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A NEW PROOF OF PÓLYA–KNOPP'S INEQUALITY WITH AN EXTENSION

ERN GUN KWON AND MIN JU JO

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Abstract. We show that a continuous version of Hölder's inequality can give a new and direct proof of the Pólya-Knopp type inequalities. We also have a single variable generalization of Pólya-Knopp's inequality.

1. Introduction

Let $Y=(Y,\nu)$ be a measure space with positive measure ν. Classical Hölder's inequality says that

$$\int_Y f_1(y)^p f_2(y)^{1-p} d\nu(y) \leqslant \left(\int_Y f_1(y)\, d\nu(y)\right)^p \left(\int_Y f_2(y)\, d\nu(y)\right)^{1-p}, \tag{1.1}$$

where f_1 and f_2 are positive functions of $L^1(\nu)$ and $0 \leqslant p \leqslant 1$.

It is well-known fact that (1.1) can be extended to the case of a multiple product of functions (see [1], [2]), and even to a continuous version ([4], [5]) as the following.

THEOREM A. *Let $X=(X,\mu)$ and $Y=(Y,\nu)$ be σ-finite measure spaces with positive measures μ and ν. If $\mu(X)=1$ and $f(x,y)$ is a positive measurable function defined on $X\times Y$, then*

$$\int_Y \exp\left(\int_X \ln f(x,y)\, d\mu(x)\right) d\nu(y) \leqslant \exp\left\{\int_X \ln\left(\int_Y f(x,y)\, d\nu(y)\right) d\mu(x)\right\}. \tag{1.2}$$

Equality holds in (1.2) as a nonzero finite value if and only if $f(x,y)=g(x)h(y)$ almost everywhere $\mu\times\nu$ for a positive μ-measurable function g with $-\infty<\int_X \ln g\, d\mu<\infty$ and a positive ν-measurable h with $\int_Y h d\nu=1$.

Inequality (1.2) is named in [4] as 'the continuous form of Hölder's inequality' because it covers well-known forms of Hölder's inequalities. It is simple to check that Theorem A is a generalization of (1.1): For $0\leqslant p\leqslant 1$ if we take

$$X=\{1,2\} \quad \text{and} \quad \mu = p\delta_1+(1-p)\delta_2,$$

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δ_j, $j=1,2$, the unit mass concentrated at j, and $f(k,y)=f_k(y)$, $k\in X$, then (1.2) reduces to (1.1):

$$\int_Y f_1(y)^p f_2(y)^{1-p}\,d\nu(y)=\int_Y \exp\left(\int_X \ln f_k(y)\,d\mu(k)\right)d\nu(y)$$

$$\leqslant \exp\left\{\int_X \ln\left(\int_Y f_k(y)\,d\nu(y)\right)d\mu(k)\right\}=\left(\int_Y f_1(y)\,d\nu(y)\right)^p\left(\int_Y f_2(y)\,d\nu(y)\right)^{1-p}.$$

Ever since its publication in [4], there have been few applications of the continuous form of Hölder's inequality.

The first purpose of this note is to give an application of (1.2) by giving a straightforward proof of Pólya-Knopp's inequality.

THEOREM B. (Pólya-Knopp's inequality [3])

$$\int_0^\infty \exp\left\{\frac{1}{x}\int_0^x \ln f(t)dt\right\}dx \leqslant e\int_0^\infty f(x)dx \tag{1.3}$$

for any function $f\geqslant 0$ *with* $\int_0^\infty f(x)dx<\infty$.

It is well-known that Pólya-Knopp's inequality (1.3) is a limiting case of Hardy's inequality and there have been several types of proofs. Our proof of (1.3) is presented in Section 2.

In Section 3, we show that (1.3) is in fact a simple consequence of the arithmetic-geometric mean inequality. We check well-known facts that the inequality (1.3) is strict (unless $f\equiv 0$ almost everywhere) and the bounding constant e is the best possible.

By a similar process we in Section 4 further give a simple proof of the following, which is known as 'Levin-Cochran-Lee's inequality'. See [6] and the references therein.

THEOREM C. *Let* $-\infty<\alpha,\beta<\infty$. *Let* $f\geqslant 0$ *with* $\int_0^\infty f(x)\,x^{\alpha-1}dx<\infty$. *If* $\beta>0$, *then*

$$\int_0^\infty \exp\left\{\frac{\beta}{x^\beta}\int_0^x t^{\beta-1}\ln f(t)dt\right\}x^{\alpha-1}dx\leqslant e^{\alpha/\beta}\int_0^\infty f(x)\,x^{\alpha-1}dx. \tag{1.4}$$

If $\beta<0$, *then*

$$\int_0^\infty \exp\left\{\frac{-\beta}{x^\beta}\int_x^\infty t^{\beta-1}\ln f(t)dt\right\}x^{\alpha-1}dx\leqslant e^{\alpha/\beta}\int_0^\infty f(x)\,x^{\alpha-1}dx.$$

We next pass to another direction. To establish an easy one variable generalization of (1.3) and (1.4) is our second purpose of this note.

THEOREM 1.1. *Let* $-\infty<\delta<\infty$, $\delta\neq 0$, *and* $-\infty\leqslant a<b\leqslant\infty$. *Let* w *be a positive, increasing, and differentiable function on* (a,b) *with* $\omega(a_+)=0$. *Then*

$$\int_a^b \exp\left\{\frac{1}{w(x)}\int_a^x \ln f(t)\,dw(t)\right\}d|w^\delta|(x)\leqslant e^\delta\int_a^b f(x)\,d|w^\delta|(x) \tag{1.5}$$

for all $f\geqslant 0$ *with* $\int_a^b f(x)\,d|w^\delta|(x)<\infty$.

Note that denoting $d|w^{\delta}|(x)$ is nothing but notational convenience: $d|w^{\delta}|(x) = dw^{\delta}(x)$ if $\delta \geqslant 0$ and $d|w^{\delta}|(x) = -dw^{\delta}(x)$ if $\delta \leqslant 0$. Note that (1.3) and (1.4) are special cases of (1.5).

Theorem 1.1 follows directly from, and as a limiting case of, the following

THEOREM 1.2. *Let* $1 < p < \infty$, $-\infty < \delta < \infty$, $\delta \neq 0$, $p > \delta$, *and* $-\infty \leqslant a < b \leqslant \infty$. *Let* w *be a positive, increasing, and differentiable function on* (a,b) *with* $\omega(a_+) = 0$. *Then*

$$\int_a^b \left(\frac{1}{w(x)} \int_a^x f(t)\, dw(t) \right)^p d|w^{\delta}|(x) \leqslant \left(\frac{p}{p-\delta} \right)^p \int_a^b f^p(x)\, d|w^{\delta}|(x) \tag{1.6}$$

for all $f \geqslant 0$ *with* $\int_a^b f^p(x)\, d|w^{\delta}|(x) < \infty$.

Elementary proofs of Theorem 1.1 and Theorem 1.2 are given in Section 5.

2. Proof of Theorem B

By setting $Y = (0,\infty)$, $X = (0,1)$, $d\mu(x) = dx$, $d\nu(y) = dy$, and $f(x,y) = f(xy)$, $0 < x < 1$, $0 < y < \infty$, in (1.2), we obtain (1.3). Note that the right side of (1.2) then becomes

$$\exp\left\{ \int_0^1 \ln \left(\int_0^\infty f(xy)\, dy \right) dx \right\} = \exp\left\{ \int_0^1 \ln \left(\frac{1}{x} \int_0^\infty f(y)\, dy \right) dx \right\} = e \int_0^\infty f(y)\, dy \tag{2.1}$$

while the left side of (1.2) becomes

$$\int_0^\infty \exp\left(\int_0^1 \ln f(xy)\, dx \right) dy = \int_0^\infty \exp\left(\frac{1}{y} \int_0^y \ln f(x)\, dx \right) dy. \tag{2.2}$$

3. Remarks on simplicity and best possibility

3.1.

By (2.1), the continuous form of Hölder's inequality (1.2) in fact can be regarded as an extension of Pólya-Knopp's inequality (1.3).

Because of the simplicity of the proof of Pólya-Knopp's inequality given in Section 2, one may suspect that there might be some heavy process in the proof of (1.2). But (1.2) is a simple consequence of the arithmetic-geometric mean inequality (or a simple consequence of Minkowski's inequality as in [4]). Absorbing the process we can give another proof of Pólya-Knopp's inequality (1.3):

Assuming $0 < \int_0^\infty f(y)dy < \infty$ and recalling (2.1) and (2.2),

$$\begin{aligned}\frac{\int_0^\infty \exp\left(\frac{1}{y}\int_0^y \ln f(x)\,dx\right)dy}{e\int_0^\infty f(z)\,dz} &= \frac{\int_0^\infty \exp\left(\int_0^1 \ln f(xy)\,dx\right)dy}{\exp\left\{\int_0^1 \ln\left(\int_0^\infty f(xz)\,dz\right)dx\right\}}\\ &= \int_0^\infty \exp\left(\int_0^1 \ln \frac{f(xy)}{\int_0^\infty f(xz)\,dz}\,dx\right)dy \qquad (3.1)\\ &\leqslant \int_0^\infty \int_0^1 \frac{f(xy)}{\int_0^\infty f(xz)\,dz}\,dxdy = 1,\end{aligned}$$

which gives (1.3). Note that we used the arithmetic-geometric mean inequality only.

3.2.

We can also check easily that the inequality (1.3) is strict unless $f \equiv 0$ (as is well-known):

Suppose $f \not\equiv 0$. Then $f > 0$ on a positive measured set, so that $0 < \int_0^\infty f(x)dx < \infty$. Since

$$\exp\left(\int_0^1 \ln \frac{f(xy)}{\int_0^\infty f(xz)\,dz}\,dx\right) \leqslant \int_0^1 \frac{f(xy)}{\int_0^\infty f(xz)\,dz}\,dx, \qquad (3.2)$$

the inequality in (3.1) becomes equality if and only if (3.2) becomes equality almost every $y \in (0,\infty)$, and since for each y fixed (3.2) becomes equality if and only if

$$\frac{f(xy)}{\int_0^\infty f(xz)\,dz} = h(y), \quad \text{that is} \quad f(xy) = \left(\int_0^\infty f(z)dz\right)\frac{1}{x}h(y) \quad \text{almost every } x \in (0,1)$$

for some measurable h satisfying $\int_0^\infty h(y)dy = 1$, it follows that $f(x)$ should be a constant times $\frac{1}{x}$ almost everywhere, which contradicts $\int_0^\infty f(x)dx < \infty$.

3.3.

For completeness, we check that the bounding constant e is the best possible: It is sufficient to show that for arbitrary $\varepsilon > 0$ there is $f = f_\varepsilon$ for which

$$\int_0^\infty \exp\left\{\frac{1}{x}\int_0^x \ln f(t)dt\right\}dx \geqslant (1-\varepsilon)e\int_0^\infty f(x)dx.$$

Let $\varepsilon > 0$ be given. Take $f(x) = x^{\varepsilon-1}\chi_{(0,1)}(x)$, where $\chi_{\{\cdot\}}$ denotes the characteristic function of the set $\{\cdot\}$. Then

$$\begin{aligned}\int_0^\infty \exp\left\{\frac{1}{x}\int_0^x \ln f(t)dt\right\}dx &\geqslant \int_0^1 \exp\left\{\frac{1}{x}\int_0^x \ln t^{\varepsilon-1}dt\right\}dx\\ = \int_0^1 \exp\{(\varepsilon-1)(\ln x - 1)\}dx &= e^{1-\varepsilon}\int_0^1 x^{\varepsilon-1}dx \geqslant (1-\varepsilon)e\int_0^\infty f(x)dx.\end{aligned}$$

4. Proof of Theorem C

We consider only for the case $\beta > 0$. The case $\beta < 0$ can be similarly proved.

4.1.

A special case of (1.2) reads

$$\int_0^\infty \exp\left\{\int_0^1 \beta y^{\beta-1} \ln f(xy)\, dy\right\} x^{\alpha-1} dx \leqslant \exp\left\{\int_0^1 \beta y^{\beta-1} \ln\left(\int_0^\infty f(xy) x^{\alpha-1} dx\right) dy\right\}. \tag{4.1}$$

By using change of variables, the left side of (4.1) equals

$$\int_0^\infty \exp\left\{\frac{\beta}{x^\beta}\int_0^x t^{\beta-1} \ln f(t) dt\right\} x^{\alpha-1} dx$$

and the right side of (4.1) equals

$$\begin{aligned}
&\exp\left\{\int_0^1 \beta y^{\beta-1} \ln\left(\int_0^\infty f(xy) x^{\alpha-1} dx\right) dy\right\} \\
&\quad = \exp\left\{\int_0^1 \beta y^{\beta-1} \ln\left(\frac{1}{y^\alpha}\int_0^\infty f(s) s^{\alpha-1} ds\right) dy\right\} \\
&\quad = \exp\left\{\int_0^1 \beta y^{\beta-1} \ln\left(\int_0^\infty f(s) s^{\alpha-1} ds\right) dy - \alpha\beta \int_0^1 y^{\beta-1} \ln y\, dy\right\} \\
&\quad = \left(\int_0^\infty f(s) s^{\alpha-1} ds\right) \cdot \exp\left(-\alpha\beta \int_0^1 y^{\beta-1} \ln y\, dy\right) = e^{\alpha/\beta} \int_0^\infty f(s) s^{\alpha-1} ds,
\end{aligned}$$

where we used a simple integration by parts to have

$$-\alpha\beta \int_0^1 y^{\beta-1} \ln y\, dy = \alpha \int_0^1 y^{\beta-1} dy = \frac{\alpha}{\beta}.$$

Therefore, we obtain (1.4).

4.2.

By modifying the process in 3.2, we can check that the inequality (1.4) is strict unless $f \equiv 0$.

In fact, unless $f \equiv 0$ the inequality (4.1) becomes equality if and only if $f(x)$ is a constant times $x^{-\alpha}$ almost everywhere, which contradicts the condition $\int_0^\infty f(x) x^{\alpha-1} dx < \infty$.

4.3.

By considering $f(x) = x^{\beta\varepsilon-\alpha}\chi_{(0,1)}(x)$ and modifying the process in 3.3, we can check that the bounding constant $e^{\alpha/\beta}$ is the best possible.

4.4.

Moreover, we can show the following well-known improvement [6] that for $0 < b < \infty$,

$$\int_0^b \exp\left\{\frac{\beta}{x^\beta}\int_0^x t^{\beta-1}\ln f(t)dt\right\} x^{\alpha-1}dx \leqslant e^{\frac{\alpha}{\beta}}\int_0^b \left\{1-\left(\frac{x}{b}\right)^\beta\right\} f(x)\, x^{\alpha-1}dx.$$

All we need, except following the process in 4.1, is a change of the order of the integrals.

5. Proof of Theorem 1.1 and Theorem 1.2

5.1.

We first see that (1.5) follows from (1.6).

The case f replaced by $f^{1/p}$ of (1.6) reads

$$\frac{1}{\delta}\int_a^b \left(\frac{1}{w(x)}\int_a^x f^{1/p}(t)\, dw(t)\right)^p dw^\delta(x) \leqslant \frac{1}{\delta}\left(\frac{p}{p-\delta}\right)^p \int_a^b f(x)\, dw^\delta(x). \tag{5.1}$$

Note for each $x \in (a,b)$ that Hölder's inequality yields

$$\begin{aligned}\frac{1}{w(x)}\int_a^x f^{1/p}(t)\, dw(t) &\leqslant \frac{1}{w(x)}\left(\int_a^x w^{(1-\delta)q/p}(t)\, dw(t)\right)^{1/q}\left(\int_a^x f(t)w^{\delta-1}(t)\, dw(t)\right)^{1/p}\\ &= \frac{w^{\delta/p}(x)}{w(x)}\left(\frac{p-1}{p-\delta}w^{(p-\delta)/(p-1)}(x)\right)^{1/q}\left(\frac{1}{\delta w^\delta(x)}\int_a^x f(t)\delta w^{\delta-1}(t)\, dw(t)\right)^{1/p}\\ &= \left(\frac{p-1}{p-\delta}\right)^{1/q}\left(\frac{1}{\delta w^\delta(x)}\int_a^x f(t)\, dw^\delta(t)\right)^{1/p} < \infty,\end{aligned}$$

where $1/p+1/q=1$.

Thus,

$$\left(\frac{1}{w(x)}\int_a^x f^{1/p}(t)\, dw(t)\right)^p \downarrow \exp\left(\frac{1}{w(x)}\int_a^x \ln f(t)\, dw(t)\right)$$

monotonically as $p \to \infty$ (See for example p. 74 [7]).

Therefore, noting $\left(\frac{p}{p-\delta}\right)^p \to e^\delta$ as $p \to \infty$, the monotone convergence theorem applied to (5.1) gives

$$\frac{1}{\delta}\int_a^b \exp\left(\frac{1}{w(x)}\int_a^x \ln f(t)\, dw(t)\right) dw^\delta(x) \leqslant \frac{e^\delta}{\delta}\int_a^b f(x)\, dw^\delta(x),$$

which gives (1.5).

5.2.

We next pass to the proof of (1.6).

Let $F(x)=\frac{1}{w(x)}\int_a^x f(t)\,dw(t)$ for $x\in(a,b)$. Then F is differentiable on (a,b) and $F'(x)=(f-F)(x)w'(x)/w(x)$ almost every x in (a,b). (See for example p. 176 [7]).

Integration by parts gives

$$\begin{aligned}\int_a^b F^p(x)\,dw^\delta(x) &= \Big[F^p(x)\,w^\delta(x)\Big]_{a_+}^{b_-} - p\int_a^b F^{p-1}(x)F'(x)\,w^\delta(x)dx\\ &= \Big[F^p(x)\,w^\delta(x)\Big]_{a_+}^{b_-} - p\int_a^b F^{p-1}(x)\{f(x)-F(x)\}\,\frac{w'(x)}{w(x)}w^\delta(x)dx \qquad (5.2)\\ &= \Big[F^p(x)\,w^\delta(x)\Big]_{a_+}^{b_-} - \frac{p}{\delta}\int_a^b F^{p-1}(x)f(x)dw^\delta(x)+\frac{p}{\delta}\int_a^b F^p(x)dw^\delta(x).\end{aligned}$$

If we denote $q=p/(p-1)$ then

$$\begin{aligned}\lim_{x\to a_+}\Big[F^p(x)\,w^\delta(x)\Big] &= \lim_{x\to a_+}\left[w^{\delta-p}(x)\left(\int_a^x f(t)dw(t)\right)^p\right]\\ &= \lim_{x\to a_+}\left[w^{\delta-p}(x)\left(\int_a^x f(t)\frac{dw^\delta(t)}{\delta w^{\delta-1}(t)}\right)^p\right]\\ &\leqslant \lim_{x\to a_+}\left[w^{\delta-p}(x)\left(\frac{1}{\delta}\int_a^x f^p(t)dw^\delta(t)\right)\left(\frac{1}{\delta}\int_a^x\frac{dw^\delta(t)}{w^{q(\delta-1)}(t)}\right)^{p/q}\right]\\ &= \lim_{x\to a_+}\left[w^{\delta-p}(x)\left(\frac{1}{\delta}\int_a^x f^p(t)dw^\delta(t)\right)\left(\frac{w^{(\delta-1)(1-q)+1}(x)}{(\delta-1)(1-q)+1}\right)^{p/q}\right]\\ &= \left(\frac{p-1}{p-\delta}\right)^{p/q}\lim_{x\to a_+}\left(\frac{1}{\delta}\int_a^x f^p(t)dw^\delta(t)\right)=0\end{aligned}$$

by Hölder's inequality and the assumption $\frac{1}{\delta}\int_a^b f^p(t)dw^\delta(t)<\infty$, whence

$$\Big[F^p(x)\,w^\delta(x)\Big]_{a_+}^{b_-}=\lim_{x\to b_-}\Big[F^p(x)\,w^\delta(x)\Big]-\lim_{x\to a_+}\Big[F^p(x)\,w^\delta(x)\Big]=\lim_{x\to b_-}\Big[F^p(x)\,w^\delta(x)\Big]. \qquad (5.3)$$

By (5.2) and (5.3)

$$\frac{p-\delta}{\delta}\int_a^b F^p(x)\,dw^\delta(x)=-\lim_{x\to b_-}\Big[F^p(x)\,w^\delta(x)\Big]+\frac{p}{\delta}\int_a^b F^{p-1}(x)f(x)dw^\delta(x). \qquad (5.4)$$

Applying the arithmetic geometric mean inequality to the last quantity of (5.4),

$$\begin{aligned}
\frac{p}{\delta}\int_a^b F^{p-1}(x)f(x)dw^\delta(x) &= \frac{p-\delta}{\delta}\int_a^b \frac{p}{p-\delta}F^{p-1}(x)f(x)dw^\delta(x) \\
&= \frac{p-\delta}{\delta}\int_a^b [F^p(x)]^{(p-1)/p}\left[\left(\frac{p}{p-\delta}\right)^p f^p(x)\right]^{1/p} dw^\delta(x) \qquad (5.5)\\
&\leqslant \frac{p-\delta}{\delta}\frac{p-1}{p}\int_a^b F^p(x)dw^\delta(x) + \frac{p-\delta}{\delta}\frac{1}{p}\left(\frac{p}{p-\delta}\right)^p\int_a^b f^p(x)dw^\delta(x).
\end{aligned}$$

By (5.4) and (5.5),

$$\begin{aligned}
&\frac{p-\delta}{\delta}\frac{1}{p}\int_a^b F^p(x)\, dw^\delta(x) \\
&\leqslant -\lim_{x\to b_-}\left[F^p(x)\, w^\delta(x)\right] + \frac{p-\delta}{\delta}\frac{1}{p}\left(\frac{p}{p-\delta}\right)^p\int_a^b f^p(x)dw^\delta(x)
\end{aligned}$$

provided $\frac{1}{\delta}\int_a^b F^p(x)\, dw^\delta(x) < \infty$ which we may assume. That is,

$$\frac{1}{\delta}\int_a^b F^p(x)\, dw^\delta(x) \leqslant -\frac{p}{p-\delta}\lim_{x\to b_-}\left[F^p(x)\, w^\delta(x)\right] + \left(\frac{p}{p-\delta}\right)^p \frac{1}{\delta}\int_a^b f^p(x)dw^\delta(x). \qquad (5.6)$$

Since $\lim_{x\to b_-}\left[F^p(x)\, w^\delta(x)\right] \geqslant 0$, (1.6) follows from (5.6).

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Ern Gun Kwon
Department of Mathematics Education
Andong National University
36729 South Korea
e-mail: egkwon@anu.ac.kr

Min Ju Jo
Department of Mathematics, Graduate School
Andong National University
36729 South Korea
e-mail: fatcat@hanmail.net