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A NOTE ON VARIANCE BOUNDS AND LOCATION OF EIGENVALUES

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Abstract. We discuss some extensions and refinements of the variance bounds for both real and complex numbers. The related bounds for the eigenvalues and spread of a matrix are also derived here.

1. Introduction

Let $z_1, z_2, \ldots, z_n$ denote n complex numbers. Their arithmetic mean is the number

$$\frac{1}{n}\sum_{i=1}^{n} z_i = \widetilde{z}. \tag{1.1}$$

In literature, the number

$$\frac{1}{n}\sum_{i=1}^{n} |z_i - \widetilde{z}|^2 = S_z^2 \tag{1.2}$$

or its equivalent expressions have been studied in several different contexts and notations and is termed as the variance of complex numbers at many places. For example, see Audenaert [2], Bhatia and Sharma [4,5], Merikoski and Kumar [13], and Park [17].

The number

$$\frac{1}{n}\sum_{i=1}^{n} (z_i - \widetilde{z})^2 = S^2 \tag{1.3}$$

is also important in this context. If z_i's are all real we denote them by x_i's with $a = \min x_i$ and $b = \max x_i$. The arithmetic mean by $\bar{x}$ and variance by the lower case letter s^2. In this case $S_z = |S| = S = s$ but in general S_z rather than $|S|$ is more consistent with s. For instance, $s = 0$ $(S_z = 0)$ if and only if all the x_i's $(z_i$'s$)$ are equal. This is not the case with $|S|$; for example, for three distinct complex numbers $0,\ \pm\frac{1}{2} + i\frac{\sqrt{3}}{2}$ we have $S = 0$. It however turns out that for some purposes s^2 is more consistent with

$$\sigma_z^2 = \frac{|S^2| + S_z^2}{2} \tag{1.4}$$

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than S_z^2. Note that the analogue of the Popoviciu inequality [18]

$$s^2 \leqslant \frac{(b-a)^2}{4} \tag{1.5}$$

for the complex numbers says that

$$\sigma_z^2 \leqslant \max_{i,j} \frac{|z_i - z_j|^2}{4}. \tag{1.6}$$

But it is not always true that $S_z^2 \leqslant \max_{i,j} \frac{|z_i - z_j|^2}{4}$. For example, for $z_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $z_2 = 0$ and $z_3 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, $S_z^2 = \frac{1}{3}$ and $\max_{i,j} |z_i - z_j| = 1$.

The corresponding inequality for S_z^2 is

$$S_z^2 \leqslant r_z^2 \leqslant \max_{i,j} \frac{|z_i - z_j|^2}{3}, \tag{1.7}$$

where r_z is the radius of the smallest disk containing all the numbers z_i's, see [4, 5].

A classical theorem of Jung [9] says that the complex numbers z_i's in a plane can be contained in a closed disk of radius $\max_{i,j} \frac{|z_i - z_j|}{\sqrt{3}}$. We thus have

$$\max_{i,j} \frac{|z_i - z_j|}{2} \leqslant r_z \leqslant \max_{i,j} \frac{|z_i - z_j|}{\sqrt{3}}.$$

In this context it is interesting to note a case when the given complex numbers lie on the boundary of the smallest disk containing them. We here show that if the complex numbers lie on a circle with centre at their arithmetic mean then this circle is the smallest circle enclosing these points, (see Theorem 2.1 & 3.1 below). A necessary and sufficient condition is given for which the numbers σ_z, S_z and $|S|$ are all equal, (Theorem 2.2). We obtain a complex analogue of the inequality, Mallows and Richter [11],

$$s^2 \geqslant \frac{r}{n-r} (\alpha_r - \overline{x})^2, \tag{1.8}$$

where α_r is the arithmetic mean of any subset of r numbers chosen from the real numbers $x_1, x_2, \ldots, x_n$, (Theorem 2.3).

On the other hand we find in literature that the inequality (1.5) and its complementary Nagy's inequality [13],

$$s^2 \geqslant \frac{(b-a)^2}{2n} \tag{1.9}$$

also provide bounds for the spread of a complex $n \times n$ matrix A when the eigenvalues $\lambda_i(A)$ of A are all real. The spread of a matrix A is the maximum distance between two eigenvalues of a matrix, $\operatorname{Spd}(A) = \lambda_{\max}(A) - \lambda_{\min}(A)$. We have,

$$\frac{4}{n} \operatorname{tr} B^2 \leqslant \operatorname{Spd}(A)^2 \leqslant 2 \operatorname{tr} B^2, \tag{1.10}$$

where $B = A - \frac{\mathrm{tr}A}{n} I$ and $\mathrm{tr}A$ denotes the trace of A, see [6, 23].

We show that the inequalities, [3, 21],

$$\frac{(b-a)^2}{2n} + \frac{2}{n-2}\left(\overline{x} - \frac{a+b}{2}\right)^2 \leqslant s^2 \leqslant (b-\overline{x})(\overline{x}-a), \tag{1.11}$$

provide some further refinements of the inequalities (1.5) and (1.9) and consequently we get better bounds for the spread of a matrix for some special cases, (Theorem 2.4, 2.5, 3.2). A refinement of the inequality (1.5) is obtained for Leptokurtic and Mesokurtic distributions, (Theorem 2.6). It is shown that better estimates can be obtained from the existing bounds for the eigenvalues and spread of a matrix when any one of its eigenvalue is known in advance, (Theorem 3.3, 3.4). Likewise, the bounds for the span of a polynomial are given, (Theorem 3.5).

2. Main results

THEOREM 2.1. *If the complex numbers z_i's in the complex plane lie on a circle with centre $\widetilde{z}$ and radius r_z, then r_z is the radius of the smallest disk containing all the points z_i's.*

Proof. For any complex number c, we can write (1.2) in the form

$$S_z^2 = \frac{1}{n}\sum_{i=1}^{n} |z_i - c + c - \widetilde{z}|^2 = \frac{1}{n}\sum_{i=1}^{n} |z_i - c|^2 - |\widetilde{z} - c|^2. \tag{2.1}$$

If all the complex numbers z_i's lie on the circle $|z-c| = r_z$, then

$$\frac{1}{n}\sum_{i=1}^{n} |z_i - c|^2 = r_z^2. \tag{2.2}$$

Combining (2.1) and (2.2), we get that

$$S_z^2 + |\widetilde{z} - c|^2 = r_z^2. \tag{2.3}$$

From the first inequality (1.7), $r_z \geqslant S_z$. So the minimum value of r_z is S_z. This implies that if $r_z = S_z$ then r_z is the radius of the smallest disk containing the points z_i's. For $\widetilde{z} = c$, (2.3) gives $r_z = S_z$. This proves the theorem. □

THEOREM 2.2. *Let $z_1, z_2, \ldots, z_n$ be the points in the finite complex plane and let S_z, S and σ_z be defined as in (1.2), (1.3) and (1.4), respectively. Then, $S_z = |S| = \sigma_z$ if and only if all the points $z_1, z_2, \ldots, z_n$ lie on a straight line.*

Proof. In the complex plane the convex combination of complex numbers lie in the convex hull of these numbers. It follows that if the points z_i's are collinear then $\widetilde{z}$ also lies on the straight line passing through z_i's.

From (1.2)–(1.4), we see that $S_z = |S| = \sigma_z$ if and only if

$$\left|\sum_{i=1}^{n} (z_i - \widetilde{z})^2\right| = \sum_{i=1}^{n} |z_i - \widetilde{z}|^2 . \tag{2.4}$$

The equality occurs in triangle inequality

$$\left|\sum_{i=1}^{n} a_i\right| \leqslant \sum_{i=1}^{n} |a_i|$$

if and only if the ratio of any two non-zero terms is positive that is $\frac{a_i}{a_j} > 0, \quad i, j = 1, 2, \ldots, n,$ see Ahlfors [1]. This means (2.4) holds true if and only if the ratio of any two non zero terms in (2.4) is positive, that is

$$\left(\frac{z_i - \widetilde{z}}{z_j - \widetilde{z}}\right)^2 > 0. \tag{2.5}$$

The square of a complex number z is positive if and only if z is real and therefore (2.5) implies that $\frac{z_i - \widetilde{z}}{z_j - \widetilde{z}}$ is real. Also, $\frac{z_i - \widetilde{z}}{z_j - \widetilde{z}}$ is real if and only if z_i lies on the straight line passing through z_j and $\widetilde{z}$. If $z_k - \widetilde{z} = 0$ for some k then $z_k = \widetilde{z}$ and so z_k lies on the straight line passing through z_j and $\widetilde{z}$. □

We need following lemma to extend the inequality (1.8) for complex numbers.

LEMMA 2.1. *Let* $Z_1 = \{z_1, z_2, \ldots, z_{n_1}\}$ *and* $Z_2 = \{z_{n_1+1}, z_{n_1+2}, \ldots, z_{n_1+n_2}\}$ *be two sets of complex numbers. Denote by* $\widetilde{Z}_i$ *and* $S^2_{Z_i}$ *the arithmetic mean and variance of* Z_i*'s,* $i = 1, 2,$ *respectively. Then the combined variance* $S^2_{Z_1 \cup Z_2}$ *of the set* $Z_1 \cup Z_2$ *is given by*

$$S^2_{Z_1 \cup Z_2} = \frac{n_1}{n_1 + n_2} S^2_{Z_1} + \frac{n_2}{n_1 + n_2} S^2_{Z_2} + \frac{n_1 n_2}{(n_1 + n_2)^2} \left|\widetilde{Z}_1 - \widetilde{Z}_2\right|^2. \tag{2.6}$$

Proof. The combined variance of the set $Z_1 \cup Z_2$ of $n_1 + n_2$ numbers can be written as

$$S^2_{Z_1 \cup Z_2} = \frac{1}{n_1 + n_2} \left(\sum_{j=1}^{n_1} \left|z_j - \widetilde{a}\right|^2 + \sum_{j=n_1+1}^{n_1+n_2} \left|z_j - \widetilde{a}\right|^2 \right), \tag{2.7}$$

where

$$\widetilde{a} = \frac{1}{n_1 + n_2} \sum_{j=1}^{n_1+n_2} z_j.$$

We note that

$$\begin{aligned} \left|z_j - \widetilde{a}\right|^2 &= \left|z_j - \widetilde{Z}_1 + \widetilde{Z}_1 - \widetilde{a}\right|^2 \\ &= \left|z_j - \widetilde{Z}_1\right|^2 + \left|\widetilde{Z}_1 - \widetilde{a}\right|^2 + 2\mathrm{Re}\,\overline{\left(\widetilde{Z}_1 - \widetilde{a}\right)} \left(z_j - \widetilde{Z}_1\right), \end{aligned}$$

$$\sum_{j=1}^{n_1} \left(z_j - \widetilde{Z}_1\right) = 0 \text{ and } \left|\widetilde{Z}_1 - \widetilde{a}\right| = \frac{n_2}{n_1 + n_2} \left|\widetilde{Z}_1 - \widetilde{Z}_2\right|.$$

Therefore,

$$\sum_{j=1}^{n_1} \left|z_j - \widetilde{a}\right|^2 = \sum_{j=1}^{n_1} \left|z_j - \widetilde{Z}_1\right|^2 + \frac{n_1 n_2^2}{(n_1 + n_2)^2} \left|\widetilde{Z}_1 - \widetilde{Z}_2\right|^2. \tag{2.8}$$

On using similar arguments, we have

$$\sum_{j=n_1+1}^{n_1+n_2} \left|z_j-\widetilde{a}\right|^2=\sum_{j=n_1+1}^{n_1+n_2}\left|z_j-\widetilde{Z}_2\right|^2+\frac{n_1^2 n_2}{(n_1+n_2)^2}\left|\widetilde{Z}_1-\widetilde{Z}_2\right|^2. \tag{2.9}$$

The assertions of the theorem now follow on using (2.8) and (2.9) in (2.7). □

THEOREM 2.3. *Let γ_r be the arithmetic mean of any subset of r numbers chosen from the set of n complex numbers $z_1, z_2, \ldots, z_n$ and let σ_z^2 be defined as in (1.4). Then the inequality*

$$|\gamma_r-\widetilde{z}|^2 \leqslant \frac{n-r}{r}\sigma_z^2 \tag{2.10}$$

holds true for $1 \leqslant r \leqslant n$.

Proof. Let Z_1 and Z_2 be the disjoint sets of r and $n-r$ numbers chosen from the numbers $z_1, z_2, \ldots, z_n$, respectively. Denote by $S^2_{z(r)}$ and $S^2_{z(n-r)}$ the variance of Z_1 and Z_2, respectively. We now apply Lemma 2.1 and find that

$$S_z^2=\frac{r}{n}S^2_{z(r)}+\frac{n-r}{n}S^2_{z(n-r)}+\frac{r(n-r)}{n^2}\left|\gamma_r-\gamma_{n-r}\right|^2 . \tag{2.11}$$

Further,

$$|\gamma_r-\gamma_{n-r}|=\left|\gamma_r-\frac{1}{n-r}\left(\sum_{i=1}^{n} z_i-\sum_{i=1}^{r} z_i\right)\right|=\left|\frac{n}{n-r}(\gamma_r-\widetilde{z})\right|$$

and therefore (2.11) can be written as

$$S_z^2=\frac{r}{n}S^2_{z(r)}+\frac{n-r}{n}S^2_{z(n-r)}+\frac{r}{n-r}\left|\gamma_r-\widetilde{z}\right|^2 . \tag{2.12}$$

On using similar arguments, we have

$$S^2=\frac{r}{n}S^2_{r}+\frac{n-r}{n}S^2_{n-r}+\frac{r}{n-r}\left(\gamma_r-\widetilde{z}\right)^2 . \tag{2.13}$$

On applying triangle inequality we find from (2.13) that

$$\left|S^2\right| \geqslant \frac{r}{n-r}\left|\gamma_r-\widetilde{z}\right|^2-\left|\frac{r}{n}S_r^2+\frac{n-r}{n}S^2_{n-r}\right|. \tag{2.14}$$

From (2.12) and (2.14), we get that

$$\left|S^2\right|+S_z^2 \geqslant \frac{2r}{n-r}\left|\gamma_r-\widetilde{z}\right|^2+\frac{r}{n}S^2_{z(r)}+\frac{n-r}{n}S^2_{z(n-r)}-\left|\frac{r}{n}S_r^2+\frac{n-r}{n}S^2_{n-r}\right| . \tag{2.15}$$

Again by triangle inequality, $S^2_{z(r)} \geqslant \left|S_r^2\right|$, $S^2_{z(n-r)} \geqslant \left|S^2_{n-r}\right|$ and therefore

$$\frac{r}{n}S^2_{z(r)}+\frac{n-r}{n}S^2_{z(n-r)} \geqslant \frac{r}{n}\left|S_r^2\right|+\frac{n-r}{n}\left|S^2_{n-r}\right| \geqslant \left|\frac{r}{n}S_r^2+\frac{n-r}{n}S^2_{n-r}\right|. \tag{2.16}$$

The inequality (2.10) now follows from (2.15) and (2.16). □

The inequality (2.10) is an extension of Mallows and Richter inequality [11]. For $r=1$, we obtain the generalisation of the well known Samuelson's inequality [20],

$$\sigma_z^2 \geqslant \frac{1}{n-1}\left|z_j-\widetilde{z}\right|^2.$$

Likewise, we can prove the following extension of Nagy's inequality [13],

$$\sigma_z^2 \geqslant \frac{1}{2n}\max_{j,k}\left|z_j-z_k\right|^2, \quad j,k=1,2,\ldots,n. \tag{2.17}$$

Note that for $r=1$, $S_1=0$ and therefore from (2.13) on using triangle inequality we get that

$$\left|S_{n-1}^2\right| \leqslant \frac{n}{n-1}\left|S^2\right| + \frac{n}{(n-1)^2}\left|\widetilde{z}-z_j\right|^2.$$

Similarly, from (2.12), we have

$$S_{z(n-1)}^2 = \frac{n}{n-1}S_z^2 - \frac{n}{(n-1)^2}\left|\widetilde{z}-z_j\right|^2$$

and by addition we obtain the inequality

$$\sigma_{z(n-1)}^2 = \frac{\left|S_{n-1}\right|^2 + S_{z(n-1)}^2}{2} \leqslant \frac{n}{n-1}\sigma_z^2.$$

It then follows inductively that the inequality

$$\sigma_{z(m)}^2 \leqslant \frac{n}{m}\sigma_z^2,$$

holds true for $m=1,2,\ldots,n$ and therefore for $m=2$, we have

$$\sigma_z^2 \geqslant \frac{2}{n}\sigma_{z(2)}^2 = \frac{1}{2n}\left|z_i-z_j\right|^2 \tag{2.18}$$

for all $i,j=1,2,\ldots,n,\ i\neq j$. The inequality (2.18) implies (2.17). Also, see [24].

THEOREM 2.4. *For $0\leqslant a<\overline{x}\leqslant s$, we have*

$$s^2 + \left(\frac{s^2-\overline{x}^2}{2\overline{x}}\right)^2 \leqslant \frac{(b-a)^2}{4} \tag{2.19}$$

and with $n\geqslant 3$

$$s^2 - \frac{2}{n-2}\left(\frac{s^2-\overline{x}^2}{2\overline{x}}\right)^2 \geqslant \frac{(b-a)^2}{2n}. \tag{2.20}$$

Proof. The second inequality (1.11) implies that

$$\overline{x}^2 \leqslant (a+b)\overline{x} - ab - s^2,$$

and therefore for $0 \leqslant a < \overline{x}$, we can write

$$\overline{x} \leqslant \frac{a+b}{2} - \frac{s^2 - \overline{x}^2 + ab}{2\overline{x}} \leqslant \frac{a+b}{2} - \frac{s^2 - \overline{x}^2}{2\overline{x}} = \alpha \text{ (say)}. \tag{2.21}$$

It is clear that $\alpha \leqslant \frac{a+b}{2}$ and since $f(x) = (x-a)(b-x)$ increases in the interval $\left[a, \frac{a+b}{2}\right], a < b,$ we find that

$$(\overline{x} - a)(b - \overline{x}) \leqslant (\alpha - a)(b - \alpha) = \frac{(b-a)^2}{4} - \left(\frac{s^2 - \overline{x}^2}{2\overline{x}}\right)^2. \tag{2.22}$$

Combining (2.22) and the second inequality (1.11); we immediately get (2.19).

Further, it follows from (2.21) that for $0 < \overline{x} \leqslant s$,

$$\left(\frac{a+b}{2} - \overline{x}\right)^2 \geqslant \left(\frac{s^2 - \overline{x}^2}{2\overline{x}}\right)^2. \tag{2.23}$$

Combining (2.23) with the first inequality (1.11); a little computation leads to (2.20). □

It may be noted here that the inequality (2.19) can equivalently be written as

$$\frac{m_2'}{\overline{x}} \leqslant b - a, \tag{2.24}$$

where $m_2' = s^2 + \overline{x}^2$.

We mention an alternative proof of (2.24). From the second inequality (1.11),

$$\frac{m_2'}{\overline{x}} \leqslant \frac{(a+b)\overline{x} - ab}{\overline{x}}, \ \overline{x} > 0. \tag{2.25}$$

Also, for $0 \leqslant a < \overline{x} \leqslant s$, from the inequality (1.5), we have $\overline{x} \leqslant s \leqslant \frac{b-a}{2} \leqslant \frac{b}{2}$ and for $\overline{x} \leqslant \frac{b}{2}$,

$$\frac{(a+b)\overline{x} - ab}{\overline{x}} \leqslant b - a. \tag{2.26}$$

The inequality (2.24) follows from (2.25) and (2.26).

THEOREM 2.5. *For $a < 0$ and $\overline{x} \geqslant \sqrt{\frac{n}{2}} s$, we have*

$$s^2 + \left(\frac{\overline{x}^2 - \frac{n}{2}s^2}{2\overline{x}}\right)^2 \leqslant \frac{(b-a)^2}{4} \tag{2.27}$$

and with $n \geqslant 3$,

$$s^2 - \frac{2}{n-2}\left(\frac{\overline{x}^2 - \frac{n}{2}s^2}{2\overline{x}}\right)^2 \geqslant \frac{(b-a)^2}{2n}. \tag{2.28}$$

Proof. We write (1.9) in the form

$$s^2 \geqslant \frac{(b-\overline{x}+\overline{x}-a)^2}{2n} = \frac{(b-\overline{x})^2+(\overline{x}-a)^2+2(b-\overline{x})(\overline{x}-a)}{2n}. \tag{2.29}$$

Using arithmetic mean - geometric mean inequality,

$$(b-\overline{x})^2+(\overline{x}-a)^2 \geqslant 2(b-\overline{x})(\overline{x}-a). \tag{2.30}$$

Thus, from (2.29) and (2.30),

$$s^2 \geqslant \frac{2}{n}(b-\overline{x})(\overline{x}-a). \tag{2.31}$$

It follows from (2.31) that

$$\overline{x}^2 \geqslant (a+b)\overline{x}-\frac{n}{2}s^2-ab$$

and consequently, for $a<0$ and $\overline{x}>0,$ we have

$$\overline{x} \geqslant \frac{a+b}{2}+\frac{1}{2\overline{x}}\left(\overline{x}^2-\frac{n}{2}s^2-ab\right) \geqslant \frac{a+b}{2}+\frac{1}{2\overline{x}}\left(\overline{x}^2-\frac{n}{2}s^2\right)=\beta \text{ (say)}. \tag{2.32}$$

It is clear that $\beta \geqslant \frac{a+b}{2}$ for $\overline{x} \geqslant \sqrt{\frac{n}{2}}s$ and since $f(x)=(\overline{x}-a)(b-\overline{x})$ decreases in the interval $\left[\frac{a+b}{2},b\right],a<b,$ we find that

$$(b-\overline{x})(\overline{x}-a) \leqslant \left(\frac{b-a}{2}\right)^2-\left(\frac{\overline{x}^2-\frac{n}{2}s^2}{2\overline{x}}\right)^2. \tag{2.33}$$

Combining (2.33) with the second inequality (1.11); we immediately get (2.27).

From (2.33), we also have

$$\left(\overline{x}-\frac{a+b}{2}\right)^2 \geqslant \left(\frac{\overline{x}^2-\frac{n}{2}s^2}{2\overline{x}}\right)^2. \tag{2.34}$$

The inequality (2.28) follows from (2.34) and the first inequality (1.11). □

Sharma et al. [22] have proved that

$$m_4+3m_2^2 \leqslant (b-a)^2(\overline{x}-a)(b-\overline{x}), \tag{2.35}$$

where $m_2=s^2$ and $m_4=\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^4$.

If the distribution is Leptokurtic or Mesokurtic, we have, see [10],

$$\frac{m_4}{m_2^2} \geqslant 3. \tag{2.36}$$

We prove a refinement of the inequality (1.5) in the following theorem.

THEOREM 2.6. *For a Leptokurtic and Mesokurtic distribution, we have*

$$s^2 \leqslant (b-a)\sqrt{\frac{(\overline{x}-a)(b-\overline{x})}{6}} \leqslant \frac{(b-a)^2}{2\sqrt{6}}. \tag{2.37}$$

Proof. Under the assumptions of the theorem the inequalities (2.35) and (2.36) hold true. By (2.36), $3s^4 \leqslant m_4$ and we obtain from (2.35) that

$$6s^4 \leqslant (b-a)^2(\overline{x}-a)(b-\overline{x}). \tag{2.38}$$

This gives the first inequality (2.37). The second inequality (2.37) follows from (2.38) on using arithmetic mean - geometric mean inequality, $(\overline{x}-a)(b-\overline{x}) \leqslant \frac{(b-a)^2}{4}$. □

We remark that the inequalities (2.37) also hold true for both discrete and continuous distributions.

3. Bounds for eigenvalues

Let $\mathbb{M}(n)$ denote the algebra of all complex $n \times n$ matrices. We assume that the eigenvalues $\lambda_i(A)$ of $A = (a_{ij}) \in \mathbb{M}(n)$ are all real, and may respectively define their arithmetic mean and variance to be

$$\overline{\lambda}(A) = \frac{1}{n}\sum_{i=1}^{n}\lambda_i(A) = \frac{\text{tr}A}{n} \tag{3.1}$$

and

$$s_\lambda^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\lambda_i(A) - \overline{\lambda}(A)\right)^2 = \frac{\text{tr}A^2}{n} - \left(\frac{\text{tr}A}{n}\right)^2 = \frac{\text{tr}B^2}{n}, \tag{3.2}$$

where $B = A - \frac{\text{tr}A}{n}I$.

The spread of a matrix is the greatest distance between its eigenvalues. The notion of the spread was introduced by Mirsky [14, 15] and several authors have studied bounds for the spread of a matrix, see [6, 8, 13, 24].

THEOREM 3.1. *If trace of a unitary matrix $U \in \mathbb{M}(n)$ is zero then the unit circle is the smallest circle enclosing the eigenvalues of U, and greatest lower bound on the $Spd(U)$ is $\sqrt{3}$.*

Proof. The eigenvalues of a unitary matrix U all lie on the unit circle and by assumption of the theorem $\text{tr}U = 0$. So, the eigenvalues $\lambda_i(U)$'s satisfy the conditions of the Theorem 2.1 and hence the unit circle is the smallest circle containing $\lambda_i(U)$'s. It also follows from the second inequality (1.7) that $\text{Spd}(U) \geqslant \sqrt{3}$. □

EXAMPLE 1. The basic circulant matrix C with first row $(0,1,0\ldots,0)$ is a unitary matrix and its trace is zero. By Theorem 3.1 the unit disk is the smallest disk containing eigenvalues of C and $\text{Spd}C \geqslant \sqrt{3}$. Also, for $n=3$ we have $\text{Spd}C = \sqrt{3}$.

The following theorem is a consequence of Theorem 2.4 and provides refinements of the inequalities (1.10).

THEOREM 3.2. *Let the eigenvalues of an element $A \in \mathbb{M}(n)$ be all non negative and let $0 < \mathrm{tr}A \leqslant \left(n\mathrm{tr}B^2\right)^{\frac{1}{2}}$. Then*

$$\mathrm{Spd}\,(A) \geqslant \frac{\mathrm{tr}A^2}{\mathrm{tr}A} \tag{3.3}$$

and with $n \geqslant 3$,

$$\mathrm{Spd}\,(A) \leqslant \frac{1}{\mathrm{tr}A}\left(2\mathrm{tr}B^2\,(\mathrm{tr}A)^2 - \frac{\left(n\mathrm{tr}B^2 - (\mathrm{tr}A)^2\right)^2}{n\,(n-2)}\right)^{\frac{1}{2}}. \tag{3.4}$$

Proof. Under the condition $\mathrm{tr}A \leqslant \left(n\mathrm{tr}B^2\right)^{\frac{1}{2}}$, we have

$$\overline{\lambda}\,(A) = \frac{\mathrm{tr}A}{n} \leqslant \left(\frac{1}{n}\mathrm{tr}B^2\right)^{\frac{1}{2}} = \left(\frac{\mathrm{tr}A^2}{n} - \left(\frac{\mathrm{tr}A}{n}\right)^2\right)^{\frac{1}{2}} = s_\lambda. \tag{3.5}$$

Further, the eigenvalues of A are all non-negative, therefore $0 < \lambda_{\min}(A) \leqslant \overline{\lambda}\,(A) \leqslant s_\lambda$ and $\mathrm{Spd}\,(A) = \lambda_{\max}(A) - \lambda_{\min}(A)$. So we can apply Theorem 2.4, the inequalities (3.3) and (3.4) follow on using (3.1) and (3.2) in (2.19) and (2.20), respectively. □

EXAMPLE 2. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 16 & 1 \\ 1 & 1 & 1 & 100 \end{bmatrix}.$$

From (1.10), $81.393 \leqslant \mathrm{Spd}\,(A) \leqslant 115.11$. The matrix A is positive definite and $\mathrm{tr}A \leqslant \left(n\mathrm{tr}B^2\right)^{\frac{1}{2}}$. So, from our bounds (3.3) and (3.4) we have better estimate $85 \leqslant \mathrm{Spd}\,(A) \leqslant 104.90$.

Likewise, we can obtain another refinement of the inequality (1.10) on applying Theorem 2.5. If $\lambda_{\min}(A) < 0$ and $0 < 2\,(\mathrm{tr}A)^2 \geqslant n^2\,\mathrm{tr}\,B^2$, then

$$\mathrm{Spd}\,(A) \geqslant \frac{1}{2n\mathrm{tr}A}\left(16n\mathrm{tr}B^2\,(\mathrm{tr}A)^2 + \left(2\,(\mathrm{tr}A)^2 - n^2\mathrm{tr}B^2\right)^2\right)^{\frac{1}{2}} \tag{3.6}$$

and for $n \geqslant 3$,

$$\mathrm{Spd}\,(A) \leqslant \frac{1}{\mathrm{tr}A}\left(2\mathrm{tr}B^2\,(\mathrm{tr}A)^2 - \frac{\left(2\,(\mathrm{tr}A)^2 - n^2\mathrm{tr}B^2\right)^2}{4n\,(n-2)}\right)^{\frac{1}{2}}. \tag{3.7}$$

Further, Wolkowicz and Styan [23] have shown that if the eigenvalues of $A \in \mathbb{M}(n)$ are all real and $\lambda_1(A) \leqslant \lambda_i(A) \leqslant \lambda_n(A)$, $i = 1, 2, \ldots, n$, then

$$\frac{\mathrm{tr}A}{n} - \sqrt{\frac{n-1}{n}\mathrm{tr}B^2} \leqslant \lambda_1(A) \leqslant \frac{\mathrm{tr}A}{n} - \sqrt{\frac{1}{n\,(n-1)}\mathrm{tr}B^2} \tag{3.8}$$

and

$$\frac{\operatorname{tr}A}{n}+\sqrt{\frac{1}{n(n-1)}\operatorname{tr}B^2}\leqslant\lambda_n(A)\leqslant\frac{\operatorname{tr}A}{n}+\sqrt{\frac{n-1}{n}\operatorname{tr}B^2}. \tag{3.9}$$

The inequalities (3.8) and (3.9) follow respectively from the inequalities, [7, 20],

$$\overline{x}-\sqrt{n-1}s\leqslant\min_i x_i\leqslant\overline{x}-\frac{s}{\sqrt{n-1}} \tag{3.10}$$

and

$$\overline{x}+\frac{s}{\sqrt{n-1}}\leqslant\max_i x_i\leqslant\overline{x}+\sqrt{n-1}s. \tag{3.11}$$

We now discuss extensions of these inequalities for the case when any one eigenvalue of A is known as in case of stochastic and singular matrices.

It is clear from Lemma 2.1 that if s_{n-1}^2 is the variance of $n-1$ numbers obtained by excluding a number x_j from the real numbers $x_1,x_2,\ldots,x_n$, then

$$s_{n-1}^2=\frac{n}{n-1}s^2-\frac{n}{(n-1)^2}\left(\overline{x}-x_j\right)^2. \tag{3.12}$$

THEOREM 3.3. *Let the eigenvalues of* $A\in\mathbb{M}(n)$ *be all real. Let* $\nu(A)$ *be an eigenvalue of* A *and denote the remaining eigenvalues by* $\nu_i(A)$, $\nu_1(A)\leqslant\nu_i(A)\leqslant\nu_{n-1}(A)$, $i=1,2,\ldots,n-1$. *Then, for* $n\geqslant3$,

$$\frac{\operatorname{tr}A-\nu(A)}{n-1}-\sqrt{n-2}s_\nu\leqslant\nu_1(A)\leqslant\frac{\operatorname{tr}A-\nu(A)}{n-1}-\frac{s_\nu}{\sqrt{n-2}} \tag{3.13}$$

and

$$\frac{\operatorname{tr}A-\nu(A)}{n-1}+\frac{s_\nu}{\sqrt{n-2}}\leqslant\nu_n(A)\leqslant\frac{\operatorname{tr}A-\nu(A)}{n-1}+\sqrt{n-2}s_\nu. \tag{3.14}$$

Proof. The arithmetic mean $\overline{\nu}(A)$ of $n-1$ eigenvalues $\nu_i(A)$ can be written as

$$\overline{\nu}(A)=\frac{1}{n-1}\sum_{i=1}^{n-1}\nu_i(A)=\frac{\operatorname{tr}A-\nu(A)}{n-1}. \tag{3.15}$$

By the use of (3.12) the variance of these eigenvalues is

$$\begin{aligned}s_\nu^2&=\frac{1}{n-1}\sum_{i=1}^{n-1}\left(\nu_i(A)-\overline{\nu}(A)\right)^2=\frac{n}{n-1}s_\lambda^2-\frac{n}{(n-1)^2}\left(\overline{\lambda}(A)-\nu(A)\right)^2\\&=\frac{\operatorname{tr}B^2}{n-1}-\frac{n}{(n-1)^2}\left(\frac{\operatorname{tr}A}{n}-\nu(A)\right)^2.\end{aligned} \tag{3.16}$$

On applying (3.10) to $n-1$ numbers $\nu_1(A),\nu_2(A),\ldots,\nu_{n-1}(A)$ and using (3.15) and (3.16); we immediately get (3.13). Likewise, (3.14) follows from (3.11). $\square$

THEOREM 3.4. *Under the conditions of Theorem 3.3., we have*

$$\max_{i,j}\left|\nu_i(A)-\nu_j(A)\right|^2 \leqslant 2\left(\operatorname{tr}B^2-\frac{n}{n-1}\left(\frac{\operatorname{tr}A}{n}-\nu(A)\right)^2\right) \tag{3.17}$$

and

$$\max_{i,j}\left|\nu_i(A)-\nu_j(A)\right|^2 \geqslant \frac{4}{n-1}\left(\operatorname{tr}B^2-\frac{n}{n-1}\left(\frac{\operatorname{tr}A}{n}-\nu(A)\right)^2\right). \tag{3.18}$$

Proof. On using the inequalities (1.5) and (1.9), for $n-1$ numbers $\nu_1(A),\nu_2(A),\ldots,\nu_{n-1}(A)$, we have

$$4s_\nu^2 \leqslant \max_{i,j}\left|\nu_i(A)-\nu_j(A)\right|^2 \leqslant 2(n-1)s_\nu^2. \tag{3.19}$$

Inserting (3.16) in (3.19), we immediately get (3.17) and (3.18) on simplifications. □

EXAMPLE 3. Let

$$A=\begin{bmatrix}1 & 2 & 9 & 4\\ 2 & 10 & 0 & 4\\ 9 & 0 & 5 & 2\\ 4 & 4 & 2 & 6\end{bmatrix}.$$

From the inequalities (3.8), we have $-9.0688 \leqslant \lambda_1(A) \leqslant .644$. The largest eigenvalue of A is 16 as all its row sums are 16 and A is a symmetric matrix. From (3.13) we have better estimate for the smallest root, $-7.521 \leqslant \lambda_1(A) \leqslant -2.7610$. The actual value of $\lambda_1(A)$ to four decimal places is -6.5788.

We now consider polynomials with real zeros. Let f be a monic polynomial

$$f(x)=x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_n \tag{3.20}$$

with only real zeros. Then the length $b-a$ of the smallest interval $[a,b]$ containing all the zeros of f is called span of f, see [12, 19]. Denote by D_n the span of f then

$$\frac{2}{n}\sqrt{(n-1)a_1^2-2na_2} \leqslant D_n \leqslant \sqrt{2\frac{n-1}{n}a_1^2-4a_2}. \tag{3.21}$$

See Corollary 6.1.4 and Theorem 6.1.6 in [19].

We prove a refinement of (3.21) in the following theorem.

THEOREM 3.5. *Let the zeros of the polynomial (3.20) be all non-negative and let* $2na_2 \leqslant (n-2)a_1^2$. *Then*

$$D_n \geqslant \frac{2a_2-a_1^2}{a_1} \tag{3.22}$$

and with $n \geqslant 3$,

$$D_n \leqslant \sqrt{\frac{2}{n}\left((n-1)a_1^2-2na_2\right)-\frac{1}{n(n-2)}\left(\frac{2na_2-(n-2)a_1^2}{a_1}\right)^2}. \tag{3.23}$$

Proof. Let $x_1, x_2, \ldots, x_n$ be the roots of the polynomial (3.20). Then, on using relation between roots and coefficient of polynomial, we have

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{-a_1}{n}$$

and

$$s^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \overline{x}^2 = \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2 - \frac{2}{n}\sum_{i<j}^{n} x_i x_j - \overline{x}^2$$
$$= \frac{(n-1)a_1^2 - 2na_2}{n^2}.$$

The assertions of the theorem now follow on applying Theorem 2.4. □

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