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HYERS–ULAM STABILITY FOR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS OF CARATHÉODORY TYPE

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Dedicated to Professor Jitsuro Sugie on the occasion of his 65th birthday.

(*Communicated by I. Raşa*)

Abstract. This study deals with the Hyers–Ulam stability (HUS) of the second order linear differential equations $x'' + \alpha x' + \beta x = f(t)$ without the assumption of continuity of $f(t)$. In particular, the main purpose of this study is to find a specific exact solution near the approximate solution, and the best HUS constant. Furthermore, the instability is also discussed, and a necessary and sufficient condition is obtained. Finally, a specific application example and a numerical simulation are presented.

1. Introduction

In this paper, we consider the second-order linear differential equation

$$x'' + \alpha x' + \beta x = f(t) \tag{1.1}$$

on $\mathbb{R}$, where α and β are real-valued coefficients, and $f(t)$ is a summable function on each segment contained in $\mathbb{R}$. Needless to say, our equation includes the case that $f(t)$ is continuous on $\mathbb{R}$. In the field of mechanical engineering, there are many applications that can be described by second-order linear differential equations of the form (1.1). For example, the problem determining the mass's motion of the mass-spring-damper system is one of the most important problem (see [2, 18]). In this problem, the function $f(t)$ in (1.1) usually is called an "*external force*" or an "*applied force*". Set the new variable $y = x'$. Then (1.1) is reduced to the system

$$x' = y, \quad y' = -\beta x - \alpha y + f(t), \tag{1.2}$$

or, equivalently, the equation

$$\boldsymbol{x}' = \begin{pmatrix} 0 & 1 \\ -\beta & -\alpha \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

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Note here that the discontinuity on the right hand side is allowed in this equation. If the right hand side in the equation is a summable function on each segment contained in $\mathbb{R}$, then the equation is often called "*Carathéodory equation*". Throughout this paper, let I be a nonempty open interval of $\mathbb{R}$. A real-valued function $(x(t), y(t))$ defined on I is called a "*Carathéodory solution*" of (1.2) on I if $x(t)$ and $y(t)$ are absolutely continuous on each closed interval of I and satisfies (1.2) almost everywhere on I. If $f(t)$ is continuous on $\mathbb{R}$, then the solution of (1.2) is continuously differentiable on $\mathbb{R}$. Needless to say, the continuous differentiability implies the absolute continuity, so, we can say that any classical solution is a Carathéodory solution. We can refer to the global existence and uniqueness of Carathéodory solutions of (1.2) (see [6, p. 4], [14, pp. 5–6], [19, p. 30]). Modifying the definition of Carathéodory solution for (1.1) yields: A real-valued function $x(t)$ defined on I is called a "*Carathéodory solution*" of (1.1) on I if $x(t)$ is differentiable on I, and its derivative is absolutely continuous on each closed interval of I and satisfies (1.1) almost everywhere on I. Note that $x(t)$ is also absolutely continuous on each closed interval of $\mathbb{R}$ from the absolute continuity of $x'(t)$. Moreover, it is known that absolute continuity implies almost everywhere differentiability. In many engineering and physical applications, we can find the case where the right hand side in the equation is discontinuous. In particular, Carathéodory differential equations are studied as important equations in the field of control theory (see [6]). In this field, the function $f(t)$ in (1.1) is called an "*input function*" or a "*control function*" (see [36]). For example, we consider the differential equation

$$x'' - x = \delta(t),$$

where $\delta(t)$ is the step function (on-off function) defined by

$$\delta(t) = \begin{cases} 1 & \text{if } t \geqslant 0, \\ 0 & \text{if } t < 0. \end{cases}$$

Then we can easily check that the function $x(t) = \delta(t)(\cosh t - 1)$ is a Carathéodory solution of this equation. Note here that the second derivative $x''(t) = \delta(t)\cosh t$ of the solution does not exist at $t = 0$.

Now we will give the definition of a stability for (1.1). We call that (1.1) has "*Hyers–Ulam stability*" on I if and only if there exists a constant $K > 0$ with the following property: Let $\varepsilon > 0$ be a given arbitrary constant, and let $\xi : I \to \mathbb{R}$ be differentiable on I, and its derivative be absolutely continuous on each closed interval of I. If $|\xi''(t) + \alpha\xi'(t) + \beta\xi(t) - f(t)| \leqslant \varepsilon$ holds for almost all $t \in I$, then there exists a Carathéodory solution $x : \mathbb{R} \to \mathbb{R}$ of (1.1) such that $|\xi(t) - x(t)| \leqslant K\varepsilon$ for all $t \in I$. We call such K a "*HUS constant*" for (1.1) on I.

Hyers–Ulam stability was initiated by Ulam's proposal [38, 39] for certain stability problem for functional equations. In 1941, Hyers [20, 21] gave a partial answer to it, and many researchers have been working on this problem until recently. See [3, 9, 10, 23] for the Hyers–Ulam stability of functional equations. In recent years, research on Hyers–Ulam stability for differential equations has started. The study of the Hyers–Ulam stability of differential equations is developing rapidly and has received much attention. For example, see [5, 8, 11, 28, 29, 30, 31, 32, 37, 40]. Recently, many researchers have been study Hyers–Ulam stability of various kinds of second-order linear differential equations under the assumption that $f(t)$ is continuous or it is equal to 0 (see [1, 4, 7, 12, 13, 15, 16, 17, 22, 24, 25, 26, 27, 33, 34, 41, 42]). In 2010, Li and Shen [27, Theorem 2.2] established the following theorem.

THEOREM A. *Let I be a finite nonempty closed interval of $\mathbb{R}$, and let $\varepsilon > 0$ be a given arbitrary constant. Suppose that $f(t)$ is continuous on I and that the characteristic equation $\mu^2 + \alpha\mu + \beta = 0$ for $x'' + \alpha x' + \beta x = 0$ has two different positive roots. Then there exists a constant $K > 0$ with the following property: If a twice continuously differentiable function $\xi : I \to \mathbb{R}$ satisfies $|\xi''(t) + \alpha\xi'(t) + \beta\xi(t) - f(t)| \leqslant \varepsilon$ for all $t \in I$, then there exists a solution $x : I \to \mathbb{R}$ of (1.1) such that $|\xi(t) - x(t)| \leqslant K\varepsilon$ for all $t \in I$.*

Note here that they assume that I is finite interval, the characteristic equation has two different positive roots, and ξ is twice continuously differentiable. In this paper, we will show that these assumptions can be relaxed to weaker conditions. In particular, the purpose of this paper is to find an explicit constant K on I, and the explicit solution x of (1.1) such that $|\xi(t) - x(t)| \leqslant K\varepsilon$ for all $t \in \mathbb{R}$.

In the next section, we summarize the previous study on the Carathéodory equation. In Section 3, we consider Hyers–Ulam stability of (1.1). In Section 4, we will show that the obtained HUS constant is the best one. In Section 5, we discuss the instability of (1.1), and so that we obtain a necessary and sufficient condition. Finally, for illustration of the obtained results, we will take an example with a numerical simulation.

2. Previous study on the Carathéodory equation

As a previous study on the Carathéodory equation, we can refer to the results of the author [29] for the first-order linear differential equation

$$x' = ax + f(t) \tag{2.1}$$

on $\mathbb{R}$, where a is a non-zero real number, and $f(t)$ is a summable real-valued function on each segment contained in $\mathbb{R}$. We say that (2.1) has "*Hyers–Ulam stability*" on I if and only if there exists a constant $K > 0$ with the following property: Let $\varepsilon > 0$ be a given arbitrary constant, and let $\xi : I \to \mathbb{R}$ be an absolutely continuous function on each closed interval of I. If $|\xi'(t) - a\xi(t) - f(t)| \leqslant \varepsilon$ holds for almost all $t \in I$, then there exists a Carathéodory solution $x : \mathbb{R} \to \mathbb{R}$ of (2.1) such that $|\xi(t) - x(t)| \leqslant K\varepsilon$ for all $t \in I$. In 2019, the author presented the following result.

THEOREM B. *Let $I = (\sigma, \tau)$ with $-\infty \leqslant \sigma < \tau \leqslant \infty$, and let $\varepsilon > 0$ be a given arbitrary constant. Suppose that $a \neq 0$ and that $\xi : I \to \mathbb{R}$ is an absolutely continuous function on each closed interval of I and it satisfies $|\xi'(t) - a\xi(t) - f(t)| \leqslant \varepsilon$ for almost all $t \in I$. Then one of the following holds:*

(i) if $a > 0$ and $\tau < \infty$, then $\lim_{t\to\tau-0}\xi(t)$ exists, and any Carathéodory solution $x:\mathbb{R}\to\mathbb{R}$ of (2.1) with $|\lim_{t\to\tau-0}\xi(t)-x(\tau)| < \varepsilon/a$ satisfies that $|\xi(t)-x(t)| < \varepsilon/a$ for all $t\in I$;

(ii) if $a > 0$ and $\tau = \infty$, then $\lim_{t\to\infty}(\xi(t)e^{-at} - \int f(t)e^{-at}dt)$ exists, and there exists the unique Carathéodory solution

$$x(t) = \left\{\int f(t)e^{-at}dt + \lim_{t\to\infty}\left(\xi(t)e^{-at} - \int f(t)e^{-at}dt\right)\right\}e^{at}$$

of (2.1) such that $|\xi(t)-x(t)| \leqslant \varepsilon/a$ for all $t\in I$;

(iii) if $a < 0$ and $\sigma > -\infty$, then $\lim_{t\to\sigma+0}\xi(t)$ exists, and any Carathéodory solution $x:\mathbb{R}\to\mathbb{R}$ of (2.1) with $|\lim_{t\to\sigma+0}\xi(t)-x(\sigma)| < \varepsilon/|a|$ satisfies that $|\xi(t)-x(t)| < \varepsilon/|a|$ for all $t\in I$;

(iv) if $a < 0$ and $\sigma = -\infty$, then $\lim_{t\to-\infty}(\xi(t)e^{-at} - \int f(t)e^{-at}dt)$ exists, and there exists the unique Carathéodory solution

$$x(t) = \left\{\int f(t)e^{-at}dt + \lim_{t\to-\infty}\left(\xi(t)e^{-at} - \int f(t)e^{-at}dt\right)\right\}e^{at}$$

of (2.1) such that $|\xi(t)-x(t)| \leqslant \varepsilon/|a|$ for all $t\in I$.

By using Theorem B, we can establish the following result, immediately.

COROLLARY C. *Let I be a nonempty open interval of $\mathbb{R}$. If $a \neq 0$ then (2.1) has Hyers–Ulam stability with an HUS constant $1/|a|$ on I.*

3. Hyers–Ulam stability

In this section, we consider Hyers–Ulam stability of second-order linear differential equation (1.1). First, we present a simple result as follows.

THEOREM 3.1. *Let I be a nonempty open interval of $\mathbb{R}$. Suppose that the characteristic equation $\mu^2+\alpha\mu+\beta = 0$ for $x''+\alpha x'+\beta x = 0$ has the non-zero real roots μ_1 and μ_2. Then (1.1) has Hyers–Ulam stability with an HUS constant $1/|\mu_1\mu_2|$ on I.*

Proof. Suppose that $\xi(t)$ is differentiable on I and $\xi'(t)$ is absolutely continuous on each closed interval of I, and they satisfying $|\xi''(t)+\alpha\xi'(t)+\beta\xi(t)-f(t)| \leqslant \varepsilon$ for almost all $t\in I$. Define $\eta_i(t) = \xi'(t)-\mu_i\xi(t)$ for $t\in I$ and $i\in\{1,2\}$, where μ_1 and μ_2 are non-zero real roots of characteristic equation $\mu^2+\alpha\mu+\beta = 0$. Since $\mu_i+\mu_{3-i} = \mu_1+\mu_2 = -\alpha$ and $\mu_i\mu_{3-i} = \mu_1\mu_2 = \beta$, we have

$$\begin{aligned}|\eta_i'(t)-\mu_{3-i}\eta_i(t)-f(t)| &= |\xi''(t)-(\mu_i+\mu_{3-i})\xi'(t)+\mu_i\mu_{3-i}\xi(t)-f(t)| \\ &= |\xi''(t)+\alpha\xi'(t)+\beta\xi(t)-f(t)| \leqslant \varepsilon \qquad (3.1)\end{aligned}$$

for almost all $t \in I$. Using (3.1) and Corollary C, we see that there exists a Carathéodory solution $y : \mathbb{R} \to \mathbb{R}$ of

$$y' = \mu_{3-i} y + f(t) \tag{3.2}$$

such that

$$|\xi'(t) - \mu_i \xi(t) - y(t)| = |\eta_i(t) - y(t)| \leqslant \frac{\varepsilon}{|\mu_{3-i}|} \tag{3.3}$$

for all $t \in I$. Note that $y(t)$ is absolutely continuous on each closed interval of $\mathbb{R}$ because it is a Carathéodory solution, so that $y(t)$ is differentiable almost everywhere on $\mathbb{R}$. From (3.3) and Corollary C, we see that there exists a Carathéodory solution $z : \mathbb{R} \to \mathbb{R}$ of

$$z' = \mu_i z + y(t) \tag{3.4}$$

such that $|\xi(t) - z(t)| < \varepsilon / |\mu_1 \mu_2|$ for all $t \in I$. Note that the function $z(t)$ is the classical solution of (3.4) because $y(t)$ is continuous on $\mathbb{R}$. Therefore, $z(t)$ is continuously differentiable on $\mathbb{R}$. From (3.4) and the almost everywhere differentiability of $y(t)$, $z'(t)$ is also differentiable almost everywhere on $\mathbb{R}$. It follows from (3.2) and (3.4) that

$$\begin{aligned} z''(t) + \alpha z'(t) + \beta z(t) &= z''(t) - (\mu_i + \mu_{3-i}) z'(t) + \mu_i \mu_{3-i} z(t) \\ &= (z'(t) - \mu_i z(t))' - \mu_{3-i}(z'(t) - \mu_i z(t)) \\ &= y'(t) - \mu_{3-i} y(t) = f(t) \end{aligned} \tag{3.5}$$

for almost all $t \in I$, and therefore, $z(t)$ is a Carathéodory solution of (1.1). This completes the proof of Theorem 3.1. $\square$

Theorem 3.1 gives an explicit HUS constant. A natural question now arises. Can we find the explicit Carathéodory solution $x(t)$ of (1.1) satisfying $|\xi(t) - x(t)| \leqslant \varepsilon / |\mu_1 \mu_2|$ for all $\mathbb{R}$? The answer to this question is as follows.

THEOREM 3.2. *Let $I = (\sigma, \tau)$ with $-\infty \leqslant \sigma < \tau \leqslant \infty$, and let $\varepsilon > 0$ be a given arbitrary constant. Suppose that the characteristic equation $\mu^2 + \alpha\mu + \beta = 0$ for $x'' + \alpha x' + \beta x = 0$ has the non-zero real roots μ_1 and μ_2. Suppose also that $\xi : I \to \mathbb{R}$ is differentiable on I, and its derivative is absolutely continuous on each closed interval of I, and $|\xi''(t) + \alpha\xi'(t) + \beta\xi(t) - f(t)| \leqslant \varepsilon$ holds for almost all $t \in I$. Then the following holds:*

(i) if $\mu_1 > 0$, $\mu_2 > 0$ and $\tau = \infty$, then

$$\lim_{t \to \infty} \left\{ (\xi'(t) - \mu_1 \xi(t)) e^{-\mu_2 t} - F(t) \right\}$$

and

$$\lim_{t \to \infty} \left\{ \xi(t) e^{-\mu_1 t} - \int (F(t) + c^+) e^{(\mu_2 - \mu_1)t} dt \right\}$$

exist, and there exists the unique Carathéodory solution

$$\begin{aligned} x(t) = \Big[&\int (F(t) + c^+) e^{(\mu_2 - \mu_1)t} dt \\ &+ \lim_{t \to \infty} \left\{ \xi(t) e^{-\mu_1 t} - \int (F(t) + c^+) e^{(\mu_2 - \mu_1)t} dt \right\} \Big] e^{\mu_1 t} \end{aligned}$$

of (1.1) *such that* $|\xi(t)-x(t)| \leqslant \varepsilon/(\mu_1\mu_2)$ *for all* $t \in I$, *where* $F(t) = \int f(t)e^{-\mu_2 t}dt$ *and* $c^+ = \lim_{t\to\infty}\{(\xi'(t)-\mu_1\xi(t))e^{-\mu_2 t} - F(t)\}$;

(ii) if $\mu_1 < 0,\ \mu_2 < 0$ *and* $\sigma = -\infty$, *then*

$$\lim_{t\to-\infty}\{(\xi'(t)-\mu_1\xi(t))e^{-\mu_2 t} - F(t)\}$$

and

$$\lim_{t\to-\infty}\left\{\xi(t)e^{-\mu_1 t} - \int (F(t)+c^-)e^{(\mu_2-\mu_1)t}dt\right\}$$

exist, and there exists the unique Carathéodory solution

$$\begin{aligned} x(t) = \Big[&\int (F(t)+c^-)e^{(\mu_2-\mu_1)t}dt \\ &+ \lim_{t\to-\infty}\left\{\xi(t)e^{-\mu_1 t} - \int (F(t)+c^-)e^{(\mu_2-\mu_1)t}dt\right\}\Big]e^{\mu_1 t} \end{aligned}$$

of (1.1) *such that* $|\xi(t)-x(t)| \leqslant \varepsilon/(\mu_1\mu_2)$ *for all* $t \in I$, *where* $F(t) = \int f(t)e^{-\mu_2 t}dt$ *and* $c^- = \lim_{t\to-\infty}\{(\xi'(t)-\mu_1\xi(t))e^{-\mu_2 t} - F(t)\}$;

(iii) if $\mu_1 < 0 < \mu_2,\ \sigma = -\infty$ *and* $\tau = \infty$, *then*

$$\lim_{t\to\infty}\{(\xi'(t)-\mu_1\xi(t))e^{-\mu_2 t} - F(t)\}$$

and

$$\lim_{t\to-\infty}\left\{\xi(t)e^{-\mu_1 t} - \int (F(t)+c^+)e^{(\mu_2-\mu_1)t}dt\right\}$$

exist, and there exists the unique Carathéodory solution

$$\begin{aligned} x(t) = \Big[&\int (F(t)+c^+)e^{(\mu_2-\mu_1)t}dt \\ &+ \lim_{t\to-\infty}\left\{\xi(t)e^{-\mu_1 t} - \int (F(t)+c^+)e^{(\mu_2-\mu_1)t}dt\right\}\Big]e^{\mu_1 t} \end{aligned}$$

of (1.1) *such that* $|\xi(t)-x(t)| \leqslant \varepsilon/|\mu_1\mu_2|$ *for all* $t \in I$, *where* $F(t) = \int f(t)e^{-\mu_2 t}dt$ *and* $c^+ = \lim_{t\to\infty}\{(\xi'(t)-\mu_1\xi(t))e^{-\mu_2 t} - F(t)\}$.

Proof. Let arbitrary $\varepsilon > 0$ be given. Suppose that $|\xi''(t)+\alpha\xi'(t)+\beta\xi(t) - f(t)| \leqslant \varepsilon$ for almost all $t \in I$, where $\xi(t)$ is a differentiable function on I, and $\xi'(t)$ is an absolutely continuous function on each closed interval of I. Define $\eta_i(t) = \xi'(t) - \mu_i\xi(t)$ for $t \in I$ and $i \in \{1,2\}$. Then we have inequality (3.1) for almost all $t \in I$. For the sake of simplicity, we write

$$F_i(t) = \int f(t)e^{-\mu_i t}dt$$

for $i \in \{1,2\}$.

First we prove case (i). Let $\mu_1 > 0$, $\mu_2 > 0$ and $\tau = \infty$. From (3.1) and Theorem B (ii),

$$\lim_{t\to\infty}\left\{(\xi'(t)-\mu_i\xi(t))e^{-\mu_{3-i}t}-F_{3-i}(t)\right\}=\lim_{t\to\infty}\left(\eta_i(t)e^{-\mu_{3-i}t}-F_{3-i}(t)\right)$$

exists, and there exists the unique Carathéodory solution $y(t)=(F_{3-i}(t)+c_i^+)e^{\mu_{3-i}t}$ of (3.2) satisfying (3.3) for all $t\in I$, where

$$c_i^+=\lim_{t\to\infty}\left\{(\xi'(t)-\mu_i\xi(t))e^{-\mu_{3-i}t}-F_{3-i}(t)\right\}. \tag{3.6}$$

From (3.3) and Theorem B (ii) again, we see that

$$\lim_{t\to\infty}\left\{\xi(t)e^{-\mu_i t}-\int(F_{3-i}(t)+c_i^+)e^{(\mu_{3-i}-\mu_i)t}dt\right\}$$

exists, and there exists the unique classical solution

$$\begin{aligned}x(t)=\Big[&\int(F_{3-i}(t)+c_i^+)e^{(\mu_{3-i}-\mu_i)t}dt\\&+\lim_{t\to\infty}\left\{\xi(t)e^{-\mu_i t}-\int(F_{3-i}(t)+c_i^+)e^{(\mu_{3-i}-\mu_i)t}dt\right\}\Big]e^{\mu_i t}\end{aligned}$$

of (3.4) such that $|\xi(t)-x(t)|\leqslant\varepsilon/(\mu_1\mu_2)$ for all $t\in I$. By the same calculation as (3.5), we conclude that $x(t)$ is a Carathéodory solution of (1.1) on $\mathbb{R}$.

Now we will show that $x(t)$ is the unique Carathéodory solution of (1.1) such that $|\xi(t)-x(t)|\leqslant\varepsilon/(\mu_1\mu_2)$ for all $t\in I$. Since $x(t)$ is a Carathéodory solution of second-order linear differential equation (1.1), it can be rewritten in the form:

$$x(t)=e^{\mu_i t}\int F_{3-i}(t)e^{(\mu_{3-i}-\mu_i)t}dt+d_1e^{\mu_i t}+d_2\begin{cases}e^{\mu_{3-i}t} & \text{if } \mu_i\neq\mu_{3-i},\\ te^{\mu_i t} & \text{if } \mu_i=\mu_{3-i}\end{cases}$$

for some d_1, $d_2\in\mathbb{R}$. Suppose to the contrary that there exists a Carathéodory solution $\tilde{x}(t)$ of (1.1) such that $\tilde{x}(t)\not\equiv x(t)$ and $|\xi(t)-\tilde{x}(t)|\leqslant\varepsilon/(\mu_1\mu_2)$ for all $t\in I$. Due to the uniqueness of the solution with respect to the initial value of the differential equation, it can be written as

$$\tilde{x}(t)=e^{\mu_i t}\int F_{3-i}(t)e^{(\mu_{3-i}-\mu_i)t}dt+\tilde{d}_1e^{\mu_i t}+\tilde{d}_2\begin{cases}e^{\mu_{3-i}t} & \text{if } \mu_i\neq\mu_{3-i},\\ te^{\mu_i t} & \text{if } \mu_i=\mu_{3-i}\end{cases}$$

for some $\tilde{d}_1$, $\tilde{d}_2\in\mathbb{R}$ with $\left(\tilde{d}_1,\tilde{d}_2\right)\neq(d_1,d_2)$. Then

$$x(t)-\tilde{x}(t)=\left(d_1-\tilde{d}_1\right)e^{\mu_i t}+\left(d_2-\tilde{d}_2\right)\begin{cases}e^{\mu_{3-i}t} & \text{if } \mu_i\neq\mu_{3-i},\\ te^{\mu_i t} & \text{if } \mu_i=\mu_{3-i}\end{cases}$$

for all $t\in I$, and so that

$$\begin{aligned}\frac{2\varepsilon}{\mu_1\mu_2}&\geqslant|x(t)-\xi(t)|+|\xi(t)-\tilde{x}(t)|\\&\geqslant\left|\left(d_1-\tilde{d}_1\right)e^{\mu_i t}+\left(d_2-\tilde{d}_2\right)\begin{cases}e^{\mu_{3-i}t} & \text{if } \mu_i\neq\mu_{3-i},\\ te^{\mu_i t} & \text{if } \mu_i=\mu_{3-i}\end{cases}\right|\end{aligned}$$

for all $t \in I$. We now consider the case $d_1 = \tilde{d}_1$. Then $d_2 \neq \tilde{d}_2$ and

$$\lim_{t\to\infty}\left|\left(d_2-\tilde{d}_2\right)\begin{cases} e^{\mu_{3-i}t} & \text{if } \mu_i \neq \mu_{3-i}, \\ te^{\mu_i t} & \text{if } \mu_i = \mu_{3-i}\end{cases}\right| = \infty$$

from $\mu_1 > 0$, $\mu_2 > 0$ and $\tau = \infty$. Hence, this is a contradiction to the above inequality. Next consider the case $d_1 \neq \tilde{d}_1$. Then we see that there exist $T \in I$ and $k > 0$ such that

$$\left|d_1-\tilde{d}_1+\left(d_2-\tilde{d}_2\right)\begin{cases} e^{(\mu_{3-i}-\mu_i)t} & \text{if } \mu_i \neq \mu_{3-i}, \\ t & \text{if } \mu_i = \mu_{3-i}\end{cases}\right| \geqslant k$$

for all $t \geqslant T$. This implies

$$\lim_{t\to\infty}\left|\left(d_1-\tilde{d}_1\right)e^{\mu_i t}+\left(d_2-\tilde{d}_2\right)\begin{cases} e^{\mu_{3-i}t} & \text{if } \mu_i \neq \mu_{3-i}, \\ te^{\mu_i t} & \text{if } \mu_i = \mu_{3-i}\end{cases}\right| \geqslant \lim_{t\to\infty} ke^{\mu_i t} = \infty.$$

This is a contradiction. Therefore, the uniqueness of $x(t)$ is shown.

We next consider case (ii). Let $\mu_1 < 0$, $\mu_2 < 0$ and $\sigma = -\infty$. From (3.1) and Theorem B (iv),

$$\lim_{t\to-\infty}\left\{(\xi'(t)-\mu_i\xi(t))e^{-\mu_{3-i}t}-F_{3-i}(t)\right\} = \lim_{t\to-\infty}\left(\eta_i(t)e^{-\mu_{3-i}t}-F_{3-i}(t)\right)$$

exists, and there exists the unique Carathéodory solution $y(t) = (F_{3-i}(t)+c_i^-)e^{\mu_{3-i}t}$ of (3.2) satisfying (3.3) for all $t \in I$, where

$$c_i^- = \lim_{t\to-\infty}\left\{(\xi'(t)-\mu_i\xi(t))e^{-\mu_{3-i}t}-F_{3-i}(t)\right\}.$$

From (3.3), (3.5) and Theorem B (iv) again, we see that

$$\lim_{t\to-\infty}\left\{\xi(t)e^{-\mu_i t}-\int(F_{3-i}(t)+c_i^-)e^{(\mu_{3-i}-\mu_i)t}dt\right\}$$

exists, and there exists the unique Carathéodory solution

$$\begin{aligned} x(t) = \Bigg[&\int(F_{3-i}(t)+c_i^-)e^{(\mu_{3-i}-\mu_i)t}dt \\ &+\lim_{t\to-\infty}\left\{\xi(t)e^{-\mu_i t}-\int(F_{3-i}(t)+c_i^-)e^{(\mu_{3-i}-\mu_i)t}dt\right\}\Bigg]e^{\mu_i t}\end{aligned}$$

of (1.1) such that $|\xi(t)-x(t)| \leqslant \varepsilon/|\mu_1\mu_2|$ for all $t \in I$. The uniqueness of $x(t)$ can be shown by the same argument as in case (i).

Finally, we consider case (iii). Let $\mu_1 < 0 < \mu_2$, $\sigma = -\infty$ and $\tau = \infty$, and let $i = 1$. From (3.1) and Theorem B (ii), $\lim_{t\to\infty}(\eta_1(t)e^{-\mu_2 t}-F_2(t))$ exists, and there exists the unique Carathéodory solution $y(t) = (F_2(t)+c_1^+)e^{\mu_2 t}$ of (3.2) satisfying (3.3) for all $t \in I$, where c_i^+ given by (3.6). From (3.3), (3.5) and Theorem B (iv), we see that

$$\lim_{t\to-\infty}\left\{\xi(t)e^{-\mu_1 t}-\int(F_2(t)+c_1^+)e^{(\mu_2-\mu_1)t}dt\right\}$$

exists, and there exists the unique Carathéodory solution

$$x(t) = \Big[\int (F_2(t) + c_1^+) e^{(\mu_2-\mu_1)t} dt + \lim_{t\to-\infty} \Big\{ \xi(t) e^{-\mu_1 t} - \int (F_2(t) + c_1^+) e^{(\mu_2-\mu_1)t} dt \Big\} \Big] e^{\mu_1 t}$$

of (1.1) such that $|\xi(t) - x(t)| \leqslant \varepsilon/|\mu_1\mu_2|$ for all $t \in I$. The uniqueness of $x(t)$ can be shown by the same argument as in case (i). However, it is necessary to pay attention to whether contradiction occurs at $t \to \infty$ or $t \to -\infty$. This completes the proof of Theorem 3.2. □

Under the assumption that $I = \mathbb{R}$, we obtain the following corollary from Theorem 3.2.

COROLLARY 3.3. *Suppose that the characteristic equation $\mu^2 + \alpha\mu + \beta = 0$ for $x'' + \alpha x' + \beta x = 0$ has the non-zero real roots μ_1 and μ_2. Then (1.1) has Hyers–Ulam stability with an HUS constant $1/|\mu_1\mu_2|$ on $\mathbb{R}$. Furthermore, the Carathéodory solution $x(t)$ of (1.1) satisfying $|\xi(t) - x(t)| \leqslant \varepsilon/|\mu_1\mu_2|$ for all $t \in \mathbb{R}$ is the unique, which written as*

$$x(t) = \Big[\int (F(t) + c) e^{(\mu_2-\mu_1)t} dt + \lim_{\mu_1 t\to\infty} \Big\{ \xi(t) e^{-\mu_1 t} - \int (F(t) + c) e^{(\mu_2-\mu_1)t} dt \Big\} \Big] e^{\mu_1 t},$$

where $F(t) = \int f(t) e^{-\mu_2 t} dt$, and $c = \lim_{\mu_2 t\to\infty} \{(\xi'(t) - \mu_1 \xi(t)) e^{-\mu_2 t} - F(t)\}$ and

$$\lim_{\mu_1 t\to\infty} \Big\{ \xi(t) e^{-\mu_1 t} - \int (F(t) + c) e^{(\mu_2-\mu_1)t} dt \Big\}$$

are finite constants.

Corollary 3.3 implies the following results.

COROLLARY 3.4. *Suppose that the characteristic equation $\mu^2 + \alpha\mu + \beta = 0$ for $x'' + \alpha x' + \beta x = 0$ has two different real roots μ_1 and μ_2 with $\mu_1\mu_2 \neq 0$. Then (1.1) has Hyers–Ulam stability with an HUS constant $1/|\mu_1\mu_2|$ on $\mathbb{R}$. Furthermore, the Carathéodory solution $x(t)$ of (1.1) satisfying $|\xi(t) - x(t)| \leqslant \varepsilon/|\mu_1\mu_2|$ for all $t \in \mathbb{R}$ is the unique, which written as*

$$x(t) = \Big\{ \int F(t) e^{(\mu_2-\mu_1)t} dt + \lim_{\mu_1 t\to\infty} \Big(\xi(t) e^{-\mu_1 t} - \int F(t) e^{(\mu_2-\mu_1)t} dt - \frac{c e^{(\mu_2-\mu_1)t}}{\mu_2 - \mu_1} \Big) \Big\} e^{\mu_1 t} + \frac{c}{\mu_2 - \mu_1} e^{\mu_2 t},$$

where $F(t) = \int f(t) e^{-\mu_2 t} dt$, and $c = \lim_{\mu_2 t\to\infty} \{(\xi'(t) - \mu_1 \xi(t)) e^{-\mu_2 t} - F(t)\}$ and

$$\lim_{\mu_1 t\to\infty} \Big(\xi(t) e^{-\mu_1 t} - \int F(t) e^{(\mu_2-\mu_1)t} dt - \frac{c e^{(\mu_2-\mu_1)t}}{\mu_2 - \mu_1} \Big)$$

are finite constants.

COROLLARY 3.5. *Suppose that the characteristic equation $\mu^2+\alpha\mu+\beta=0$ for $x''+\alpha x'+\beta x=0$ has exactly one real root μ with $\mu\neq 0$. Then (1.1) has Hyers–Ulam stability with an HUS constant $1/\mu^2$ on $\mathbb{R}$. Furthermore, the Carathéodory solution $x(t)$ of (1.1) satisfying $|\xi(t)-x(t)|\leqslant \varepsilon/\mu^2$ for all $t\in\mathbb{R}$ is the unique, which written as*

$$x(t)=\left\{\int F(t)dt+\lim_{\mu t\to\infty}\left(\xi(t)e^{-\mu t}-\int F(t)dt-ct\right)\right\}e^{\mu t}+cte^{\mu t},$$

where $F(t)=\int f(t)e^{-\mu t}dt$, and $c=\lim_{\mu t\to\infty}\{(\xi'(t)-\mu\xi(t))e^{-\mu t}-F(t)\}$ and

$$\lim_{\mu t\to\infty}\left(\xi(t)e^{-\mu t}-\int F(t)dt-ct\right)$$

are finite constants.

THEOREM 3.6. *Let $I=(\sigma,\tau)$ with $-\infty\leqslant\sigma<\tau\leqslant\infty$, and let $\varepsilon>0$ be a given arbitrary constant. Suppose that the characteristic equation $\mu^2+\alpha\mu+\beta=0$ for $x''+\alpha x'+\beta x=0$ has the non-zero real roots μ_1 and μ_2. Suppose also that $\xi: I\to\mathbb{R}$ is differentiable on I, and its derivative is absolutely continuous on each closed interval of I, and $|\xi''(t)+\alpha\xi'(t)+\beta\xi(t)-f(t)|\leqslant\varepsilon$ holds for almost all $t\in I$. Then the following holds:*

(i) if $\mu_1>0$, $\mu_2>0$ and $\tau<\infty$, then $\lim_{t\to\tau-0}\xi(t)$ and $\lim_{t\to\tau-0}\xi'(t)$ exist, and any Carathéodory solution $x:\mathbb{R}\to\mathbb{R}$ of (1.1) with

$$\begin{cases}\left|\lim\limits_{t\to\tau-0}(\xi'(t)-\mu_1\xi(t))-(x'(\tau)-\mu_1x(\tau))\right|<\dfrac{\varepsilon}{\mu_2},\\[2mm] \left|\lim\limits_{t\to\tau-0}\xi(t)-x(\tau)\right|<\dfrac{\varepsilon}{\mu_1\mu_2}\end{cases}\tag{3.7}$$

satisfies that $|\xi(t)-x(t)|<\varepsilon/(\mu_1\mu_2)$ for all $t\in I$;

(ii) if $\mu_1<0$, $\mu_2<0$ and $\sigma>-\infty$, then $\lim_{t\to\sigma+0}\xi(t)$ and $\lim_{t\to\sigma+0}\xi'(t)$ exist, and any Carathéodory solution $x:\mathbb{R}\to\mathbb{R}$ of (1.1) with

$$\begin{cases}\left|\lim\limits_{t\to\sigma+0}(\xi'(t)-\mu_1\xi(t))-(x'(\sigma)-\mu_1x(\sigma))\right|<\dfrac{\varepsilon}{|\mu_2|},\\[2mm] \left|\lim\limits_{t\to\sigma+0}\xi(t)-x(\sigma)\right|<\dfrac{\varepsilon}{\mu_1\mu_2}\end{cases}\tag{3.8}$$

satisfies that $|\xi(t)-x(t)|<\varepsilon/(\mu_1\mu_2)$ for all $t\in I$;

(iii) if $\mu_1<0<\mu_2$ and $-\infty<\sigma<\tau<\infty$, then $\lim_{t\to\sigma+0}\xi(t)$, $\lim_{t\to\sigma+0}\xi'(t)$, $\lim_{t\to\tau-0}\xi(t)$ and $\lim_{t\to\tau-0}\xi'(t)$ exist, and any Carathéodory solution $x:\mathbb{R}\to\mathbb{R}$ of (1.1) with

$$\begin{cases}\left|\lim\limits_{t\to\tau-0}(\xi'(t)-\mu_1\xi(t))-(x'(\tau)-\mu_1x(\tau))\right|<\dfrac{\varepsilon}{\mu_2},\\[2mm] \left|\lim\limits_{t\to\sigma+0}\xi(t)-x(\sigma)\right|<\dfrac{\varepsilon}{|\mu_1\mu_2|}\end{cases}\tag{3.9}$$

$$\left(\textit{resp.,} \quad \begin{cases} \left| \lim_{t\to\sigma+0} (\xi'(t) - \mu_2\xi(t)) - (x'(\sigma) - \mu_2 x(\sigma)) \right| < \dfrac{\varepsilon}{|\mu_1|}, \\ \left| \lim_{t\to\tau-0} \xi(t) - x(\tau) \right| < \dfrac{\varepsilon}{|\mu_1\mu_2|} \end{cases} \right)$$

satisfies that $|\xi(t) - x(t)| < \varepsilon/|\mu_1\mu_2|$ *for all* $t \in I$;

(iv) if $\mu_1 < 0 < \mu_2$, $\sigma = -\infty$ *and* $\tau < \infty$, *then* $\lim_{t\to\tau-0}\xi(t)$ *and*

$$\lim_{t\to-\infty} \left\{ (\xi'(t) - \mu_2\xi(t))e^{-\mu_1 t} - F_1(t) \right\}$$

exist, and any Carathéodory solution $x : \mathbb{R} \to \mathbb{R}$ *of* (1.1) *with*

$$x'(\tau) - \mu_2 x(\tau) = (F_1(\tau) + c_1)e^{\mu_1\tau} \quad \text{and} \quad \left| \lim_{t\to\tau-0} \xi(t) - x(\tau) \right| < \frac{\varepsilon}{|\mu_1\mu_2|} \tag{3.10}$$

satisfies that $|\xi(t) - x(t)| < \varepsilon/|\mu_1\mu_2|$ *for all* $t \in I$, *where* $F_1(t) = \int f(t)e^{-\mu_1 t}dt$ *and* $c_1 = \lim_{t\to-\infty}\{(\xi'(t) - \mu_2\xi(t))e^{-\mu_1 t} - F_1(t)\}$;

(v) if $\mu_1 < 0 < \mu_2$, $\sigma > -\infty$ *and* $\tau = \infty$, *then* $\lim_{t\to\sigma+0}\xi(t)$ *and*

$$\lim_{t\to\infty} \left\{ (\xi'(t) - \mu_1\xi(t))e^{-\mu_2 t} - F_2(t) \right\}$$

exist, and any Carathéodory solution $x : \mathbb{R} \to \mathbb{R}$ *of* (1.1) *with*

$$x'(\sigma) - \mu_1 x(\sigma) = (F_2(\sigma) + c_2)e^{\mu_2\sigma} \quad \text{and} \quad \left| \lim_{t\to\sigma+0} \xi(t) - x(\sigma) \right| < \frac{\varepsilon}{|\mu_1\mu_2|}$$

satisfies that $|\xi(t) - x(t)| < \varepsilon/|\mu_1\mu_2|$ *for all* $t \in I$, *where* $F_2(t) = \int f(t)e^{-\mu_2 t}dt$ *and* $c_2 = \lim_{t\to\infty}\{(\xi'(t) - \mu_1\xi(t))e^{-\mu_2 t} - F_2(t)\}$.

Proof. Define $\eta_i(t) = \xi'(t) - \mu_i\xi(t)$ for $t \in I$ and $i \in \{1,2\}$. Then inequality (3.1) holds for almost all $t \in I$.

First, we prove case (i). Let $\mu_1 > 0$, $\mu_2 > 0$ and $\tau < \infty$. From (3.1) and (i) in Theorem B, we see that

$$\lim_{t\to\tau-0} \big(\xi'(t) - \mu_i\xi(t)\big) = \lim_{t\to\tau-0} \eta_i(t)$$

exists, and any Carathéodory solution $y : \mathbb{R} \to \mathbb{R}$ of (3.2) with $|\lim_{t\to\tau-0}\eta_i(t) - y(\tau)| < \varepsilon/\mu_{3-i}$ satisfies that inequality (3.3) for all $t \in I$. Moreover, from (3.3) and (i) in Theorem B, $\lim_{t\to\tau-0}\xi(t)$ exists, and any Carathéodory solution $z : \mathbb{R} \to \mathbb{R}$ of (3.4) with $|\lim_{t\to\tau-0}\xi(t) - z(\tau)| < \varepsilon/(\mu_1\mu_2)$ satisfies that $|\xi(t) - z(t)| < \varepsilon/(\mu_1\mu_2)$ for all $t \in I$. Note here that the function $y(t)$ in (3.4) requires both that it is a Carathéodory solution of (3.2) on $\mathbb{R}$ and that it satisfies $|\lim_{t\to\tau-0}\eta_i(t) - y(\tau)| < \varepsilon/\mu_{3-i}$. Since $\lim_{t\to\tau-0}\xi(t)$ and $\lim_{t\to\tau-0}\eta_i(t)$ exist, $\lim_{t\to\tau-0}\xi'(t)$ also exists. For the sake of simplicity, let

$$\xi_\tau = \lim_{t\to\tau-0} \xi(t) \quad \text{and} \quad \xi'_\tau = \lim_{t\to\tau-0} \xi'(t).$$

We now consider any Carathéodory solution $x:\mathbb{R}\to\mathbb{R}$ of (1.1) with (3.7). Define $w(t)=x'(t)-\mu_i x(t)$ for all $t\in\mathbb{R}$ and $i\in\{1,2\}$. From (3.7), $w(t)$ satisfies

$$\left|\lim_{t\to\tau-0}\eta_i(t)-w(\tau)\right|=\left|\xi'_\tau-\mu_i\xi_\tau-w(\tau)\right|<\frac{\varepsilon}{\mu_{3-i}}.$$

Since $w(t)$ is absolutely continuous on each closed interval of $\mathbb{R}$ and it satisfies

$$w'(t)-\mu_{3-i}w(t)-f(t)=x''(t)+\alpha x'(t)+\beta x(t)-f(t)=0$$

for almost all $t\in\mathbb{R}$, $w(t)$ is a Carathéodory solution of (3.2) with $|\lim_{t\to\tau-0}\eta_i(t)-w(\tau)|<\varepsilon/\mu_{3-i}$ on $\mathbb{R}$. From this and (3.7), $x(t)$ is a Carathéodory solution of (3.4) with $|\xi_\tau-x(\tau)|<\varepsilon/(\mu_1\mu_2)$ and $y(t)=w(t)$. Therefore, we conclude that $|\xi(t)-x(t)|<\varepsilon/(\mu_1\mu_2)$ holds for all $t\in I$.

Next, we prove case (ii). Suppose that μ_1 and μ_2 are negative and $\sigma>-\infty$. Let $s=-t$, $\gamma(s)=\xi(-s)$, $\nu_1=-\mu_1$, $\nu_2=-\mu_2$, $g(s)=f(-s)$, and $\dot\gamma=d\gamma/ds$. Then we can transform $|\xi''(t)-(\mu_1+\mu_2)\xi'(t)+\mu_1\mu_2\xi(t)-f(t)|\leqslant\varepsilon$ for almost all $t\in I$ into the inequality

$$|\ddot\gamma(s)-(\nu_1+\nu_2)\dot\gamma(s)+\nu_1\nu_2\gamma(s)-g(s)|\leqslant\varepsilon$$

for almost all $s\in(-\tau,-\sigma)$. Clearly, ν_1 and ν_2 are positive roots of the characteristic equation $\mu^2+\alpha\mu+\beta=0$. Using (i) in Theorem 3.6, we see that the following. $\lim_{s\to-\sigma-0}\gamma(s)$ and $\lim_{s\to-\sigma-0}\dot\gamma(s)$ exist, and any Carathéodory solution $X:\mathbb{R}\to\mathbb{R}$ of $\ddot X-(\nu_1+\nu_2)\dot X+\nu_1\nu_2X=g(s)$ with initial condition

$$\left|\lim_{s\to-\sigma-0}(\dot\gamma(s)-\nu_1\gamma(s))-\big(\dot X(-\sigma)-\nu_1X(-\sigma)\big)\right|<\frac{\varepsilon}{\nu_2}$$

and

$$\left|\lim_{s\to-\sigma-0}\gamma(s)-X(-\sigma)\right|<\frac{\varepsilon}{\nu_1\nu_2}$$

satisfies that $|\gamma(s)-X(s)|<\varepsilon/(\nu_1\nu_2)$ for all $s\in(-\tau,-\sigma)$. Let $x(t)=X(-t)$. Thus, taking notice that $\xi(t)=\gamma(-t)$, $\xi'(t)=-\dot\gamma(-t)$ and

$$\lim_{t\to\sigma+0}\xi'(t)=\lim_{s\to-\sigma-0}(-\dot\gamma(s))\quad\text{and}\quad\lim_{t\to\sigma+0}\xi(t)=\lim_{s\to-\sigma-0}\gamma(s),$$

we can conclude that any Carathéodory solution of (1.1) with (3.8) satisfies that $|\xi(t)-x(t)|<\varepsilon/(\mu_1\mu_2)$ for all $t\in I$.

We will prove case (iii). Let $\mu_1<0<\mu_2$ and $-\infty<\sigma<\tau<\infty$. From (3.1) and (i) (resp., (iii)) in Theorem B, we see that

$$\lim_{t\to\tau-0}\big(\xi'(t)-\mu_1\xi(t)\big)=\lim_{t\to\tau-0}\eta_1(t)\quad\left(\text{resp., }\lim_{t\to\sigma+0}\big(\xi'(t)-\mu_2\xi(t)\big)=\lim_{t\to\sigma+0}\eta_2(t)\right)$$

exists, and any Carathéodory solution $y:\mathbb{R}\to\mathbb{R}$ of (3.2) with $|\lim_{t\to\tau-0}\eta_1(t)-y(\tau)|<\varepsilon/\mu_2$ and $i=1$ (resp., $|\lim_{t\to\sigma+0}\eta_2(t)-y(\sigma)|<\varepsilon/|\mu_1|$ and $i=2$) satisfies that inequality (3.3) with $i=1$ (resp., $i=2$) for all $t\in I$. Moreover, from (3.3) with $i=1$

(resp., $i=2$) and (iii) (resp., (i)) in Theorem B, $\lim_{t\to\sigma+0}\xi(t)$ (resp., $\lim_{t\to\tau-0}\xi(t)$) exists, and any Carathéodory solution $z:\mathbb{R}\to\mathbb{R}$ of (3.4) with $|\lim_{t\to\sigma+0}\xi(t)-z(\sigma)|<\varepsilon/|\mu_1\mu_2|$ and $i=1$ (resp., $|\lim_{t\to\tau-0}\xi(t)-z(\tau)|<\varepsilon/|\mu_1\mu_2|$ and $i=2$) satisfies that $|\xi(t)-z(t)|<\varepsilon/|\mu_1\mu_2|$ for all $t\in I$. Therefore,

$$\lim_{t\to\sigma+0}\xi(t),\quad \lim_{t\to\sigma+0}\xi'(t),\quad \lim_{t\to\tau-0}\xi(t)\quad\text{and}\quad \lim_{t\to\tau-0}\xi'(t)$$

exist. Define

$$\xi_\sigma=\lim_{t\to\sigma+0}\xi(t),\ \ \xi'_\sigma=\lim_{t\to\sigma+0}\xi'(t),\ \ \xi_\tau=\lim_{t\to\tau-0}\xi(t)\quad\text{and}\quad \xi'_\tau=\lim_{t\to\tau-0}\xi'(t).$$

We have only to prove the case $i=1$ because the proof of the case $i=2$ is the same.

We now consider any Carathéodory solution $x:\mathbb{R}\to\mathbb{R}$ of (1.1) with (3.9). Define $w(t)=x'(t)-\mu_1x(t)$ for all $t\in\mathbb{R}$. Using (3.9), we see that $w(t)$ is a Carathéodory solution of (3.2) with $i=1$ and

$$\left|\lim_{t\to\tau-0}\eta_1(t)-w(\tau)\right|=\left|\xi'_\tau-\mu_1\xi_\tau-w(\tau)\right|<\frac{\varepsilon}{\mu_2}$$

on $\mathbb{R}$. From this and (3.9), $x(t)$ is a Carathéodory solution of (3.4) with $|\xi_\sigma-x(\sigma)|<\varepsilon/|\mu_1\mu_2|$, $y(t)=w(t)$ and $i=1$. Therefore, we obtain $|\xi(t)-x(t)|<\varepsilon/|\mu_1\mu_2|$ for all $t\in I$.

Next we prove case (iv). Let $\mu_1<0<\mu_2$, $\sigma=-\infty$ and $\tau<\infty$. From (3.1) and Theorem B (iv), we see that

$$\lim_{t\to-\infty}\left(\eta_2(t)e^{-\mu_1t}-\int f(t)e^{-\mu_1t}dt\right)$$

exists, and there exists the unique Carathéodory solution

$$y(t)=\left\{\int f(t)e^{-\mu_1t}dt+\lim_{t\to-\infty}\left(\eta_2(t)e^{-\mu_1t}-\int f(t)e^{-\mu_1t}dt\right)\right\}e^{\mu_1t}$$

of (3.2) with $i=2$ satisfying (3.3) with $i=2$ for all $t\in I$. Moreover, from (3.3) with $i=2$ and (i) in Theorem B, $\lim_{t\to\tau-0}\xi(t)$ exists, and any Carathéodory solution $z:\mathbb{R}\to\mathbb{R}$ of (3.4) with $|\lim_{t\to\tau-0}\xi(t)-z(\tau)|<\varepsilon/|\mu_1\mu_2|$ satisfies that $|\xi(t)-z(t)|<\varepsilon/|\mu_1\mu_2|$ for all $t\in I$.

We consider any Carathéodory solution $x:\mathbb{R}\to\mathbb{R}$ of (1.1) with (3.10). Define $w(t)=x'(t)-\mu_2x(t)$ and $F(t)=\int f(t)e^{-\mu_1t}dt$ for all $t\in\mathbb{R}$. Then, we see that $w(t)$ is a Carathéodory solution of (3.2) with $i=2$, and it satisfies

$$w(\tau)=x'(\tau)-\mu_2x(\tau)=\left\{F(\tau)+\lim_{t\to-\infty}\left(\eta_2(t)e^{-\mu_1t}-F(t)\right)\right\}e^{\mu_1\tau}=y(\tau)$$

from (3.10). By means of the uniqueness of Carathéodory solutions of (3.2) with $i=2$, we conclude that $w(t)=y(t)$ for all $t\in\mathbb{R}$. From this and (3.10), $x(t)$ is a Carathéodory solution of (3.4) with $|\lim_{t\to\tau-0}\xi(t)-x(\tau)|<\varepsilon/|\mu_1\mu_2|$, $y(t)=w(t)$ and $i=2$. Therefore, we obtain $|\xi(t)-x(t)|<\varepsilon/|\mu_1\mu_2|$ for all $t\in I$.

Finally, using the same transformation as in the proof of case (ii) and using assertion (iv), we can prove case (v). This completes the proof of Theorem 3.6. □

REMARK 3.1. The conditions

$$\left|\lim_{t\to\tau-0}\xi'(t)-x'(\tau)\right|+\mu_1\left|\lim_{t\to\tau-0}\xi(t)-x(\tau)\right|<\frac{\varepsilon}{\mu_2}$$

and

$$\left|\lim_{t\to\sigma+0}\xi'(t)-x'(\sigma)\right|+|\mu_1|\left|\lim_{t\to\sigma+0}\xi(t)-x(\sigma)\right|<\frac{\varepsilon}{|\mu_2|}$$

imply conditions (3.7) and (3.8), respectively. Then, initial conditions (3.7) and (3.8) in Theorem 3.6 can be changed to simple conditions.

4. Best HUS constant

In the previous section, we discussed the Hyers–Ulam stability for (1.1) and obtained the an exact HUS constant given by $1/|\mu_1\mu_2|$. Needless to say, any value greater than this constant is one of the HUS constants. Now the question arises. Is this HUS constant the minimum of HUS constants? Section 4 answers this question. If there exists the minimum of HUS constant, we call it the "*best HUS constant*" for (1.1) on I. Recently, the best constants have been derived for various equations. For example, see [5, 7, 15, 16, 29, 30]. Now, we present a result as follows.

THEOREM 4.1. *Let $I=(\sigma,\tau)$ with $-\infty\leqslant\sigma<\tau\leqslant\infty$. Suppose that the characteristic equation $\mu^2+\alpha\mu+\beta=0$ for $x''+\alpha x'+\beta x=0$ has the non-zero real roots μ_1 and μ_2. Then the following holds:*

(i) if $\mu_1>0$, $\mu_2>0$ and $\tau=\infty$, then (1.1) has Hyers–Ulam stability with the best HUS constant $1/(\mu_1\mu_2)$ on I;

(ii) if $\mu_1<0$, $\mu_2<0$ and $\sigma=-\infty$, then (1.1) has Hyers–Ulam stability with the best HUS constant $1/(\mu_1\mu_2)$ on I;

(iii) if $\mu_1<0<\mu_2$, $\sigma=-\infty$ and $\tau=\infty$, then (1.1) has Hyers–Ulam stability with the best HUS constant $1/|\mu_1\mu_2|$ on I.

Proof. Let μ_1 and μ_2 be the non-zero real roots of $\mu^2+\alpha\mu+\beta=0$. Define the function ψ by

$$\psi(t):=e^{\mu_1 t}\int\left(e^{(\mu_2-\mu_1)t}\int f(t)e^{-\mu_2 t}dt\right)dt+\frac{\varepsilon}{\mu_1\mu_2}$$

on I, where $f(t)$ is a summable function on each segment contained in $\mathbb{R}$. Since

$$\psi'(t)=\mu_1\psi(t)+e^{\mu_2 t}\int f(t)e^{-\mu_2 t}dt-\frac{\varepsilon}{\mu_2}$$

holds, we have

$$\psi''(t)=\mu_1\psi'(t)+\mu_2\left(\psi'(t)-\mu_1\psi(t)+\frac{\varepsilon}{\mu_2}\right)+f(t)=-\alpha\psi'(t)-\beta\psi(t)+\varepsilon+f(t),$$

so that $\psi(t)$ satisfies the equation

$$\left|\psi''(t)+\alpha\psi'(t)+\beta\psi(t)-f(t)\right|=\varepsilon$$

for all $t\in I$.

First, we consider case (i). Suppose $\mu_1>0$, $\mu_2>0$ and $\tau=\infty$. Now we will use Theorem 3.2 (i). From

$$(\psi'(t)-\mu_1\psi(t))e^{-\mu_2 t}=\int f(t)e^{-\mu_2 t}dt-\frac{\varepsilon}{\mu_2}e^{-\mu_2 t}=F(t)-\frac{\varepsilon}{\mu_2}e^{-\mu_2 t},$$

we get

$$\lim_{t\to\infty}\left\{(\psi'(t)-\mu_1\psi(t))e^{-\mu_2 t}-F(t)\right\}=-\frac{\varepsilon}{\mu_2}\lim_{t\to\infty}e^{-\mu_2 t}=0=c^+.$$

Moreover, by

$$\psi(t)e^{-\mu_1 t}-\int(F(t)+c^+)e^{(\mu_2-\mu_1)t}dt=\frac{\varepsilon}{\mu_1\mu_2}e^{-\mu_1 t},$$

we have

$$\lim_{t\to\infty}\left\{\psi(t)e^{-\mu_1 t}-\int(F(t)+c^+)e^{(\mu_2-\mu_1)t}dt\right\}=0.$$

Hence, using Theorem 3.2 (i), we see that there exists the unique Carathéodory solution

$$x(t)=e^{\mu_1 t}\int\left(e^{(\mu_2-\mu_1)t}\int f(t)e^{-\mu_2 t}dt\right)dt$$

of (1.1) such that $|\psi(t)-x(t)|\leqslant\varepsilon/(\mu_1\mu_2)$ for all $t\in I$. More precisely, the last inequality will be equality as follows:

$$|\psi(t)-x(t)|=\frac{\varepsilon}{\mu_1\mu_2}$$

for all $t\in I$. This says that the minimum HUS constant on I is at least $1/(\mu_1\mu_2)$.

Using the same argument we have assertions (ii) and (iii). This completes the proof. □

If $I=\mathbb{R}$ then we obtain the following result, immediately.

COROLLARY 4.2. *Suppose that the characteristic equation $\mu^2+\alpha\mu+\beta=0$ for $x''+\alpha x'+\beta x=0$ has the non-zero real roots μ_1 and μ_2. Then (1.1) has Hyers–Ulam stability with the best HUS constant $1/|\mu_1\mu_2|$ on $\mathbb{R}$.*

5. Instability

In this section, we deal with the instability for (1.1). As a result, the necessary and sufficient condition is finally obtained.

THEOREM 5.1. *Let $I=(\sigma,\tau)$ with $-\infty\leqslant\sigma<\tau\leqslant\infty$. Suppose that $\sigma=-\infty$ or $\tau=\infty$. Suppose also that the characteristic equation $\mu^2+\alpha\mu+\beta=0$ for $x''+\alpha x'+\beta x=0$ has real roots μ_1 and μ_2. If $\mu_1\mu_2=0$, then (1.1) does not have Hyers–Ulam stability on I.*

Proof. First, we consider the case that the characteristic equation $\mu^2+\alpha\mu+\beta=0$ for $x''+\alpha x'+\beta x=0$ has exactly one real root $\mu_1=\mu_2=0$. Now, we consider the function

$$\xi(t)=\varepsilon\frac{t^2}{2}+\int\left(\int f(t)dt\right)dt,$$

where $\varepsilon>0$ is a given arbitrary constant and $f(t)$ is a summable real-valued function on each segment contained in I. Then $\xi(t)$ satisfies $|\xi''(t)-f(t)|=\varepsilon$ for almost all $t\in I$. Clearly, the characteristic equation for $x''=0$ has exactly one real root $\mu_1=\mu_2=0$. Consider the case $\tau=\infty$. Since any Carathéodory solution of $x''=f(t)$ is given by

$$x(t)=c_1t+c_2+\int\left(\int f(t)dt\right)dt,$$

where c_1 and c_2 are arbitrary constants, we have

$$\lim_{t\to\infty}|\xi(t)-x(t)|=\lim_{t\to\infty}\left|\frac{\varepsilon t^2}{2}-c_1t-c_2\right|=\infty.$$

Similarly, the case $\sigma=-\infty$ leads to $\lim_{t\to-\infty}|\xi(t)-x(t)|=\infty$. Hence, $x''=f(t)$ does not have Hyers–Ulam stability on I.

Next, we consider the case that the characteristic equation $\mu^2+\alpha\mu+\beta=0$ for $x''+\alpha x'+\beta x=0$ has two different real roots $\mu_1\neq\mu_2=0$. We consider the function

$$\xi(t)=-\frac{\varepsilon t}{\mu_1}+e^{\mu_1t}\int\left(e^{-\mu_1t}\int f(t)dt\right)dt,$$

where $\varepsilon>0$ is a given arbitrary constant and $f(t)$ is a summable real-valued function on each segment contained in I. Then $\xi(t)$ satisfies $|\xi''(t)-\mu_1\xi'(t)-f(t)|=\varepsilon$ for almost all $t\in I$. Clearly, the characteristic equation for $x''-\mu_1x'=0$ has two different real roots $\mu_1\neq0$ and 0. Consider the case $\tau=\infty$. Since any Carathéodory solution of $x''-\mu_1x'=f(t)$ is given by

$$x(t)=c_1+c_2e^{\mu_1t}+e^{\mu_1t}\int\left(e^{-\mu_1t}\int f(t)dt\right)dt,$$

where c_1 and c_2 are arbitrary constants, we have

$$\lim_{t\to\infty}|\xi(t)-x(t)|=\lim_{t\to\infty}\left|-\frac{\varepsilon t}{\mu_1}-c_1-c_2e^{\mu_1t}\right|=\infty.$$

Using the same argument, we see that the case $\sigma = -\infty$ leads to $\lim_{t\to-\infty} |\xi(t) - x(t)| = \infty$. Therefore, $x'' - \mu_1 x' = f(t)$ does not have Hyers–Ulam stability on I. $\square$

Under the assumption that $I = \mathbb{R}$, we obtain the following result from Theorem 5.1, immediately.

COROLLARY 5.2. *Suppose that the characteristic equation* $\mu^2 + \alpha\mu + \beta = 0$ *for* $x'' + \alpha x' + \beta x = 0$ *has real roots* μ_1 *and* μ_2*. If* $\mu_1\mu_2 = 0$*, then* (1.1) *does not have Hyers–Ulam stability on* $\mathbb{R}$.

Theorems 3.1 and 5.1 imply the following result.

THEOREM 5.3. *Let* $I = (\sigma, \tau)$ *with* $-\infty \leqslant \sigma < \tau \leqslant \infty$*. Suppose that* $\sigma = -\infty$ *or* $\tau = \infty$*. Suppose also that the characteristic equation* $\mu^2 + \alpha\mu + \beta = 0$ *for* $x'' + \alpha x' + \beta x = 0$ *has real roots* μ_1 *and* μ_2*. Then* (1.1) *has Hyers–Ulam stability on* I *if and only if* $\mu_1\mu_2 \neq 0$.

6. Example and numerical simulation

In this section, we will present an example with a numerical simulation.

EXAMPLE 6.1. Let us consider the forced equation

$$x'' + \alpha x' + \beta x = f(t) + e(t) \tag{6.1}$$

on $\mathbb{R}$, where α and β are real-valued constants, and $f(t)$ and $e(t)$ are summable real-valued functions on each segment contained in $\mathbb{R}$. We assume that the characteristic equation $\mu^2 + \alpha\mu + \beta = 0$ has the non-zero real roots μ_1 and μ_2. Then, by means of Corollary 4.2, (6.1) has Hyers–Ulam stability with the best HUS constant $1/|\mu_1\mu_2|$ on $\mathbb{R}$.

Now we regard (1.1) and (6.1) as the mathematical model and the real model, respectively. In addition, we can regard $e(t)$ as the error between mathematical and real models. Note here that we can easily find the Carathéodory solution of mathematical model (1.1), however, the solution of real model (6.1) is unknown because this model includes unknown error $e(t)$. Let $\xi(t)$ be a Carathéodory solution of (1.1) (math. model). That is, $\xi(t)$ is an approximate solution of (6.1) (real model). We may assume without loss of generality that the error is small, that is, $|e(t)| \leqslant \varepsilon$ holds for all $t \in \mathbb{R}$. Using this, Carathéodory solution $\xi(t)$ of (1.1) satisfies

$$|\xi''(t) + \alpha\xi'(t) + \beta\xi(t) - f(t) - e(t)| = |e(t)| \leqslant \varepsilon$$

for almost all $t \in \mathbb{R}$. Since (6.1) has Hyers–Ulam stability with the best HUS constant $1/|\mu_1\mu_2|$ on $\mathbb{R}$, there exists a Carathéodory solution $x(t)$ of real model (6.1) near to $\xi(t)$ for all $t \in \mathbb{R}$. To be precise, we see that $|\xi(t) - x(t)| \leqslant \varepsilon/|\mu_1\mu_2|$ for all $t \in \mathbb{R}$. This says that, for all $t \in \mathbb{R}$, the error between solutions of mathematical and real models is at most $\varepsilon/|\mu_1\mu_2|$. In this way, the best HUS constant represents the maximum value of the error that occurs between an approximate solution and an exact solution, and plays an important role in application.

Now we will present a numerical simulation. Let $I = (0,\infty)$, $\alpha = 3$ and $\beta = 2$. Then characteristic equation has the non-zero real roots $\mu_1 = -2$ and $\mu_2 = -1$. We consider the Carathéodory solution $\xi(t)$ of the initial value problem (1.1) with $(\xi(0), \xi'(0)) = (\xi_0, \xi_0')$ on $\mathbb{R}$. Using (ii) in Theorem 3.6, we conclude that any Carathéodory solution $x(t)$ of (6.1) with $x'(0) = \xi_0'$ and $|\xi_0 - x(0)| < \varepsilon/2$ satisfies that $|\xi(t) - x(t)| < \varepsilon/2$ for all $t \in I$. This means that any solution $x(t)$ of (6.1) starting in the neighborhood of ξ_0 stays in the neighborhood of $\xi(t)$ for all $t \in I$ when $x'(0) = \xi_0'$. To present a numerical simulation, we give some information. Define the step function (on-off function) δ by

$$\delta(t) = \begin{cases} 1 & \text{if } 2n \leqslant t < 2n+1, \\ 0 & \text{if } 2n+1 \leqslant t < 2(n+1), \end{cases} \qquad n \in \mathbb{Z}.$$

Let $f(t) = 10\delta(t)$ and $e(t) = 4\sin t$. A solution curve of (1.1) with $(\xi(0), \xi'(0)) = (8, 0)$ is given in Figure 1 (red curve). Moreover, dashed curves are graphs of $\xi(t) - 2$ and $\xi(t) + 2$, respectively. Each solution curve of (6.1) with $(x(0), x'(0)) = (6.1, 0)$ or $(x(0), x'(0)) = (9.9, 0)$ is also given in Figure 1 (blue curves).

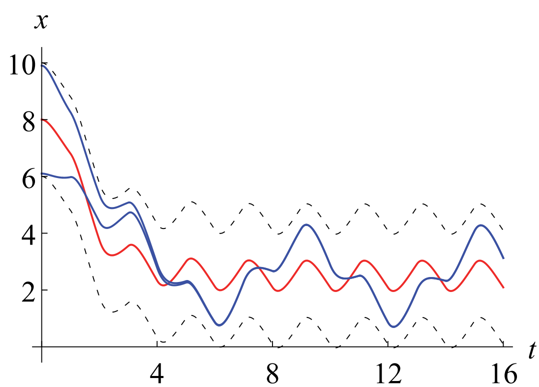

Figure 1: *Solution curves of* (1.1) *and* (6.1) *with* $\alpha = 3$, $\beta = 2$, $f(t) = 10\delta(t)$ *and* $e(t) = 4\sin t$.

7. Conclusions

The purpose of this study was to deal with Hyers–Ulam stability for second-order linear differential equations. In particular, this work clarified the following: finding an explicit HUS constant K on I, and the explicit solution x of (1.1) such that $|\xi(t) - x(t)| \leqslant K\varepsilon$ for all $t \in \mathbb{R}$. It have given some theorems that describe the exact behavior of the solutions for the various situations of σ and τ in $I = (\sigma, \tau)$. Moreover, it was shown that the obtained HUS constant is the best one. On the other hand, the instability was also considered and a necessary and sufficient condition was obtained. In the end, in order to assert the importance of Hyers–Ulam stability, an example was presented. It clarified that the best HUS constant means the maximum value of the error between an approximate solution and an exact solution.

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