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A SOLUTION TO AN OPEN PROBLEM FOR WILKER–TYPE INEQUALITIES

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Abstract. This paper is to solve the open problem for Wilker-Type inequality: what are the best possible for the constants c_1 and c_2 such that the double inequality $c_1x^{3a}\tan x < (\frac{\sin x}{x})^{2a} + (\frac{\tan x}{x})^a - 2 < c_2x^{3a}\tan x$ holds?

1. Introduction

J. B. Wilker proposed two open problems in [1], using the following statements:
(a) If $x \in (0, \frac{\pi}{2})$, then

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2. \tag{1}$$

(b) There exists a largest constant c for $x \in (0, \frac{\pi}{2})$ such that

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3\tan x. \tag{2}$$

J. S. Summer et al affirmed the truth of the inequalities above and obtained an extended result as follow in [2].

THEOREM 1. *For* $0 < x < \frac{\pi}{2}$,

$$c_1x^3\tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 < c_2x^3\tan x \tag{3}$$

holds, and the values $\frac{16}{\pi^4}$ *and* $\frac{8}{45}$ *are the best possible for constants* c_1 *and* c_2 *respectively.*

In [3], Zhu proved an exponential generalization of a Wilker-type inequality as follow.

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THEOREM 2. *Let* $0 < x < \frac{\pi}{2}$,

$$f(a) = \left(\frac{\sin x}{x}\right)^{2a} + \left(\frac{\tan x}{x}\right)^{a} - 2 \tag{4}$$

then the inequality $f(a) > 0$ *holds for* $a \geqslant 1$.

These inequalities are of great practical importance and were extended in different forms in the recent past, and they are used for research of trigonometric and hyperbolic functions [4, 5, 6].

Based on Zhu's works, an open problem was raised in [7]: for $x \in (0, \frac{\pi}{2})$, what are the best possible for constants c_1 and c_2 such that the double inequality

$$c_1 x^{3a} \tan x < \left(\frac{\sin x}{x}\right)^{2a} + \left(\frac{\tan x}{x}\right)^{a} - 2 < c_2 x^{3a} \tan x \tag{5}$$

holds?

In this paper, we conclude that for $0 < a \leqslant 1$, the best possible for c_1 and c_2 such that inequality (5) holds are 0 and $\frac{8}{45}$ respectively; for $a > 1$, the best possible for c_1 is $\frac{16}{\pi^4}$ and the constant c_2 such that inequality (5) holds doesn't exist.

2. The main results

In order to derive the main results, some lemmas are given.

LEMMA 1. *For* $x \in (0, \frac{\pi}{2})$, $\frac{\sin^2 x \tan x}{x^3} > 1$ *holds.*

Proof. Let $k(x) = \frac{\sin^3 x}{\cos x} - x^3$ for $x \in (0, \frac{\pi}{2})$, its third derivative

$$k'''(x) = \frac{14 \sin^4 x}{\cos^2 x} + \frac{6 \sin^6 x}{\cos^4 x} > 0,$$

which reveals the second derivative $k''(x) = -6x + 6 \sin x \cos x + \frac{4 \sin^3 x}{\cos x} + \frac{2 \sin^5 x}{\cos^3 x}$ is strictly increasing on $(0, \frac{\pi}{2})$. Therefore, $k''(x) > k''(0) = 0$ for $x \in (0, \frac{\pi}{2})$. In the same manner, the first derivative of k(x) is strictly increasing on $(0, \frac{\pi}{2})$, and

$$k'(x) = -3x^2 + 3 \sin^2 x + \frac{\sin^4 x}{\cos^2 x} > k'(0) = 0.$$

Then $k(x)$ is strictly increasing on $(0, \frac{\pi}{2})$, so $k(x) = \frac{\sin^3 x}{\cos x} - x^3 > k(0) = 0$, thus $\frac{\sin^2 x \tan x}{x^3}$ > 1 holds on $(0, \frac{\pi}{2})$. □

LEMMA 2. *For* $x \in (0, \frac{\pi}{2})$, $\frac{\tan x}{x} > 1$ *holds.*

Proof. Let $w(x) = \tan x - x$ for $x \in (0, \frac{\pi}{2})$, the differentiation yields that $w'(x) = (\sec x)^2 - 1 > 0$ on $(0, \frac{\pi}{2})$. So $w(x)$ is increasing on $(0, \frac{\pi}{2})$, then $w(x) > w(0) = 0$, thus $\frac{\tan x}{x} > 1$ holds on $(0, \frac{\pi}{2})$. □

LEMMA 3. *For $x \in (0, \frac{\pi}{2})$, there exists one and only one real root for the trigonometric equation*

$$\frac{\sin^4 x}{x^4} - x^6 = 0.$$

Proof. As $\left(\frac{\sin x}{x}\right)' = \frac{x\cos x - \sin x}{x^2}$, by Lemma 2, $x\cos x - \sin x < 0$ holds on $(0, \frac{\pi}{2})$, so $\frac{\sin x}{x}$ is strictly decreasing on $(0, \frac{\pi}{2})$. It is clear that $\frac{\sin^4 x}{x^4}$ and $\frac{\sin^4 x}{x^4} - x^6$ are also strictly decreasing on $(0, \frac{\pi}{2})$.

Let $f(x) = \frac{\sin^4 x}{x^4} - x^6$, then $\lim_{x \to 0^+} f(x) = 1 > 0$ and $f(\frac{\pi}{2}) = (\frac{2}{\pi})^4 - (\frac{\pi}{2})^6 < 0$. we conclude that $f(x)$ has one and only one real root on $(0, \frac{\pi}{2})$. □

LEMMA 4. *For $x \in (0, \frac{\pi}{2})$, $\frac{\tan x}{x} - x^3 > 0$ holds.*

Proof. As the Taylor expansion of $\tan x$ at 0

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \frac{1382}{155925}x^{11} + \cdots,$$

then for $x \in (0, \frac{\pi}{2})$,

$$\tan x - x^4 > x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \frac{1382}{155925}x^{11} - x^4.$$

Let

$$g(x) = \frac{x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \frac{1382}{155925}x^{11} - x^4}{x^4}$$

$$= \frac{1}{x^3} + \frac{1}{3x} + \frac{2}{15}x + \frac{17}{315}x^3 + \frac{62}{2835}x^5 + \frac{1382}{155925}x^7 - 1.$$

To complete the proof of the lemma, we need only to show that $g(x) > 0$ holds on $(0, \frac{\pi}{2})$. Obviously, $g(x) > 0$ holds on $(0, 1]$, so it remains to prove that $g(x) > 0$ holds on $(1, \frac{\pi}{2})$.

Let

$$g_1(x) = \frac{1}{x^3} + \frac{1}{3x},$$

$$g_2(x) = \frac{2}{15}x + \frac{17}{315}x^3 + \frac{62}{2835}x^5 + \frac{1382}{155925}x^7.$$

Then $g(x) = g_1(x) + g_2(x) - 1$. For $0 < x_1 < x_2$, let $V(x_1, x_2) = g_1(x_2) + g_2(x_1) - 1$, then $g(x) > V(x_1, x_2)$ on $[x_1, x_2]$.

The elementary calculation of rational numbers proposes

$$V\left(1, \frac{19}{16}\right) = \frac{102572248}{1069489575} > 0,$$

$$V\left(\frac{19}{16}, \frac{21}{16}\right) = \frac{26781048047921}{1025467062681600} > 0,$$

$$V\Big(\frac{21}{16},\frac{23}{16}\Big)=\frac{13744536928453}{1347247354675200}>0,$$
$$V\Big(\frac{23}{16},\frac{13}{8}\Big)=\frac{1692477562044649}{45978594626764800}>0.$$

That is to say, $g(x)>0$ holds on $\big[1,\frac{19}{16}\big],\big[\frac{19}{16},\frac{21}{16}\big],\big[\frac{21}{16},\frac{23}{16}\big]$ and $\big[\frac{23}{16},\frac{13}{8}\big]$.

Therefore, $g(x)>0$ holds on $[1,\frac{13}{8}]\supset[1,\frac{\pi}{2})$. □

THEOREM 3. *For each fixed* $x\in(0,\frac{\pi}{2})$,

$$f(a,x)=\frac{(\frac{\sin x}{x})^{2a}+(\frac{\tan x}{x})^{a}-2}{x^{3a}\tan x} \tag{6}$$

is increasing by a on $[0,+\infty)$.

Proof. The differentiation yields

$$f_1(a,x)=\frac{df(a,x)}{da}=\frac{2(\frac{\sin x}{x})^{2a}\ln\frac{\sin x}{x}+(\frac{\tan x}{x})^{a}\ln\frac{\tan x}{x}-3((\frac{\sin x}{x})^{2a}+(\frac{\tan x}{x})^{a}-2)\ln x}{x^{3a}\tan x}.$$

The denominator is demonstrably positive, so we focus on its numerator and let

$$f_{11}(a,x)=2\Big(\frac{\sin x}{x}\Big)^{2a}\ln\frac{\sin x}{x}+\Big(\frac{\tan x}{x}\Big)^{a}\ln\frac{\tan x}{x}-3\Big(\Big(\frac{\sin x}{x}\Big)^{2a}+\Big(\frac{\tan x}{x}\Big)^{a}-2\Big)\ln x.$$

Then the inequality

$$f_{11}(0,x)=2\ln\frac{\sin x}{x}+\ln\frac{\tan x}{x}=\ln\frac{\sin^2 x\tan x}{x^3}>0 \tag{7}$$

can be verified easily due to Lemma 1.

Differentiation again yields

$$\begin{aligned}f_2(a,x)=\frac{df_{11}(a,x)}{da}&=4\Big(\frac{\sin x}{x}\Big)^{2a}\Big(\ln\frac{\sin x}{x}\Big)^2+\Big(\frac{\tan x}{x}\Big)^{a}\Big(\ln\frac{\tan x}{x}\Big)^2\\&-6\ln x\Big(\frac{\sin x}{x}\Big)^{2a}\ln\frac{\sin x}{x}-3\ln x\Big(\frac{\tan x}{x}\Big)^{a}\ln\frac{\tan x}{x}.\end{aligned}$$

For simplicity, let $f_{21}(a,x)=\frac{f_2(a,x)}{(\frac{\tan x}{x})^a}$, then

$$\begin{aligned}f_{21}(a,x)=4\Big(\frac{\sin x\cos x}{x}\Big)^{a}\Big(\ln\frac{\sin x}{x}\Big)^2+\Big(\ln\frac{\tan x}{x}\Big)^2\\-6\ln x\Big(\frac{\sin x\cos x}{x}\Big)^{a}\ln\frac{\sin x}{x}-3\ln x\ln\frac{\tan x}{x}.\end{aligned} \tag{8}$$

Combining similar terms, we have

$$f_{21}(a,x)=\Big(\frac{\sin x\cos x}{x}\Big)^{a}\ln\frac{\sin x}{x}\Big(4\ln\frac{\sin x}{x}-6\ln x\Big)+\ln\frac{\tan x}{x}\Big(\ln\frac{\tan x}{x}-3\ln x\Big).$$

By Lemma 3, there exists one and only one real root for $\frac{\sin^4 x}{x^4} - x^6 = 0$ on $(0, \frac{\pi}{2})$, denote the root as x_0. It is obvious that $\frac{\sin^4 x}{x^4} - x^6 > 0$ holds on $(0, x_0)$ and $\frac{\sin^4 x}{x^4} - x^6 < 0$ holds on $\left(x_0, \frac{\pi}{2}\right)$. Furthermore, $(x_0)^{10} = \sin^4 x_0 < 1$, so $x_0 < 1$.

For $x \in (x_0, \frac{\pi}{2})$, $\frac{\sin^4 x}{x^4} - x^6 < 0$, so $4\ln\frac{\sin x}{x} - 6\ln x < 0$, thus

$$\ln\frac{\sin x}{x}\left(4\ln\frac{\sin x}{x} - 6\ln x\right) > 0.$$

$\ln\frac{\tan x}{x} - 3\ln x > 0$ on $(x_0, \frac{\pi}{2})$ due to Lemma 4, $\ln\frac{\tan x}{x} > 0$ on $(x_0, \frac{\pi}{2})$ due to Lemma 2, we conclude that $\ln\frac{\tan x}{x}(\ln\frac{\tan x}{x} - 3\ln x) > 0$, and then $f_{21}(a,x) > 0$ holds for $a \in [0, +\infty)$ and $x \in \left(x_0, \frac{\pi}{2}\right)$.

Now we discuss the situation that $x \in (0, x_0)$. Differentiation of $f_{21}(a,x)$ yields

$$\begin{aligned} f_3(a,x) = \frac{df_{21}(a,x)}{da} &= 4\left(\frac{\sin x\cos x}{x}\right)^a\left(\ln\frac{\sin x}{x}\right)^2\ln\frac{\sin x\cos x}{x} \\ &\quad -6\left(\frac{\sin x\cos x}{x}\right)^a\ln x\ln\frac{\sin x\cos x}{x}\ln\frac{\sin x}{x}. \end{aligned}$$

Let $f_{31}(a,x) = \frac{f_3(a,x)}{(\frac{\sin x\cos x}{x})^a\ln\frac{\sin x}{x}\ln\frac{\sin x\cos x}{x}}$, then $f_{31}(a,x) = 4\ln\frac{\sin x}{x} - 6\ln x$.

Let $f_{32}(a,x) = (\frac{\sin x}{x})^4 - x^6$. It is obvious that $\operatorname{sgn}(f_3(a,x)) = \operatorname{sgn}(f_{31}(a,x)) = \operatorname{sgn}(f_{32}(a,x))$ on $(0, x_0)$.

As $\frac{(\sin x)^4}{x^4} - x^6 > 0$ holds for $x \in (0, x_0)$, so $f_3(a,x) > 0$ holds on $(0, x_0)$, hence, $f_{21}(a,x)$ is increasing by a for $x \in (0, x_0)$. That is to say, for $a \geqslant 0$ and $x \in (0, x_0)$, $f_{21}(a,x) > f_{21}(0,x)$.

For $x \in (0, x_0)$, let $p_1(x) = f_{21}(0,x)$, then

$$p_1(x) = 4\left(\ln\frac{\sin x}{x}\right)^2 + \left(\ln\frac{\tan x}{x}\right)^2 - 6\ln x\ln\frac{\sin x}{x} - 3\ln x\ln\frac{\tan x}{x}. \tag{9}$$

Let

$$\begin{aligned} p_{11}(x) &= -6\ln x\ln\frac{\sin x}{x} - 3\ln x\ln\frac{\tan x}{x} \\ &= -3\ln x\left(2\ln\frac{\sin x}{x} + \ln\frac{\tan x}{x}\right) = -3\ln x p_{12}(x), \end{aligned}$$

where

$$p_{12}(x) = 2\ln\frac{\sin x}{x} + \ln\frac{\tan x}{x} = \ln\frac{\sin^2 x\tan x}{x^3}.$$

As $x \in (0, x_0)$ and $x_0 < 1$, so $\ln x < 0$, then $\operatorname{sgn}(p_{11}(x)) = \operatorname{sgn}(p_{12}(x))$.

By Lemma 1, $\frac{\sin^2 x\tan x}{x^3} > 1$ on $\left(0, \frac{\pi}{2}\right)$, so $p_{12}(x) > 0$, thus $p_{11}(x) > 0$. Then $p_1(x) = f_{21}(0,x) > 0$, which implies that for $a \in [0, +\infty)$, $f_{21}(a,x) > f_{21}(0,x) > 0$ on $(0, x_0)$.

Hence, $f_{21}(a,x) > 0$ holds for all $x \in \left(0, \frac{\pi}{2}\right)$ and $a \in [0, +\infty)$, so $f_{11}(a,x)$ is increasing by a on $[0, +\infty)$ for $x \in \left(0, \frac{\pi}{2}\right)$, which reveals that $f_{11}(a,x) > f_{11}(0,x)$. By (7), $f_{11}(0,x) > 0$, so we obtain that $f_{11}(a,x) > 0$, which implies that $f_1(a,x) > 0$ holds due to the fact that $\operatorname{sgn}(f_1(a,x)) = \operatorname{sgn}(f_{11}(a,x))$.

So $f(a,x)$ is increasing by a on $[0, +\infty)$ for $x \in \left(0, \frac{\pi}{2}\right)$. □

THEOREM 4. *If* $x \in (0, \frac{\pi}{2})$, *for* $0 < a \leqslant 1$, *the best possible values for constants* c_1 *and* c_2 *such that inequality (5) holds are* 0 *and* $\frac{8}{45}$ *respectively; for* $a > 1$, *the best possible value for constant* c_1 *is* $\frac{16}{\pi^4}$ *and the constant* c_2 *such that inequality (5) holds doesn't exist.*

Proof. We differentiate two cases if $a \in (0,1]$ or $a \in (1,+\infty)$.

(1) $a \in (0,1]$. By Theorem 3,

$$f(0,x) \leqslant f(a,x) = \frac{(\frac{\sin x}{x})^{2a} + (\frac{\tan x}{x})^{a} - 2}{x^{3a}\tan x} \leqslant f(1,x)$$

holds for $x \in \left(0, \frac{\pi}{2}\right)$. Thanks to Theorem 1, we have

$$f(1,x) = \frac{(\frac{\sin x}{x})^{2} + \frac{\tan x}{x} - 2}{x^{3}\tan x} < \frac{8}{45}.$$

Let us notice that $f(0,x) = 0$, thus $0 = f(0,x) < f(a,x) \leqslant f(1,x) < \frac{8}{45}$. We get the boundaries for $f(a,x)$.

As $\lim_{a\to 0^+} f(a,x) = f(0,x) = 0$ for each $x \in (0, \frac{\pi}{2})$, so the best possible for c_1 is 0.

As $\lim_{a\to 1^-} f(a,x) = f(1,x)$ for each $x \in (0, \frac{\pi}{2})$ and it can be obtained by L'Hospital's Rule that

$$\lim_{x\to 0^+} f(1,x) = \lim_{x\to 0^+} \frac{\sin^2 x\cos x + x\sin x - 2x^2\cos x}{x^5 \sin x} = \frac{8}{45},$$

so $\frac{8}{45}$ is the best possible for c_2.

(2) $a \in (1,+\infty)$. For each a,

$$\lim_{x\to(\frac{\pi}{2})^-} \frac{(\frac{\sin x}{x})^{2a} + (\frac{\tan x}{x})^{a} - 2}{x^{3a}\tan x} = \lim_{x\to(\frac{\pi}{2})^-} \frac{(\frac{\sin x}{x})^{2a}}{x^{3a}\tan x} + \lim_{x\to(\frac{\pi}{2})^-} \frac{(\frac{\tan x}{x})^{a}}{x^{3a}\tan x} - \lim_{x\to(\frac{\pi}{2})^-} \frac{2}{x^{3a}\tan x}$$

$$= \lim_{x\to(\frac{\pi}{2})^-} \frac{(\tan x)^{a-1}}{x^{4a}} = +\infty,$$

which implies that the upper bound of $f(a,x)$ is infinite, so there is no constant c_2 such that the inequality

$$\left(\frac{\sin x}{x}\right)^{2a} + \left(\frac{\tan x}{x}\right)^{a} - 2 < c_2 x^{3a}\tan x$$

is fulfilled.

By Theorem 3 we get that

$$f(a,x) > f(1,x) = \frac{(\frac{\sin x}{x})^{2} + \frac{\tan x}{x} - 2}{x^{3}\tan x} > \frac{16}{\pi^4}.$$

So, $c_1 \geqslant \frac{16}{\pi^4}$. Furthermore, $\frac{16}{\pi^4}$ is the best possible for c_1, which follows from $\lim_{a\to 1^-} f(a,x) = f(1,x)$ and

$$\lim_{x\to(\frac{\pi}{2})^-} f(1,x) = \lim_{x\to(\frac{\pi}{2})^-} \frac{\sin^2 x\cos x + x\sin x - 2x^2\cos x}{x^5 \sin x} = \frac{16}{\pi^4}. \quad \square$$

3. Remarks

The best possible constants for the exponential generalization of Wilker-type inequality are found on two given intervals in this paper, and we expect that the method is exemplary for the same type of problems.

It is worth noting in particular that Theorem 3 itself is also very useful, for example, Theorem 2 is its direct inference. In fact, due to Theorem 3 it is clear that Theorem 2 still holds after "$a \geqslant 1$" is replaced by "$a > 0$".

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