Journal of
Mathematical
Inequalities
Volume 14, Number 4 (2020), 1383–1388 doi:10.7153/jmi-2020-14-90

AN EIGENVALUE INEQUALITY FOR POSITIVE SEMIDEFINITE $k \times k$ BLOCK MATRICES

FENG ZHANG AND JINLI XU*

(*Communicated by J. Pečarić*)

Abstract. In this paper, we give some generalized results on matrix eigenvalue majorization inequality for positive semidefinite block matrices under a condition, which is a natural extended result given by Lin [4].

1. Introduction

First, we recall the definition of majorization. Given a real vector $x = (x_1, x_2, \dots, x_n) \in R^n$, we rearrange its components as $x_{[1]} \geqslant x_{[2]} \geqslant \dots \geqslant x_{[n]}$. For $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in R^n$, if

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]}, \ k = 1, 2 \dots, n,$$

then we say that x is weakly majorized by y and denote $x \prec_w y$. If $x \prec_w y$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ hold, then we say that x is majorized by y and denote $x \prec y$.

As usual, the set of $m \times n$ complex matrices is denoted by $M_{m,n}$. For $A \in M_{n,n}$, we use $s_i(A)$ to present the singular values of A with $s_1(A) \geqslant \dots \geqslant s_n(A)$. Let $s(A) = (s_1(A), \dots, s_n(A))$. If $A \in M_{n,n}$ is Hermitian, then all eigenvalues of A are real and ordered as $\lambda_1(A) \geqslant \dots \geqslant \lambda_n(A)$ and set $\lambda(A) = (\lambda_1(A), \dots, \lambda_n(A))$. Note that $s_i(A) = \lambda_i(|A|)$, where $|A|$ is the modulus of A, i.e. $|A| = (A^*A)^{\frac{1}{2}}$ and A^* is the conjugate transpose of A. $A \geqslant 0$ means that A is positive semidefinite. In this paper, we use $A \oplus B$ to present the block matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

The study of eigenvalues is of central importance in matrix analysis. In 1923, Schur [1] showed that the diagonal entries of a Hermitian matrix are majorized by its eigenvalues, i.e.

$$\mathrm{diag}\,(H) \prec \lambda\,(H).$$

Mathematics subject classification (2010): 15A42, 06A06.
Keywords and phrases: Positive semidefinite matrices, eigenvalue inequality, majorization.
* Corresponding author.

Let $H = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$ be a partitioned Hermitian matrix, where $A, B \in M_{n,n}$. Ky Fan extended Schur's result to block Hermitian matrices, i.e.

$$\lambda(A \oplus B) \prec \lambda(H).$$

Lin and Wolkowicz [4] gave a reverse majorization result of above:

$$\lambda(H) \prec \lambda((A+B) \oplus 0) \tag{1}$$

holds under the conditions that C is Hermitian and H is a positive semidefinite matrix. In 2012, Turkmen, Paksoy and Zhang [7] proved (1), where C is skew-Hermitian and H is a positive semidefinite matrix. Zhang [8] showed that

$$\lambda(H) \prec \frac{1}{2}\lambda([A+B+\sqrt{-1}(zC^* - z^*C)] \oplus 0) + \frac{1}{2}\lambda([A+B+\sqrt{-1}(z^*C - zC^*)] \oplus 0),$$

where $|z| = 1$. One may see [9] and its references for more results on majorization inequalities.

Motivated by the above, we generalize (1) to following:

THEOREM 1. *Let* $A_{ij} \in M_{n,n}$, $i, j = 1, 2, \ldots, s$ $(s \geqslant 2)$, *Let* $(i \neq j)$ *be skew-Hermitian matrices. Let* $H = [A_{ij}] \in M_{sn,sn}$ *be positive semidefinite matrix. Then*

$$\lambda(H) \prec \lambda((\sum_{i=1}^{s} A_{ii}) \oplus 0).$$

2. Proofs of the main results and corollaries

Before we prove the main results, we first recall some well known results on majorization:

LEMMA 2. [10] *Let* $A, B \in M_{n,n}$ *be Hermitian matrices. Then we have* $\lambda(A+B) \prec \lambda(A) + \lambda(B)$.

LEMMA 3. (Lemma1.3 of [4]) *Let* $A \in M_{m,n}$ *and* $m \geqslant n$. *Then* $\lambda(AA^*) = \lambda((A^*A) \oplus 0)$.

LEMMA 4. (Theorem 2.3.3 of [3]) *Suppose* $f(t)$ *is a monotonically increasing and convex function,* $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$. *Then* $x \prec_w y$ *implies*

$$(f(x_1), \ldots, f(x_n)) \prec_w (f(y_1), \ldots, f(y_n)).$$

Let $\mathrm{Span}\{P_1, P_2, \ldots, P_s\} = \{k_1, \ldots, k_s \in R | k_1P_1 + k_2P_2 + \ldots + k_sP_s\}$. We assume that $P_i^*P_j = -P_j^*P_i (i \neq j)$. Now we use mathematical induction to deduce the following lemma.

LEMMA 5. *Let* $P_1,P_2,\dots,P_s\in M_{m,n}$ $(s\geqslant 2)$ *satisfying* $P_i^*P_j=-P_j^*P_i(i\neq j)$. *Then there exist* 2^{s-1} *matrices* $V_1,\dots,V_{2^{s-1}}\in \mathrm{Span}\{P_1,P_2,\dots,P_s\}$ *such that*

$$\sum_{i=1}^{2^{s-1}} V_iV_i^* = 2^{s-1}(\sum_{i=1}^{s} P_iP_i^*)$$

and for all $j=1,\dots,2^{s-1}$

$$V_j^*V_j=\sum_{i=1}^{s}P_i^*P_i.$$

Proof. When $s=2$, let $V_1=P_1+P_2$, $V_2=P_1-P_2$, we obtain

$$V_1V_1^*+V_2V_2^*=2(P_1P_1^*+P_2^*P_2)$$

and

$$V_j^*V_j=\sum_{i=1}^{2}P_i^*P_i$$

for $j=1,2$. Then the inequality holds.

Suppose that the Lemma holds for $s=t$, that is, there exist 2^{t-1} matrices $U_1,\dots,U_{2^{t-1}}\in \mathrm{Span}\{P_1,P_2,\dots,P_t\}$ satisfying

$$\sum_{i=1}^{2^{t-1}} U_iU_i^* = 2^{t-1}(\sum_{i=1}^{t} P_iP_i^*) \tag{2}$$

and

$$U_j^*U_j=\sum_{i=1}^{t}P_i^*P_i \tag{3}$$

for $j=1,\dots,2^{t-1}$.

Then for $s=t+1$, set $B_i=U_i+P_{t+1}$, $C_i=U_i-P_{t+1}$, $1\leqslant i\leqslant 2^{t-1}$,

$$\begin{aligned}\sum_{i=1}^{2^{t-1}} B_iB_i^*+\sum_{i=1}^{2^{t-1}} C_iC_i^* &= 2(\sum_{i=1}^{2^{t-1}} U_iU_i^*)+2^tP_{t+1}P_{t+1}^* \\ &= 2^t(\sum_{i=1}^{t}P_iP_i^*)+2^tP_{t+1}P_{t+1}^* \\ &= 2^t(\sum_{i=1}^{t+1}P_iP_i^*).\end{aligned} \tag{4}$$

The equality (4) follows from $P_i^*P_j=-P_j^*P_i$ and equality (2).

Let $V_i=B_i$, $V_{2^{t-1}+i}=C_i$, $(1\leqslant i\leqslant 2^{t-1})$. Then $V_1,\dots,V_{2^t}\in \mathrm{Span}\{P_1,P_2,\dots,P_t,P_{t+1}\}$ and

$$\sum_{i=1}^{2^t} V_iV_i^* = 2^t(\sum_{i=1}^{t+1}P_iP_i^*).$$

By $P_i^*P_j = -P_j^*P_i$, we have

$$V_j^*V_j = (U_j+P_{t+1})^*(U_j+P_{t+1}) = U_j^*U_j + P_{t+1}^*P_{t+1} \\ = \sum_{i=1}^{t} P_i^*P_i + P_{t+1}^*P_{t+1} = \sum_{i=1}^{t+1} P_i^*P_i$$

for $1 \leqslant j \leqslant 2^{t-1}$ and

$$V_j^*V_j = (U_j-P_{t+1})^*(U_j-P_{t+1}) = U_j^*U_j + P_{t+1}^*P_{t+1} \\ = \sum_{i=1}^{t} P_i^*P_i + P_{t+1}^*P_{t+1} = \sum_{i=1}^{t+1} P_i^*P_i$$

for $2^{t-1}+1 \leqslant j \leqslant 2^t$.

That is, the equality

$$V_j^*V_j = \sum_{i=1}^{t+1} P_i^*P_i$$

holds for $1 \leqslant j \leqslant 2^t$. Thus we have finished the proof. □

Proof of Theorem 1. Since H is positive semidefinite, it follows that there exists a matrix $P \in M_{sn,sn}$ such that $H = P^*P$. Let $P = [P_1, P_2, \dots, P_s]$, where $P_1, P_2, \dots, P_s \in M_{sn,n}$. An easy computation shows that

$$A_{ij} = P_i^*P_j,\ i,j = 1,2,\dots,s.$$

It follows from A_{ij} $(i \neq j)$ is skew-Hermitian that

$$P_i^*P_j + P_j^*P_i = 0,\ i \neq j.$$

By Lemma 5, there exist 2^{s-1} matrices $V_1, \dots, V_{2^{s-1}} \in M_{sn,n}$ such that

$$\sum_{i=1}^{2^{s-1}} V_iV_i^* = 2^{s-1}\Big(\sum_{i=1}^{s} P_iP_i^*\Big)$$

and for $j = 1, \dots, 2^{s-1}$

$$V_j^*V_j = \sum_{i=1}^{s} P_i^*P_i.$$

Therefore, by the property of H and the Lemma 2

$$\begin{aligned}\lambda(H)&=\lambda(P^*P)=\lambda(PP^*)=\lambda(\sum_{j=1}^{s}P_jP_j^*)\\&=\frac{1}{2^{s-1}}\lambda(\sum_{j=1}^{2^{s-1}}V_jV_j^*)\prec\frac{1}{2^{s-1}}\sum_{j=1}^{2^{s-1}}\lambda(V_jV_j^*)=\frac{1}{2^{s-1}}\sum_{j=1}^{2^{s-1}}\lambda((V_j^*V_j)\oplus 0)\\&=\frac{1}{2^{s-1}}\sum_{j=1}^{2^{s-1}}\lambda(\sum_{i=1}^{s}P_i^*P_i\oplus 0)=\lambda(\sum_{i=1}^{s}P_i^*P_i\oplus 0)\\&=\lambda((\sum_{i=1}^{s}A_{ii})\oplus 0).\end{aligned}$$

The proof is completed. □

When $s=2$, we notice $\lambda(\begin{bmatrix}A_{11} & \sqrt{-1}A_{12}\\ \sqrt{-1}A_{12} & A_{22}\end{bmatrix})=\lambda(\begin{bmatrix}A_{11} & A_{12}\\ -A_{12} & A_{22}\end{bmatrix})\prec\lambda((A_{11}+A_{22})\oplus 0)$. Theorem 1 shows that Lin's result (Theorem 2.1 of [4]) is a special case of our result.

COROLLARY 6. *Let* $A_1,A_2,\dots,A_k\in M_{n,n}$ *with* $A_i^*A_j=-A_j^*A_i$, $(1\leqslant i<j\leqslant k)$. *Then*

$$\lambda(\sum_{i=1}^{k}A_iA_i^*)\prec\lambda(\sum_{i=1}^{k}A_i^*A_i).$$

Proof. Let $P=[A_1,A_2,\dots,A_k]$. Then $H=P^*P$ is a positive semidefinite matrix and satisfy the condition of Theorem 1. Hence

$$\lambda(H)\prec\lambda((\sum_{i=1}^{k}A_i^*A_i)\oplus 0).$$

It follows from this and Lemma 3 that

$$\lambda((\sum_{i=1}^{k}A_iA_i^*)\oplus 0)=\lambda(PP^*\oplus 0)=\lambda(P^*P)=\lambda(H)\prec\lambda((\sum_{i=1}^{k}A_i^*A_i)\oplus 0).\qquad\square$$

Next we show that the condition $A_i^*A_j$ is skew-Hermitian in Corollary 6 is necessary.

REMARK 7. Let $A_1=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $A_2=\begin{bmatrix}1&1\\0&0\end{bmatrix}$. Then

$$\begin{bmatrix}1&1\\0&0\end{bmatrix}=A_1^*A_2\neq A_2^*A_1=\begin{bmatrix}1&0\\1&0\end{bmatrix}$$

and

$$A_1A_1^* + A_2A_2^* = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_1^*A_1 + A_2^*A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

A trivial verification shows that $\lambda(A_1A_1^* + A_2A_2^*) \not\prec \lambda(A_1^*A_1 + A_2^*A_2)$.

Acknowledgement. The author would like to thank the referees for their valuable comments which have improve this paper significantly. Jinli Xu is supported by the Fundamental Research Funds for the Central Universities (No. 2572019BC07), the National Natural Science Foundation of China (No. 11701075) and the Foundation of Talent Introduction and the Double First-Rate for the Northeast Forestry University (No. 1020160016).

REFERENCES

[1] I. SCHUR, *Über eine klasse von mittelbildungen mit anwendungen die determinanten-theorie sitzungsber*, Berlin. Math. Gesellschaft. **22**, 1923, 9–20.
[2] S. FURUICHI, M. LIN, *A matrix trace inequality and its application*, Linear Algebra Appl. **433**, 2010, 1324–1328.
[3] R. BHATIA, *Matrix analysis*, New York, Springer-Verlag, 1997.
[4] M. LIN, H. WOLKOWICZ, *An eigenvalue majorization inequality for positive semidefinite block matrices*, Linear Multilinear A. **60**, 2012, 1365–1368.
[5] R. A. HORN AND C. R. JOHNSON, *Matrix analysis*, Cambridge, Cambridge University Press, 1985.
[6] X. ZHAN, *Matrix inequalities*, New York, Springer-Verlag, 2002.
[7] R. TURKMEN, V. PAKSOY, F. ZHANG, *Some inequalities of majorization type*, Linear Algebra Appl. **437**, 2012, 1305–1316.
[8] Y. ZHANG, *Eigenvalue majorization inequalities for positive semidefinite block matrices and their blocks*, Linear Algebra Appl. **446**, 2014, 216–223.
[9] A. W. MARSHALL, I. OLKIN, B. C. ARNOLD, *Inequalities: Theory of majorization and its applications*, Springer Science Business Media, 2011.
[10] K. FAN, *On a theorem of Weyl concerning eigenvalues of linear transformations*, I. Proc. Natl. Acad. Sci. USA **35**, 1949, 652–655.

(Received April 16, 2020)

Feng Zhang
Department of Mathematics
Northeast Forestry University
Harbin 150040, China

Jinli Xu
Department of Mathematics
Northeast Forestry University
Harbin 150040, China