P_j, \quad i, j = 1, 2, \ldots, s.
\]
It follows from \( A_{ij} \ (i \neq j) \) is skew-Hermitian that
\[
P_i^*P_j + P_j^*P_i = 0, \quad i \neq j.
\]
By Lemma 5, there exist \( 2^{s-1} \) matrices \( V_1, \ldots, V_{2^{s-1}} \in M_{sn,n} \) such that
\[
\sum_{i=1}^{2^{s-1}} V_i V_i^* = 2^{s-1}(\sum_{i=1}^s P_i P_i^*)
\]
and for \( j = 1, \ldots, 2^{s-1} \)
\[
V_j^*V_j = \sum_{i=1}^s P_i^*P_i.
\]
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Therefore, by the property of $H$ and the Lemma 2

$$\lambda(H) = \lambda(P^*P) = \lambda(PP^*) = \lambda\left(\sum_{j=1}^{s} P_j P_j^*\right)$$

$$= \frac{1}{2^{s-1}} \lambda\left(\sum_{j=1}^{2^{s-1}} V_j V_j^*\right) \prec \frac{1}{2^{s-1}} \sum_{j=1}^{2^{s-1}} \lambda(V_j V_j^*) = \frac{1}{2^{s-1}} \sum_{j=1}^{2^{s-1}} \lambda((V_j V_j)^\oplus 0)$$

$$= \frac{1}{2^{s-1}} \sum_{j=1}^{2^{s-1}} \lambda\left(\sum_{i=1}^{s} P_i^* P_i^\oplus 0\right) = \lambda\left(\sum_{i=1}^{s} P_i^* P_i^\oplus 0\right)$$

$$= \lambda\left(\left(\sum_{i=1}^{s} A_{ii}\right)^\oplus 0\right).$$

The proof is completed. □

When $s = 2$, we notice $\lambda\left(\begin{bmatrix} A_{11} & \sqrt{-1} A_{12} \\ \sqrt{-1} A_{12} & A_{22} \end{bmatrix}\right) = \lambda\left(\begin{bmatrix} A_{11} & A_{12} \\ -A_{12} & A_{22} \end{bmatrix}\right) \prec \lambda((A_{11} + A_{22})^\oplus 0)$. Theorem 1 shows that Lin’s result (Theorem 2.1 of [4]) is a special case of our result.

**COROLLARY 6.** Let $A_1, A_2, \ldots, A_k \in M_{n,n}$ with $A_i^* A_j = -A_j^* A_i$, ($1 \leq i < j \leq k$). Then

$$\lambda\left(\sum_{i=1}^{k} A_i A_i^*\right) \prec \lambda\left(\sum_{i=1}^{k} A_i^* A_i\right).$$

**Proof.** Let $P = [A_1, A_2, \ldots, A_k]$. Then $H = P^*P$ is a positive semidefinite matrix and satisfy the condition of Theorem 1. Hence

$$\lambda(H) \prec \lambda\left(\left(\sum_{i=1}^{k} A_i^* A_i\right)^\oplus 0\right).$$

It follows from this and Lemma 3 that

$$\lambda\left(\left(\sum_{i=1}^{k} A_i A_i^*\right)^\oplus 0\right) = \lambda(P P^* \oplus 0) = \lambda(P^*P) = \lambda(H) \prec \lambda\left(\left(\sum_{i=1}^{k} A_i^* A_i\right)^\oplus 0\right).$$

Next we show that the condition $A_i^* A_j$ is skew-Hermitian in Corollary 6 is necessary.

**REMARK 7.** Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Then

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = A_1^* A_2 \neq A_2^* A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$
and
\[ A_1 A_1^* + A_2 A_2^* = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1^* A_1 + A_2^* A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}. \]

A trivial verification shows that \( \lambda(A_1 A_1^* + A_2 A_2^*) \nless \lambda(A_1^* A_1 + A_2^* A_2) \).

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