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A NEW FRACTIONAL BOUNDARY VALUE PROBLEM AND LYAPUNOV–TYPE INEQUALITY

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Abstract. Throughout this paper, we study a new modified version of fractional boundary value problem (BVP) of the form

$$({}_a^C D^\alpha y)(t) + p(t)y'(t) + q(t)y(t) = 0, \qquad a < t < b, \qquad 2 < \alpha \leqslant 3,$$

with $y(a) = y'(a) = y(b) = 0$, where $p \in C^1([a,b])$ and $q \in C([a,b])$. Using the vector Green function we obtain a Lyapunov-type inequality for the BVP subject to Dirichlet-type boundary conditions. Moreover, we utilize the new inequality to infer a criteria for the nonexistence of real zeros of some certain Mittag-Leffler functions using the generalized Wright functions.

1. Introduction

In a celebrated paper of Russian mathematician Lyapunov [5] the following remarkable result has been proved.

THEOREM 1. *If $y(t)$ is a nontrivial solution of the second order differential equation*

$$y''(t) + q(t)y(t) = 0, \qquad y(a) = y(b) = 0 \tag{1}$$

where $a, b \in \mathbb{R}$ with $a < b$ be consecutive zeros, $q(t)$ is a real-valued continuous function and $y(t) \neq 0$ for $t \in (a,b)$, then the so-called Lyapunov inequality holds:

$$\int_a^b |q(s)|ds > \frac{4}{b-a}. \tag{2}$$

As we know that Theorem 1 has applications in the study of various properties of solutions in many directions such as oscillation theory, disconjugacy and eigenvalue problems of (1), several proofs and generalizations or improvements have appeared in the literature. Several authors including Reid ([7]–[8]), Hartman [9], Hochstadt [10], Eliason [11], Singh [12], Kwong [13] and Cheng [14] have contributed the above result.

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Since the theory of fractional differential equations has been extensively investigated in various results (see for example [1]–[2]) the second order differential equation mentioned in Theorem 1 has been recently considered as the following fractional boundary value problem

$$({}_a^CD^\alpha y)(t)+q(t)y(t)=0,\quad a<t<b,\quad 1<\alpha\leqslant 2,\quad y(a)=y(b)=0 \tag{3}$$

by substituting the classical derivative y'' in (1) with the Caputo fractional operator ${}_a^CD^\alpha y$. Recently, Ferreira ([3], [4]) proved that for any nontrivial continuous solution of Eq. (3) the following inequality holds:

$$\int_a^b |q(s)|ds>\frac{\Gamma(\alpha)\alpha^\alpha}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{4}$$

In both papers [3, 4], the author has presented nice applications to obtain intervals where certain Mittag-Leffler functions have no real zeros.

Very recently, Jleli and Samet [20] established some Lyapunov-type inequalities for fractional boundary value problem (3) under Sturm-Liouville boundary conditions $pu(a)-ru'(a)=u(b)=0$ where $p>0,r>0$ and considered two cases to study. Throughout this paper, we initially deal with a fractional boundary value problem including the usual derivative as follows

$$({}_a^CD^\alpha y)(t)+p(t)y'(t)+q(t)y(t)=0,\quad a<t<b,\quad 2<\alpha\leqslant 3, \tag{5}$$

with $y(a)=y'(a)=y(b)=0$, where $p\in C^1([a,b])$ and $q\in C([a,b])$.

To the best of the authors knowledge, there is no result available in the literature concerning with the problem of existence of nontrivial solutions for the boundary value problem (5). In 1999, Parhi and Panigrahi [19] have derived a series of novel results for Liapunov-type inequality of the special case of the BVP (5) given by

$$y'''+p(t)y=0,$$

where p is a real-valued continuous function on $[0,\infty)$. As is well-known, the goal of finding nontrivial solutions is of great significance in various fields of science and engineering.

For the completeness, in this section, we gather some definitions and fundamental facts of Caputo's derivatives of fractional order which can also be found in ([15], [16], [17]).

DEFINITION 1. Let $\alpha\geqslant 0$ and f be a real function defined on $[a,b]$. The Riemann-Liouville fractional integral of order α is defined by $({}_aI^0f)(t)=f(t)$ and

$$({}_aI^\alpha f)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}f(s)ds,\qquad \alpha>0,\qquad t\in[a,b].$$

DEFINITION 2. The Caputo fractional derivative of order $\alpha\geqslant 0$ is given by

$$({}_a^CD^0f)(t)=f(t)\ \text{ and }\ ({}_a^CD^\alpha f)(t)=({}_aI^{m-\alpha}D^mf)(t),\ \text{ for }\ \alpha>0,$$

where m is the smallest integer greater or equal to α. That is,

$$({}_a^C D^\alpha f)(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^m(s)}{(t-s)^{\alpha+1-m}} ds, \qquad m = [\alpha]+1, \qquad t \in [a,b].$$

In the next section, analogy with boundary value problem for differential equations of fractional order, we derive the corresponding Green function named by a vector Green function. Consequently, a sufficient condition for the existence of nontrivial solution of problem (5) is obtained. Finally, in Section 3, we give a criteria for the nonexistence of real zeros of some certain Mittag-Leffler functions.

2. Main result

The Green function for the BVP (5) can be considered as form of a vector by using a simple but crucial lemma obtained by Zhang [6] as follows:

LEMMA 1. *Let $\alpha > 0$, then the differential equation*

$$D_{0+}^\alpha u(t) = 0$$

has solutions $u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0,1,\cdots,n$, $n = [\alpha]+1$ where here, D_{0+}^α is the Caputo's fractional derivative.

Moreover, it has been proved that $I_{0+}^\alpha D_{0+}^\alpha u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^{n-1}$ for some $c_i \in \mathbb{R}$, $i = 0,1,\cdots,n$, $n = [\alpha]+1$ (see Lemma 2.3 in [6]). Here, the notations I_{0+}^α and D_{0+}^α are obtained by putting $a = 0$ in ${}_a^C D^\alpha$ and ${}_a I^\alpha$, respectively (See Definitions 1,2).

Before presenting our next result we need to clarify about the hypotheses of the following lemma. We remark that all the components $G_i(s,t)$ are defined as piecewise functions by two sub-functions $g_{i1}(s,t), g_{i2}(s,t)$ as follows:

$$G_i(s,t) = \begin{cases} g_{i1}(s,t), \ a \leqslant s \leqslant t \leqslant b \\ g_{i2}(s,t), \ a \leqslant t \leqslant s \leqslant b \end{cases}$$

for $i = 1,2,3$ where

$$\begin{bmatrix} \{g_{11}(s,t), \ g_{12}(s,t)\} \\ \{g_{21}(s,t), \ g_{22}(s,t)\} \\ \{g_{31}(s,t), \ g_{32}(s,t)\} \end{bmatrix} \tag{6}$$

$$:= \begin{bmatrix} \left\{ (\alpha-1)\left(\frac{(t-a)^2}{(b-a)^2}(b-s)^{\alpha-2} - (t-s)^{\alpha-2}\right), \quad \frac{(\alpha-1)(t-a)^2}{(b-a)^2}(b-s)^{\alpha-2} \right\} \\ \left\{ \frac{(t-a)^2}{(b-a)^2}(b-s)^{\alpha-1} - (t-s)^{\alpha-1}, \quad \frac{(t-a)^2}{(b-a)^2}(b-s)^{\alpha-1} \right\} \\ \left\{ \frac{-(t-a)^2}{(b-a)^2}(b-s)^{\alpha-1} + (t-s)^{\alpha-1}, \quad -\frac{(t-a)^2}{(b-a)^2}(b-s)^{\alpha-1} \right\} \end{bmatrix}.$$

where the first entry of each pair holds for $a \leqslant s \leqslant t \leqslant b$ and the second for $a \leqslant t \leqslant s \leqslant b$.

Besides,

$$|g_{i1}(s,t)| \leqslant |g_{i2}(s,t)|$$

for all $i = 1,2,3$. Now, we present the following lemma as discussed above.

LEMMA 2. *$y \in C^1([a,b])$ is a solution of the boundary value problem (5) if and only if y satisfies the integral equation*

$$y(t) = \int_a^b G(s,t)H(s)y(s)ds$$

where

$$G = [G_1, G_2, G_3], \qquad H = \begin{bmatrix} p \\ q \\ p' \end{bmatrix}$$

and

$$G(s,t) = \begin{bmatrix} G_1(s,t) \\ G_2(s,t) \\ G_3(s,t) \end{bmatrix} = \frac{1}{\Gamma(\alpha)} \begin{bmatrix} \{g_{11}(s,t),\, g_{12}(s,t)\} \\ \{g_{21}(s,t),\, g_{22}(s,t)\} \\ \{g_{31}(s,t),\, g_{32}(s,t)\} \end{bmatrix} \tag{7}$$

in which the sub-functions $g_{i1}(s,t)$ and $g_{i2}(s,t)$ are given by (6) for $i = 1,2,3$.

Proof. From the property of Caputo's derivative adopted by Lemma 1 and the fact mentioned right after that together with the Riemann-Liouville fractional integral ${}_aI^{\alpha}$ we can reduce the equation of problem (5) to an equivalent integral equation

$$\begin{aligned} y(t) &= c_0 + c_1(t-a) + c_2(t-a)^2 - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \Big(p(s)y'(s) + q(s)y(s) \Big) ds \\ &= c_0 + c_1(t-a) + c_2(t-a)^2 \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \Big(p'(s) - \frac{\alpha-1}{t-s} p(s) - q(s) \Big) y(s)ds. \end{aligned} \tag{8}$$

To give more details about the recent equality, we note that applying the integrating by part yields that

$$\begin{aligned} &\int_a^t (t-s)^{\alpha-1} p(s)y'(s)ds \\ &\quad = (t-s)^{\alpha-1}p(s)y(s)\Big|_a^t - \int_a^t \Big[(t-s)^{\alpha-1}p'(s) - (\alpha-1)(t-s)^{\alpha-2}p(s) \Big] y(s)ds. \end{aligned}$$

Following the boundary conditions we easily infer that $c_0 = 0$. Also, since $y'(a) = 0$, so differentiating from both sides of (8) implies that $c_1 = 0$. Thus,

$$c_2 = \frac{-1}{(b-a)^2\Gamma(\alpha)} \int_a^b \Big((b-s)^{\alpha-1}(p'(s) - q(s)) - (\alpha-1)(b-s)^{\alpha-2}p(s) \Big) y(s)ds.$$

Consequently,

$$y(t) = -\frac{(t-a)^2}{(b-a)^2\Gamma(\alpha)}\int_a^b \Big((b-s)^{\alpha-1}(p'(s)-q(s)) - (\alpha-1)(b-s)^{\alpha-2}p(s)\Big)y(s)ds + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\Big(p'(s) - \frac{\alpha-1}{t-s}p(s) - q(s)\Big)y(s)ds. \tag{9}$$

Considering the coefficients matrix $H = (p,q,p')^T$ in integral equation $y(t) = \int_a^b G(s,t)H(s)y(s)ds$ and comparing with (9) we easily find out $G(s,t)$ is as form of (7) and the consequence follows. □

LEMMA 3. *All the functions G_i for $i = 1,2,3$ given in Lemma 2 satisfy the below inequalities:*

$$|G_1(s,t)| \leqslant \frac{1}{\Gamma(\alpha)}(\alpha-1)(b-a)^{\alpha-2}\max\{g(\alpha),h(\alpha)\}$$

where

$$\max\{g(\alpha),h(\alpha)\} = \begin{cases} g(\alpha), & 2 < \alpha \leqslant \alpha_0 \\ h(\alpha), & \alpha_0 \leqslant \alpha \leqslant 3 \end{cases}, \quad (g-h)(\alpha_0) = 0,\ \alpha_0 \cong 2.427$$

and

$$|G_2(s,t)| = |G_3(s,t)| \leqslant \frac{1}{\Gamma(\alpha)}(b-a)^{\alpha-1}\max\{g(\alpha+1),h(\alpha+1),A(\alpha+1)\} \tag{10}$$

where

$$g(\alpha) = \frac{1}{4}(4-\alpha)^2,$$
$$h(\alpha) = \left(\frac{\alpha-2}{2}\right)^{\frac{(\alpha-2)(3-\alpha)}{4-\alpha}} - \left(\frac{\alpha-2}{2}\right)^{\frac{2-(\alpha-2)^2}{4-\alpha}},$$
$$A(\alpha) = 4\alpha^{-\alpha}(\alpha-2)^{\alpha-2}.$$

Proof. First, by differentiating of $g_{12}(s,s)$ on the interval (a,b) and some simple calculation we obtain

$$g_{12}'(s,s) = (\alpha-1)\frac{(b-s)^{\alpha-3}}{(b-a)^2}\Big(2(s-a)(b-s) - (\alpha-2)(s-a)^2\Big)$$

which has the zero as follows

$$g_{12}'(s,s) = 0 \quad \Longleftrightarrow \quad s = s^* = \frac{2b + a(\alpha-2)}{\alpha}.$$

Evidently, we have

$$\begin{cases} g'_{12}(s,s) > 0, \ s \in (a,s^*) \\ g'_{12}(s,s) < 0, \ s \in (s^*,b) \end{cases}$$

which implies $\max_{s\in[a,b]} g_{12}(s,s) = g_{12}(s^*,s^*)$. On the other hand, we see that

$$\begin{aligned} 0 &\leqslant g_{12}(s,t) \leqslant g_{12}(s,s) \leqslant g_{12}(s^*,s^*) \\ &= \frac{(\alpha-1)}{(b-a)^2}\Big(\frac{2b+a(\alpha-2)}{\alpha}-a\Big)^2\Big(b-\frac{2b+a(\alpha-2)}{\alpha}\Big)^{\alpha-2} \\ &= \frac{4(\alpha-1)\big((\alpha-2)(b-a)\big)^{\alpha-2}}{\alpha^\alpha}. \end{aligned} \tag{11}$$

From now on, since we are not interested to consider the classic form (i.e., $\alpha = 2$) suppose that $2 < \alpha < 3$. Now, drawing our attention to the function $g_{11}(s,t)$ and considering its differentiation related to t we infer

$$\frac{\partial}{\partial t} g_{11}(s,t) = (\alpha-1)\left(\frac{2(t-a)(b-s)^{\alpha-2}}{(b-a)^2} - (\alpha-2)(t-s)^{\alpha-3}\right), \qquad s < t,$$

$$\frac{\partial^2}{\partial t^2} g_{11}(s,t) = (\alpha-1)\left(\frac{2(b-s)^{\alpha-2}}{(b-a)^2} + (\alpha-2)(3-\alpha)(t-s)^{\alpha-3}\right) > 0, \qquad s < t.$$

Since $g_{11}(s,s) = \dfrac{(s-a)^2}{(b-a)^2}(b-s)^{\alpha-2} > 0$ and $g_{11}(s,b) = 0$, based on the sign of

$$\frac{\partial}{\partial t} g_{11}(s,b) = \left[\frac{2}{b-a}(b-s) - (\alpha-2)\right](b-s)^{\alpha-3},$$

it would be two possible cases as follows:

Case 1. Suppose that $\dfrac{\partial}{\partial t} g_{11}(s,b) < 0$, then

$$\frac{\partial}{\partial t} g_{11}(s,b) < 0 \iff 2(b-s) - (\alpha-2)(b-a) < 0 \iff s > \frac{2b-(\alpha-2)(b-a)}{2}.$$

Obviously, in this case $g_{11}(s,t)$ is nonnegative and decreasing with respect to $t \in [s,b]$. Furthermore,

$$\max_{t\in[s,b]} g_{11}(s,t) = g_{11}(s,s) = \frac{(\alpha-1)(s-a)^2}{(b-a)^2}(b-s)^{\alpha-2}, \quad s \in \Big(\frac{2b-(\alpha-2)(b-a)}{2}, b\Big).$$

From the discussion before the lemma, since the inequality $|g_{11}(s,t)| \leqslant |g_{12}(s,t)|$ holds for $a \leqslant s \leqslant t \leqslant b$, so by (11) we have

$$\begin{aligned} |g_{11}(s,t)| \leqslant g_{12}(s,t) &\leqslant g_{12}(s^*,s^*) \\ &= \frac{4(\alpha-1)\Big((\alpha-2)(b-a)\Big)^{\alpha-2}}{\alpha^\alpha} \\ &:= (\alpha-1)(b-a)^{\alpha-2}A(\alpha) \end{aligned} \tag{12}$$

where $\frac{2b-(\alpha-2)(b-a)}{2} < s \leqslant t \leqslant b$. We remark that $s^* > \frac{2b-(\alpha-2)(b-a)}{2}$ since we observe that

$$s^* = \frac{2b+a(\alpha-2)}{\alpha} > \frac{2b-(\alpha-2)(b-a)}{2} \iff (b-a)(\alpha-2)^2 > 0.$$

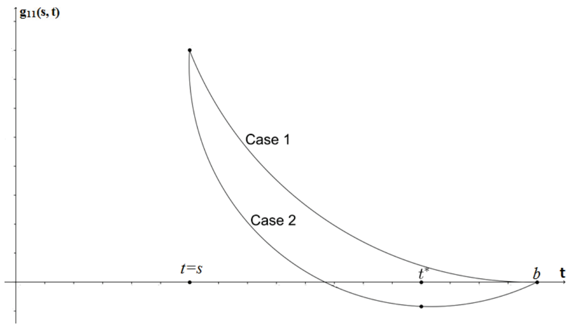

Figure 1: *Graph of sub-function* $g_{11}(s,t)$ *with respect to* t *on* $[s,b]$

Case 2. Now, let us consider $\frac{\partial}{\partial t} g_{11}(s,b) > 0$, then following the fact that $\frac{\partial^2}{\partial t^2} g_{11}(s,t) > 0$ for any $t \in (s,b)$ we easily see that there is a local minimum for sub-function $g_{11}(s,t)$, say it $t = t^*$. By Figure 1, since $s < c := \frac{2b-(\alpha-2)(b-a)}{2}$ for $a \leqslant s \leqslant t \leqslant b$ we get

$$\max_{t\in[s,b]} |g_{11}(s,t)| \leqslant \max_{t\in[s,b]} \max_{s\in[a,c)} \{|g_{11}(s,s)|, |g_{11}(s,t^*)|\} = \max_{s\in[a,c)} \{|g_{11}(s,s)|, |g_{11}(s,t^*)|\}.$$

Since

$$\begin{aligned} |g_{11}(s,s)| &= (\alpha-1)\left(\frac{(s-a)^2}{(b-a)^2}(b-s)^{\alpha-2}\right) \\ &\leqslant (\alpha-1)\left(\frac{(\frac{2b-(\alpha-2)(b-a)}{2}-a)^2}{(b-a)^2}(b-a)^{\alpha-2}\right) \\ &= \frac{1}{4}(\alpha-1)(4-\alpha)^2(b-a)^{\alpha-2} \\ &:= (\alpha-1)(b-a)^{\alpha-2} g(\alpha), \end{aligned} \tag{13}$$

$$|g_{11}(s,t^*)| = (\alpha-1)\left|\frac{(t^*-a)^2}{(b-a)^2}(b-s)^{\alpha-2} - (t^*-s)^{\alpha-2}\right|.$$

On the other hand, since $0 < \frac{t^*-s}{b-a} \leqslant \frac{t^*-a}{b-a} < 1$ and $0 < \alpha - 2 < 1$ we get

$$\left(\frac{t^*-a}{b-a}\right)^2 \left(\frac{b-s}{b-a}\right)^{\alpha-2} \leqslant \left(\frac{t^*-a}{b-a}\right)^2 \leqslant \left(\frac{t^*-s}{b-a}\right)^{\alpha-2}.$$

This shows that

$$
\begin{aligned}
|g_{11}(s,t^*)| &= (\alpha-1)(b-a)^{\alpha-2}\left[\left(\frac{t^*-s}{b-a}\right)^{\alpha-2} - \left(\frac{t^*-a}{b-a}\right)^{2}\left(\frac{b-s}{b-a}\right)^{\alpha-2}\right] \\
&\leqslant (\alpha-1)(b-a)^{\alpha-2}\left[\left(\frac{t^*-a}{b-a}\right)^{\alpha-2} - \left(\frac{t^*-a}{b-a}\right)^{2}\left(\frac{b-\frac{2b-(\alpha-2)(b-a)}{2}}{b-a}\right)^{\alpha-2}\right] \\
&\leqslant (\alpha-1)(b-a)^{\alpha-2}\left[\left(\frac{t^*-a}{b-a}\right)^{\alpha-2} - \left(\frac{t^*-a}{b-a}\right)^{2}\left(\frac{\alpha-2}{2}\right)^{\alpha-2}\right].
\end{aligned}
$$

Now, considering the function $f(x) = x^{\alpha-2} - (\frac{\alpha-2}{2})^{\alpha-2}x^2$ we see that f attains the maximum at $x = (\frac{\alpha-2}{2})^{\frac{3-\alpha}{4-\alpha}}$. This implies that

$$
\begin{aligned}
|g_{11}(s,t^*)| &\leqslant (\alpha-1)(b-a)^{\alpha-2}\left[\left(\frac{\alpha-2}{2}\right)^{\frac{(\alpha-2)(3-\alpha)}{4-\alpha}} - \left(\frac{\alpha-2}{2}\right)^{\frac{2-(\alpha-2)^2}{4-\alpha}}\right] \\
&:= (\alpha-1)(b-a)^{\alpha-2}h(\alpha).
\end{aligned} \tag{14}
$$

From the calculus, comparing the functions f,g in (13) and (14) we infer that there exists an $\alpha_0 \in (2,\frac{5}{2})$ such that $(g-h)(\alpha_0) = 0$ and $h(\alpha) < g(\alpha)$ for $\alpha \in (2,\alpha_0)$ and $g(\alpha) < h(\alpha)$ for $\alpha \in (\alpha_0,3)$. Also, it is worth mentioning that following the numerical methods we find that $\alpha_0 \cong 2.427$ (see also Figure 2).

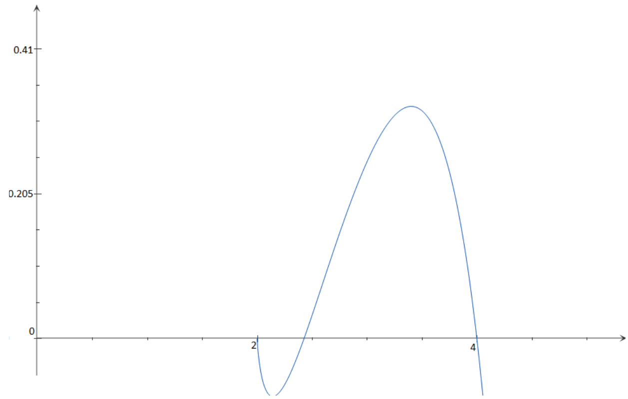

Figure 2: *Graph of function* $(h-g)(x)$

Moreover, a comparison of both functions $g(\alpha), h(\alpha)$ with $A(\alpha)$ shows that $A(\alpha) < g(\alpha)$ and $A(\alpha) < h(\alpha)$ for $\alpha \in (2,3)$ (see also Figure 3).

Now, considering the arguments as above and (12), (13) and (14) we conclude that

$$
|G_1(s,t)| \leqslant \frac{1}{\Gamma(\alpha)}(\alpha-1)(b-a)^{\alpha-2}\max\{g(\alpha),h(\alpha)\}
$$

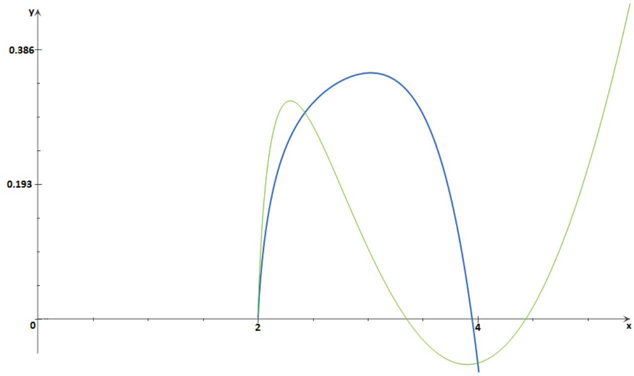

Figure 3: *Graphs of functions* $(g-A)(x)$ *(with green line) and* $(h-A)(x)$ *(with blue line)*

where

$$\max\{g(\alpha),h(\alpha)\} = \begin{cases} g(\alpha),\ 2<\alpha\leqslant\alpha_0 \\ h(\alpha),\ \alpha_0\leqslant\alpha\leqslant 3 \end{cases}.$$

We note that one can follow the process of proof for the upper bound of $|G_1(s,t)|$ and obtain the upper bound for G_2, G_3. Indeed, since the definition of G_1 is sort of similar to the structures of G_2, G_3 and only the order α is replaced by $(\alpha-1)$ and the existing coefficient $(\alpha-1)$ given in G_1 is disappeared in G_2, G_3 we easily conclude that

$$|G_2(s,t)| = |G_3(s,t)| \leqslant \frac{1}{\Gamma(\alpha)}(b-a)^{\alpha-1}\max\{g(\alpha+1),h(\alpha+1),A(\alpha+1)\}.$$

To give more details concerning with the term $A(\alpha+1)$ in recent relation we want the reader to focus on Figure 3 and the fact that $3<\alpha+1<4$. This completes the proof. □

THEOREM 2. *If a nontrivial continuously differentiable solution of the following fractional boundary value problem (FBVP) exists*

$$({}_a^C D^\alpha y)(t)+p(t)y'(t)+q(t)y(t)=0, \qquad a<t<b, \qquad 2<\alpha\leqslant 3,$$

$y(a)=y'(a)=y(b)=0$ *where* $p\in C^1([a,b])$ *and* $q\in C([a,b])$, *then*

$$\int_a^b \Big(|p(s)|+|q(s)|+|p'(s)|\Big)ds \geqslant \frac{\Gamma(\alpha)(b-a)^{1-\alpha}}{\max\{g(\alpha),h(\alpha),A(\alpha+1)\}}$$

if $\alpha\leqslant b-a+1$ *and*

$$\int_a^b \Big(|p(s)|+|q(s)|+|p'(s)|\Big)ds \geqslant \frac{\Gamma(\alpha)(b-a)^{2-\alpha}}{(\alpha-1)\max\{g(\alpha),h(\alpha),A(\alpha+1)\}}$$

if $\alpha\geqslant b-a+1$ *and* g,h *and* A *are as given in Lemma 3.*

Proof. From the discussion mentioned as before, a non-zero solution y to the FBVP satisfies the integral equation

$$y(t)=\int_a^b G(s,t)H(s)y(s)ds,\ \ t\in[a,b].$$

Obviously, we have

$$|y(t)|\leqslant\int_a^b |G(s,t)H(s)|\ |y(s)|ds,\ \ t\in[a,b].$$

By considering supremum norm for $y(t)$ on $[a,b]$ together with the preceding lemma we have

$$\begin{aligned}
&\|y\|\\
&\leqslant\int_a^b\Big(\max_{s,t\in[a,b]}|G_1(s,t)|\ |p(s)|+\max_{s,t\in[a,b]}|G_2(s,t)|\ |q(s)|+\max_{s,t\in[a,b]}|G_3(s,t)|\ |p'(s)|\Big)ds\|y\|\\
&\leqslant\frac{(b-a)^{\alpha-2}}{\Gamma(\alpha)}S(\alpha)\int_a^b\Big(|p(s)|+|q(s)|+|p'(s)|\Big)ds\|y\|
\end{aligned}$$

where

$$S(\alpha)=\max\Big\{(\alpha-1)\max\{g(\alpha),h(\alpha)\},(b-a)\max\{g(\alpha+1),h(\alpha+1),A(\alpha+1)\}\Big\}.$$

This shows that

$$\frac{\Gamma(\alpha)}{S(\alpha)(b-a)^{\alpha-2}}\leqslant\int_a^b\Big(|p(s)|+|q(s)|+|p'(s)|\Big)ds.$$

Now, since $g(\alpha),h(\alpha)$ are strictly decreasing on $(2,3)$, if $\alpha\leqslant b-a+1$, then

$$\frac{\Gamma(\alpha)(b-a)^{1-\alpha}}{\max\{g(\alpha),h(\alpha),A(\alpha+1)\}}\leqslant\int_a^b\Big(|p(s)|+|q(s)|+|p'(s)|\Big)ds.$$

Otherwise, we get

$$\frac{\Gamma(\alpha)(b-a)^{2-\alpha}}{(\alpha-1)\max\{g(\alpha),h(\alpha),A(\alpha+1)\}}\leqslant\int_a^b\Big(|p(s)|+|q(s)|+|p'(s)|\Big)ds.\qquad\square$$

Here, we give an immediate consequence as follows.

3. Real zeros of some Mittag-Leffler functions

Before we present the last result and discuss on the solution of fractional differential equation (5), we need to recall two classes of functions (one of which may be considered to be a special case of the other) and study the solutions by the terms of these functions. These functions will turn out to be of fundamental importance in the further context and applicable in so many areas in the literature. Suppose that $n > 0$. The function E_n defined by

$$E_n(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(jn+1)}$$

whenever the series converges is called the Mittag-Leffler function of order n. We immediately find that $E_1(z)$ is just the well known exponential function $\exp(z)$. The more general class of functions for $n_1, n_2 > 0$ is given as follows.

$$E_{n_1,n_2}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(jn_1+n_2)}. \tag{15}$$

The function $E_{n_1,n_2}(z)$ whenever the series converges is called the two-parameter Mittag-Leffler function with parameters n_1 and n_2.

Based on the Mittag-Leffler function and generalized Wright function ${}_p\Psi_q$ we can apply the last result to conclude an interval in which (15) with certain coefficients n_1, n_2 has no real zeros. For simplicity let now $a = 0$ and $b = 1$ and consider the following fractional eigenvalue problem:

$$({}_0^C D^{\alpha} y)(t) + \lambda y'(t) + \mu y(t) = 0, \qquad 0 < t < 1 \tag{16}$$

where $y(0) = y(1) = 0$ and $\lambda, \mu \in \mathbb{R}$. From the theory of fractional differential equations we know that (16) has the solutions as follows (see [15], Theorem 5.13):

$$y_0(t) = \sum_{k=0}^{\infty} \frac{(-\mu)^k}{k!} t^{\alpha k} \, {}_1\Psi_1 \left[\begin{matrix} (k+1,1) \\ (\alpha k+1, \alpha-1) \end{matrix} \middle| -\lambda t^{\alpha-1} \right]$$
$$+\lambda \sum_{k=0}^{\infty} \frac{(-\mu)^k}{k!} t^{\alpha(k+1)-1} \, {}_1\Psi_1 \left[\begin{matrix} (k+1,1) \\ (\alpha(k+1), \alpha-1) \end{matrix} \middle| -\lambda t^{\alpha-1} \right],$$

$$y_j(t) = \sum_{k=0}^{\infty} \frac{(-\mu)^k}{k!} t^{\alpha k+j} \, {}_1\Psi_1 \left[\begin{matrix} (k+1,1) \\ (\alpha k+j+1, \alpha-1) \end{matrix} \middle| -\lambda t^{\alpha-1} \right], \qquad j = 1,2.$$

Bring into the mind that the function ${}_p\Psi_q$ was introduced by Wright [18] and is called the generalized Wright function which is defined for $z \in \mathbb{C}$, $a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R}(\alpha_i, \beta_j \neq 0; i = 1,2,\ldots,p; j = 1,2,\ldots,q)$ by the series

$${}_p\Psi_q(z) \equiv {}_p\Psi_q \left[\begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{z^k}{k!}.$$

In particular, provided by the recent series the solutions of the problem

$$({}_0^C D^{\alpha} y)(t) + \lambda y'(t) = 0, \qquad 0 < t < 1 \tag{17}$$

with boundary conditions $y(0) = y(1) = 0$ are given by

$$\begin{aligned} y_0(t) &= E_{\alpha-1,1}(-\lambda t^{\alpha-1}) + \lambda t^{\alpha-1} E_{\alpha-1,\alpha}(-\lambda t^{\alpha-1}), \\ y_j(t) &= t^j E_{\alpha-1,j+1}(-\lambda t^{\alpha-1}), \qquad j = 1,2. \end{aligned} \tag{18}$$

THEOREM 3. *Let* $2 < \alpha \leqslant 3$. *Then, the Mittag-Leffler function* $E_{\alpha-1,j+1}(x)(j = 0,1)$ *has no real zeros for*

$$x \in \left(\frac{-\Gamma(\alpha)}{(\alpha-1)\max\{g(\alpha),h(\alpha),A(\alpha+1)\}}, 0 \right),$$

where g,h,A *are given in Lemma 3. Moreover, we have the following relation*

$$E_{\alpha-1,1}(x) = xE_{\alpha-1,\alpha}(x). \tag{19}$$

Proof. Using the boundary condition we see that the eigenvalues $\lambda \in \mathbb{R}$ of the problem (17) are the solutions of $E_{\alpha-1,j+1}(-\lambda) = 0$ for $j = 0,1$. This together with the fact that the series in (15) must be convergent implies that $\lambda \in \mathbb{R}^+$. Following Theorem 2, since $\alpha - 1 \geqslant b - a + 1 = 2$ we see that

$$\lambda = |\lambda| \geqslant \frac{\Gamma(\alpha)}{(\alpha-1)\max\{g(\alpha),h(\alpha),A(\alpha+1)\}}.$$

Therefore, the solution of $E_{\alpha-1,j+1}(-\lambda) = 0$ should hold in

$$x = -\lambda \leqslant \frac{-\Gamma(\alpha)}{(\alpha-1)\max\{g(\alpha),h(\alpha),A(\alpha+1)\}}.$$

This completes the first part of the claim. Following the relations in (18), the equality (19) is also obvious and the consequence follows. □

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