Journal of
Mathematical
Inequalities
Volume 15, Number 4 (2021), 1519–1532 doi:10.7153/jmi-2021-15-104

QUANTITATIVE VORONOVSKAYA TYPE RESULTS FOR A SEQUENCE OF STANCU TYPE OPERATORS

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(*Communicated by I. Raşa*)

Abstract. In this paper, we give some approximation properties of a sequence of operators defined by Stancu [21]. Its quantitative Voronovskaya type results are obtained with the aid of the second moduli of continuity. In order to study non-multiplicativity of these operators some Grüss-Voronovskaya type theorems are established.

1. Introduction

For $f \in C[0,1]$, the space of all continuous functions on $[0,1]$ endowed with the norm $\|f\|_{\infty} = \sup_{x\in[0,1]} |f(x)|$, Stancu [21] investigated the remainder of the approximation formula by means of a sequence of generalized Bernstein operators, depending on two parameters $r,s \in \mathbb{N}\cup\{0\}$ and defined as:

$$S_{n,r,s}(f;x) = \sum_{\alpha=0}^{n-sr} p_{n-sr,\alpha}(x) \sum_{\beta=0}^{s} p_{s,\beta}(x) f\left(\frac{\alpha+\beta r}{n}\right), \text{ where } sr \leqslant n. \tag{1.1}$$

Note that if $r = s = 0$, the operators (1.1) include the Bernstein polynomials $B_n(f;x)$ given by

$$B_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right),$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $0 \leqslant x \leqslant 1$ is the Bernstein basis.

Bernstein polynomials were modified by Kantorovich [16] as follows

$$K_n(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(x) \int_{k/n+1}^{k+1/n+1} f(t)\,dt, \ \ x \in [0,1]$$

in order to approximate $f \in L_p[0,1], 1 \leqslant p < \infty$. Ditzian and Zhou [9], studied the saturation and direct-converse theorems for these Bernstein-Kantorovich operators.

Mathematics subject classification (2020): 41A10, 41A25, 41A30, 26A15.
Keywords and phrases: Kantorovich operators, Voronovskaya type results, Grüss Voronovskaya theorem.

Paper JMI-15-104

In the literature, many researchers have studied approximation behavior of Kantorovich variants of several sequences of linear positive operators (see [7], [10], [17], [23], [24] etc.). In [19], Miclaus studied some approximation properties of Bernstein-Stancu type operators based on Polya distribution. In [3], Agrawal et al. introduced a Kantorovich modification of the operators considered by Lupas and Lupas [18] based on Polya distribution along with bivariate generalizations of these operators and studied their approximation properties. Cheng [8] approximated functions of bounded variation using the sequence of Bernstein polynomials. Also, Bojanic and Cheng ([5], [6]) used probabilistic approach to study the rate of approximation of Bernstein polynomials for functions of derivatives of bounded variation. Many researchers have discussed the degree of approximation with derivatives of bounded variation (cf. [4], [14], [15] etc.) and the references therein.

In [2], Acu and Gonska obtained a quantitative Voronovskaya-type result in terms of second moduli of continuity for the Kantorovich operators and also discussed the non-multiplicativity property. In [22], for $f\in C[0,1]$, endowed with the sup-norm, M. Talpau Dimitriu introduced a Kantorovich variant of the operators (1.1) and proved some global smoothness preservation results. Denote by $K_{n,r,s}$ the operator introduced in [22]:

$$K_{n,r,s}(f;x)=\sum_{\alpha=0}^{n-sr}p_{n-sr,\alpha}(x)\sum_{\beta=0}^{s}p_{s,\beta}(x)\int_0^1 f\left(\frac{\alpha+\beta r+t}{n+1}\right)dt. \tag{1.2}$$

In this article, we study a quantitative Voronovskaya type result for the operators (1.2) in terms of the second moduli of continuity. To show their non-multiplicative character, Chebyshev-Grüss inequality is obtained and then two Grüss-Voronovskaya theorems are discussed.

In our further consideration, C denotes a constant which may not be the same at each occurence.

2. Auxiliary results

Let $e_i(t)=t^i, i=\{0,1,2,3,4\}$.

LEMMA 2.1. *For the operators $S_{n,r,s}(f;x)$, we have*

$$\begin{aligned}
&S_{n,r,s}(e_0;x)=1;\\
&S_{n,r,s}(e_1;x)=x;\\
&S_{n,r,s}(e_2;x)=x^2+\frac{x(1-x)(r^2s-rs+n)}{n^2};\\
&S_{n,r,s}(e_3;x)=x^3+\frac{1}{n^3}\left\{3x^2(1-x)n^2+x(1-x)(3r^2sx-3rsx-2x+1)n\right.\\
&\qquad\left.+x(2r^3sx^2-3r^3sx+r^3s-2rsx^2+3rsx-rs)\right\};
\end{aligned}$$

$$S_{n,r,s}(e_4;x) = x^4 + \frac{1}{4}\{6x^3(1-x)n^3 - x^2(1-x)(6r^2sx - 6rsx - 11x + 7)n^2$$
$$-x(1-x)(8r^3sx^2 - 4r^3sx + 6r^2sx^2 - 6r^2sx - 14rsx^2 + 10rsx - 6x^2 + 6x - 1)n$$
$$+xsr(1-r)(1-x)(3r^2sx^2 - 3r^2sx - 6r^2x^2 - 3rsx^2 + 6r^2x + 3rsx$$
$$-6rx^2 - r^2 + 6rx - 6x^2 - r + 6x - 1)\}.$$

As a consequence of Lemma 2.1, we have

LEMMA 2.2. *For the operators $S_{n,r,s}(f;x)$, there hold the following equalities:*

$$(i) \quad S_{n,r,s}\left((t-x)^2;x\right) = \frac{x(1-x)(r^2s - rs + n)}{n^2};$$

$$(ii) \quad S_{n,r,s}\left((t-x)^3;x\right) = \frac{(r^3s - rs + n)x(1-2x)(1-x)}{n^3};$$

$$(iii) \quad S_{n,r,s}\left((t-x)^4;x\right) = \frac{x(1-x)}{n^4}\{3x(1-x)n^2 + (6x(1-x)(r^2s - rs - 1) + 1)n$$
$$+sr(1-r)(3r^2sx^2 - 3r^2sx - 6r^2x^2 - 3rsx^2 + 6r^2x + 3rsx - 6rx^2 - r^2 + 6rx$$
$$-6x^2 - r + 6x - 1)\}.$$

Consequently,

LEMMA 2.3. *For each $x \in [0,1]$, we have*

$$\lim_{n\to\infty} nS_{n,r,s}\left((t-x)^2;x\right) = x(1-x);$$
$$\lim_{n\to\infty} n^2S_{n,r,s}\left((t-x)^3;x\right) = x(1-x)(1-2x);$$
$$\lim_{n\to\infty} n^2S_{n,r,s}\left((t-x)^4;x\right) = 3x^2(1-x)^2.$$

REMARK 2.4. For $n \in \mathbb{N}$ and $x \in [0,1]$, from Lemma 2.2 we have

$$S_{n,r,s}\left((t-x)^2;x\right) \leqslant \frac{\varphi(r,s)}{n},$$

where $\varphi(r,s) = \frac{(1+rs|r-1|)}{4}$.

LEMMA 2.5. *For the operators $K_{n,r,s}(f;x)$, we have*

$$K_{n,r,s}(e_0;x) = 1;$$
$$K_{n,r,s}(e_1;x) = \frac{2nx+1}{2(n+1)};$$
$$K_{n,r,s}(e_2;x) = \frac{1}{(n+1)^2}\left(n^2x^2 - x(x-2)n - r^2sx^2 + r^2sx + rsx^2 - rsx + \frac{1}{3}\right);$$

$$K_{n,r,s}(e_3;x) = \frac{1}{4(n+1)^3}\{4n^3x^3 - 6x^2(2x-3)n^2$$
$$-2x\left(6r^2sx^2 - 6r^2sx - 6rsx^2 + 6rsx - 4x^2 + 9x - 7\right)n + 8r^3sx^3$$
$$-12r^3sx^2 + 4r^3sx - 6r^2sx^2 - 8rsx^3 + 6r^2sx + 18rsx^2 - 10rsx + 1\};$$
$$K_{n,r,s}(e_4;x) = \frac{1}{5(n+1)^4}\{-10x^3(3x-4)n^3$$
$$-5x^2\left(6r^2sx^2 - 6r^2sx - 6rsx^2 + 6rsx - 11x^2 + 24x - 15\right)n^2$$
$$+5x(8r^3sx^3 - 12r^3sx^2 + 6r^2sx^3 + 4r^3sx - 18r^2sx^2 - 14rsx^3 + 12r^2sx$$
$$+30rsx^2 - 16rsx - 6x^3 + 16x^2 - 15x + 6)n + 15r^4s^2x^4 - 30r^4s^2x^3$$
$$-30r^4sx^4 - 30r^3s^2x^4 + 5n^4x^4 + 15r^4s^2x^2 + 60r^4sx^3 + 60r^3s^2x^3 + 15r^2s^2x^4$$
$$-35r^4sx^2 - 30r^3s^2x^2 + 20r^3sx^3 - 30r^2s^2x^3 + 5r^4sx - 30r^3sx^2 + 15r^2s^2x^2$$
$$+30rsx^4 + 10r^3sx - 10r^2sx^2 - 80rsx^3 + 10r^2sx + 75rsx^2 - 25rsx + 1\}.$$

As a consequence of Lemma 2.5, we have

LEMMA 2.6. *For the operators $K_{n,r,s}(f;x)$, there hold the following equalities:*

(i) $K_{n,r,s}((t-x);x) = \dfrac{1-2x}{2(n+1)}$;

(ii) $K_{n,r,s}\left((t-x)^2;x\right) = \dfrac{1}{(n+1)^2}\left\{x(1-x)(r^2s - rs + n - 1) + \dfrac{1}{3}\right\}$;

(iii) $K_{n,r,s}\left((t-x)^3;x\right) = \dfrac{1}{4(n+1)^3}\{10x(1-x)(1-2x)n + 8r^3sx^3 - 12r^3sx^2 + 12r^2sx^3$
$+4r^3sx - 18r^2sx^2 - 20rsx^3 + 6r^2sx + 30rsx^2 - 10rsx + (1-2x)(2x^2 - 2x + 1)\}$;

(iv) $K_{n,r,s}\left((t-x)^4;x\right) = \dfrac{1}{5(n+1)^4}\{15x^2(1-x)^2n^2 - 5x(1-x)(6r^2sx^2 - 6r^2sx - 6rsx^2$
$+6rsx - 20x^2 + 20x - 5)n + 15r^4s^2x^4 - 30r^4s^2x^3 - 30r^4sx^4 - 30r^3s^2x^4 + 15r^4s^2x^2$
$+60r^4sx^3 + 60r^3s^2x^3 - 40r^3sx^4 + 15r^2s^2x^4 - 35r^4sx^2 - 30r^3s^2x^2 + 80r^3sx^3$
$-30r^2s^2x^3 - 30r^2sx^4 + 5r^4sx - 50r^3sx^2 + 15r^2s^2x^2 + 60r^2sx^3 + 100rsx^4 + 10r^3sx$
$-40r^2sx^2 - 200rsx^3 + 10r^2sx + 125rsx^2 - 25rsx + 10x^2 - 5x + 1 + 5x^4 - 10x^3\}$.

Consequently,

LEMMA 2.7. *For each $x \in [0,1]$, we have*

$$\lim_{n\to\infty} nK_{n,r,s}((t-x);x) = \frac{1-2x}{2};$$
$$\lim_{n\to\infty} nK_{n,r,s}\left((t-x)^2;x\right) = x(1-x);$$
$$\lim_{n\to\infty} n^2K_{n,r,s}\left((t-x)^3;x\right) = \frac{5}{2}x(1-x)(1-2x);$$
$$\lim_{n\to\infty} n^2K_{n,r,s}\left((t-x)^4;x\right) = 3x^2(1-x)^2.$$

REMARK 2.8. For $n \in \mathbb{N}$ and $x \in [0,1]$, from Lemma 2.6 we have

$$K_{n,r,s}\left((t-x)^2;x\right) \leqslant \frac{\rho(r,s)}{n+1},$$

where $\rho(r,s) = \frac{(7+3rs|r-1|)}{12}$.

Let I be a compact interval and $\omega_k(\varphi;h) := \sup\left\{\left|\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\varphi(x+ih)\right| : |\delta| \leqslant h, x, x+ih \in I\right\}$ the modulus of smoothness of order k. Păltănea [20] gave a very remarkable result regarding the order of approximation by any sequence of linear positive operators for a function $f \in C[0,1]$, in terms of the moduli of continuity of first and second orders as stated below:

THEOREM 2.9. [20] *If $L_n : C[0,1] \to C[0,1]$ is a sequence of positive linear operators, then for $f \in C[0,1]$, $x \in [0,1]$ and each $0 < h \leqslant \frac{1}{2}$, the following inequality holds:*

$$\begin{aligned}|L_n(f;x) - f(x)| \leqslant{} & |L_n(e_0;x) - 1| \cdot |f(x)| + \tfrac{1}{h}|L_n(e_1 - x;x)|\,\omega_1(f;h) \\ & + \left[(L_n e_0)(x) + \tfrac{1}{2h^2} L_n\left((e_1-x)^2;x\right)\right]\omega_2(f;h).\end{aligned}$$

As a consequence of Theorem 2.9, we get

THEOREM 2.10. *For all $f \in C[0,1]$ and all $n \geqslant 4$, we have*

i) $\|S_{n,r,s}f - f\|_\infty \leqslant \left(1 + \frac{1+rs|r-1|}{8}\right)\omega_2\left(f; \frac{1}{\sqrt{n+1}}\right)$,

ii) $\|K_{n,r,s}f - f\|_\infty \leqslant \frac{1}{2\sqrt{n+1}}\omega_1\left(f; \frac{1}{\sqrt{n+1}}\right) + \left(1 + \frac{7+3rs|r-1|}{24}\right)\omega_2\left(f; \frac{1}{\sqrt{n+1}}\right)$.

Proof. Using Theorem 2.9 and taking $h = n^{-1/2}$, respectively $h = (n+1)^{-1/2}$, the proof is straight forward. Hence, we omit the details. □

3. Grüss Voronovskaya type theorems for the operators $S_{n,r,s}$

In this section, we investigate the non-multiplicative character of the Stancu type operators defined in first section.

THEOREM 3.1. *Let $f, g \in C^2[0,1]$. Then*

$$\lim_{n\to\infty} n\left[S_{n,r,s}(fg;x) - S_{n,r,s}(f;x) \cdot S_{n,r,s}(g;x)\right] = x(1-x)f'(x)g'(x).$$

Proof. Let $X := x(1-x)$. Denote $A_{n,r,s}f := S_{n,r,s}f - f - \frac{X}{2n}f''$. We have

$$\begin{aligned} & n\left[S_{n,r,s}(fg;x) - S_{n,r,s}(f;x)\cdot S_{n,r,s}(g;x)\right] \\ = & n\Bigg[A_{n,r,s}(fg;x) - g(x)A_{n,r,s}(f;x) - S_{n,r,s}(f;x)A_{n,r,s}(g;x) + \frac{x(1-x)}{n}f'(x)g'(x) \\ & +g''(x)\frac{x(1-x)}{2n}\left(f(x) - S_{n,r,s}(f;x)\right)\Bigg]. \end{aligned} \tag{3.1}$$

But $\lim_{n\to\infty} nA_{n,r,s}(f;x) = 0$. Passing to the limit in (3.1) we get

$$\lim_{n\to\infty} n\left[S_{n,r,s}(fg;x) - S_{n,r,s}(f;x)\cdot S_{n,r,s}(g;x)\right] = x(1-x)f'(x)g'(x). \quad \square$$

Let $\psi \in C^2[0,1]$. Denote

$$P(\psi,n) = C\left\{\frac{1}{\sqrt{n}}\omega_1\left(\psi'';\frac{1}{\sqrt{n}}\right) + \omega_2\left(\psi'';\frac{1}{\sqrt{n}}\right) + \frac{1}{n}\|f''\|_\infty\right\}.$$

THEOREM 3.2. *Let $f,g \in C^2[0,1]$. Then, for all $x \in [0,1]$ and $n \in \mathbb{N}$,*

$$\begin{aligned} & \left\|n\left\{S_{n,r,s}(fg) - S_{n,r,s}f\cdot S_{n,r,s}g\right\} - Xf'g'\right\|_\infty \\ & \leqslant C\left\{P(fg;n) + \|f\|_\infty P(g;n) + \|g\|_\infty P(f;n) + \frac{1}{n}\right\}, \end{aligned}$$

where $X = x(1-x)$.

Proof. Denote $A_{n,r,s}f := S_{n,r,s}f - f - \frac{X}{2n}f''$. We can write

$$\begin{aligned} & S_{n,r,s}(fg) - S_{n,r,s}f\cdot S_{n,r,s}g - \frac{X}{n}f'g' \\ & = A_{n,r,s}(fg) - fA_{n,r,s}g - gA_{n,r,s}f + [g - S_{n,r,s}g][S_{n,r,s}f - f]. \end{aligned} \tag{3.2}$$

Using the generalized Voronovskaya type theorem [13, Theorem 3], for any $\psi \in C^2[0,1]$, we have

$$\begin{aligned} & \left|S_{n,r,s}(\psi;x) - \psi(x) - \frac{1}{2}S_{n,r,s}\left((t-x)^2;x\right)\psi''(x)\right| \\ & \leqslant S_{n,r,s}\left((t-x)^2;x\right)\left\{\frac{\left|S_{n,r,s}\left((t-x)^3;x\right)\right|}{S_{n,r,s}\left((t-x)^2;x\right)}\frac{5}{6h}\omega_1\left(\psi'';h\right)\right. \\ & \left. + \left(\frac{3}{4} + \frac{S_{n,r,s}\left((t-x)^4;x\right)}{S_{n,r,s}\left((t-x)^2;x\right)}\cdot\frac{1}{16h^2}\right)\omega_2\left(\psi'';h\right)\right\}. \end{aligned}$$

Using Lemma 2.2 and choosing $h = \frac{1}{\sqrt{n}}$, we get

$$\begin{aligned}&\left|S_{n,r,s}(\psi;x)-\psi(x)-\frac{x(1-x)(r^2s-rs+n)}{2n^2}\psi''(x)\right|\\ &\leqslant \frac{C}{n}\left\{\frac{1}{\sqrt{n}}\omega_1\left(\psi'';\frac{1}{\sqrt{n}}\right)+\omega_2\left(\psi'';\frac{1}{\sqrt{n}}\right)\right\},\end{aligned}$$

where C is a positive constant depending on r and s, but independent of n.

Multiply by n both sides,

$$\begin{aligned}&\left|n\left[S_{n,r,s}(\psi;x)-\psi(x)\right]-\frac{x(1-x)(r^2s-rs+n)}{2n}\psi''(x)\right|\\ &\leqslant C\left\{\frac{1}{\sqrt{n}}\omega_1\left(\psi'';\frac{1}{\sqrt{n}}\right)+\omega_2\left(\psi'';\frac{1}{\sqrt{n}}\right)\right\}.\end{aligned}$$

Therefore,

$$\begin{aligned}|nA_{n,r,s}(f;x)| &= \left|n\left[S_{n,r,s}-\psi(x)\right]-\frac{x(1-x)}{2}\Psi''(x)\right|\\ &= \left|n\left[S_{n,r,s}-\psi(x)\right]-\frac{x(1-x)(r^2s-rs+n)}{2n}\Psi''(x)\right|\\ &\quad+\left|\frac{x(1-x)rs(r-1)}{2n}\psi''(x)\right|\\ &\leqslant C\left\{\frac{1}{\sqrt{n}}\omega_1\left(\psi'';\frac{1}{\sqrt{n}}\right)+\omega_2\left(\psi'';\frac{1}{\sqrt{n}}\right)+\frac{\|f''\|_\infty}{n}\right\}=P(\psi,n).\end{aligned}$$

From Theorem 2.10 we get

$$\|S_{n,r,s}\psi-\psi\|_\infty \leqslant \left(1+\frac{1+rs|r-1|}{8}\right)\frac{1}{n}\|f''\|_\infty \leqslant \frac{C}{n}.$$

From relation (3.2) we obtain

$$\begin{aligned}&\left\|n\left\{S_{n,r,s}(fg)-S_{n,r,s}f\cdot S_{n,r,s}g\right\}-Xf'g'\right\|_\infty\\ &\leqslant C\left\{P(fg;n)+\|f\|_\infty P(g;n)+\|g\|_\infty P(f;n)+\frac{1}{n}\right\}. \quad \square\end{aligned}$$

In order to prove the next result, we shall need the following important theorem:

THEOREM 3.3. [1] *Let $D: C[a,b] \to C[a,b]$ be positive linear and satisfy $De_0 = e_0$. Then, for $f, g \in C[a,b]$ and $x \in [a,b]$ arbitrary but fixed, one has*

$$\begin{aligned}&|D(fg;x)-D(f;x)D(g;x)|\\ &\leqslant \frac{1}{4}\tilde{\omega}\left(f;2\sqrt{H\left((e_1-x)^2;x\right)}\right)\cdot\tilde{\omega}\left(g;2\sqrt{H\left((e_1-x)^2;x\right)}\right),\end{aligned}$$

where $\tilde{\omega}(f;t) := \sup\left\{\frac{(t-u)\omega_1(f;v)+(v-t)\omega_1(f;u)}{v-u} : 0 \leqslant u \leqslant t \leqslant v \leqslant b-a, u \neq v\right\}$ *is the least concave majorant of the first order modulus* ω_1.

Consequently, using Lemma 2.5 and the Remark 2.8, we obtain

THEOREM 3.4. *For the operators* $S_{n,r,s}, K_{n,r,s} : C[0,1] \to C[0,1]$, *the following uniform inequality holds:*

i) $\|S_{n,r,s}(fg) - S_{n,r,s}fS_{n,r,s}g\|_\infty \leqslant \frac{1}{4}\tilde{\omega}\left(f;2\sqrt{\frac{\varphi(r,s)}{n}}\right)\tilde{\omega}\left(g;2\sqrt{\frac{\varphi(r,s)}{n}}\right), \ n \geqslant 1,$

ii) $\|K_{n,r,s}(fg) - K_{n,r,s}fK_{n,r,s}g\|_\infty \leqslant \frac{1}{4}\tilde{\omega}\left(f;2\sqrt{\frac{\rho(r,s)}{n+1}}\right)\tilde{\omega}\left(g;2\sqrt{\frac{\rho(r,s)}{n+1}}\right),$
for all $f,g \in C[0,1]$.

4. Grüss Voronovskaya theorems for Kantorovich variant of Stancu operators

In the following result, we establish Grüss Voronovskaya theorem for the operators defined in (1.2), where $f,g \in C^2[0,1]$, the space of twice continuously differentiable functions on $[0,1]$. Let for any $\psi \in C^2[0,1]$,

$$Q(\psi,n) = \frac{1}{\sqrt{n+1}}\omega_1\left(\psi'',\frac{1}{\sqrt{n+1}}\right) + \omega_2\left(\psi'',\frac{1}{\sqrt{n+1}}\right) + \frac{1}{2(n+1)}\|\psi'\|_\infty + \frac{1}{n+1}\|\psi''\|_\infty.$$

THEOREM 4.1. *Let* $f,g \in C^2[0,1]$. *Then,*

$$\left\|n\{K_{n,r,s}(fg) - K_{n,r,s}f \cdot K_{n,r,s}g\} - Xf'g'\right\|_\infty \leqslant C\{Q(fg,n) + \|f\|_\infty Q(g,n) + \|g\|_\infty Q(f,n) + 1/n\},$$

where $X := x(1-x)$ *and* C *is any constant depending on* r *and* s *but independent of* n.

Proof. We have

$$\frac{1}{2}(Xf')' = \frac{1}{2}Xf'' + \frac{1}{2}X'f'.$$

Denote

$$L_{n,r,s}f := K_{n,r,s}f - f - \frac{1}{2n}(Xf')'.$$

For $f,g \in C^2[0,1]$, we may write

$$K_{n,r,s}(fg) - K_{n,r,s}f \cdot K_{n,r,s}g - \frac{1}{n}Xf'g' = L_{n,r,s}(fg) - fL_{n,r,s}g - gL_{n,r,s}f + [g - K_{n,r,s}g][K_{n,r,s}f - f]$$

$$-K_{n,r,s}f\cdot K_{n,r,s}g-\frac{1}{n}Xf'g'+fg+\frac{1}{2n}\left(X\left(fg\right)'\right)'$$
$$+fL_{n,r,s}g+gL_{n,r,s}f-\left[g-K_{n,r,s}g\right]\left[K_{n,r,s}f-f\right]. \tag{4.1}$$

By simple computation can be proved that the last two lines of the above relation are equal 0. Hence, (4.1) reduces to

$$K_{n,r,s}\left(fg\right)-K_{n,r,s}f\cdot K_{n,r,s}g-\frac{1}{n}Xf'g'$$
$$=L_{n,r,s}(fg)-fL_{n,r,s}g-gL_{n,r,s}f+\left[g-K_{n,r,s}g\right]\left[K_{n,r,s}f-f\right]. \tag{4.2}$$

Using the generalized estimate [13, Theorem 3], of a quantitative version of the well-known Voronovskaya theorem for a positive linear operators, for any $\psi\in C^2\left[0,1\right]$, we have

$$\left|K_{n,r,s}\left(\psi;x\right)-\psi\left(x\right)-K_{n,r,s}\left(t-x;x\right)\psi'\left(x\right)-\frac{1}{2}K_{n,r,s}\left(\left(t-x\right)^2;x\right)\psi''\left(x\right)\right|$$
$$\leqslant K_{n,r,s}\left(\left(t-x\right)^2;x\right)\left\{\frac{\left|K_{n,r,s}\left(\left(t-x\right)^3;x\right)\right|}{K_{n,r,s}\left(\left(t-x\right)^2;x\right)}\frac{5}{6h}\omega_1\left(\psi'';h\right)\right.$$
$$\left.+\left(\frac{3}{4}+\frac{K_{n,r,s}\left(\left(t-x\right)^4;x\right)}{K_{n,r,s}\left(\left(t-x\right)^2;x\right)}\cdot\frac{1}{16h^2}\right)\omega_2\left(\psi'';h\right)\right\}.$$

Hence, using Lemma 2.6 and choosing $h=\frac{1}{\sqrt{n+1}}$, we have

$$\left|K_{n,r,s}\left(\psi;x\right)-\psi\left(x\right)-\frac{1-2x}{2(n+1)}\psi'\left(x\right)\right.$$
$$\left.-\frac{1}{2(n+1)^2}\left[x(1-x)(r^2s-rs+n-1)+\frac{1}{3}\right]\psi''\left(x\right)\right|$$
$$\leqslant\frac{C}{n+1}\left\{\frac{1}{\sqrt{n+1}}\omega_1\left(\psi'';\frac{1}{\sqrt{n+1}}\right)+\omega_2\left(\psi'';\frac{1}{\sqrt{n+1}}\right)\right\},$$

where C is some positive constant depending on r and s but independent of n.

Multiplying by n both sides,

$$\left|n[K_{n,r,s}\left(\psi;x\right)-\psi\left(x\right)]-\frac{(1-2x)n}{2(n+1)}\psi'\left(x\right)\right.$$
$$\left.-\frac{n}{2(n+1)^2}\left(x(1-x)(r^2s-rs+n-1)+\frac{1}{3}\right)\psi''\left(x\right)\right|$$
$$\leqslant C\left\{\frac{1}{\sqrt{n+1}}\omega_1\left(\psi'';\frac{1}{\sqrt{n+1}}\right)+\omega_2\left(\psi'';\frac{1}{\sqrt{n+1}}\right)\right\}. \tag{4.3}$$

Also,

$$\begin{aligned}
&u\left|n\left[K_{n,r,s}\left(\psi;x\right)-\psi\left(x\right)\right]-\frac{1}{2}\left(X\psi'\right)'\right| \\
&\leqslant \left|n\left[K_{n,r,s}\left(\psi;x\right)-\psi\left(x\right)\right]-\frac{n(1-2x)}{2(n+1)}\psi'\left(x\right)\right. \\
&\left.-\frac{n}{2(n+1)^2}\left(x(1-x)(r^2s-rs+n-1)+\frac{1}{3}\right)\psi''\left(x\right)\right| \\
&+\left|\frac{1-2x}{2(n+1)}\psi'\left(x\right)+\frac{x\left(1-x\right)}{2(n+1)^2}(3n+1-nrs(r-1))\psi''\left(x\right)-\frac{n}{6(n+1)^2}\psi''\left(x\right)\right|.
\end{aligned} \tag{4.4}$$

Using (4.3) and (4.4), we have

$$\begin{aligned}
\left\|nL_{n,r,s}\psi\right\| &\leqslant C\left\{\frac{1}{\sqrt{n+1}}\omega_1\left(\psi'';\frac{1}{\sqrt{n+1}}\right)\right. \\
&\left.+\omega_2\left(\psi'';\frac{1}{\sqrt{n+1}}\right)+\frac{1}{2(n+1)}\left\|\psi'\right\|_\infty+\frac{1}{n+1}\left\|\psi''\right\|_\infty\right\} \\
&=CQ\left(\psi,n\right).
\end{aligned} \tag{4.5}$$

Applying Theorem 2.10, we get for $\psi\in C^2[0,1]$

$$\left\|K_{n,r,s}\psi-\psi\right\|_\infty\leqslant\frac{1}{2(n+1)}\left\|\psi'\right\|_\infty+\frac{1}{n+1}\left(1+\frac{7+3rs\left|r-1\right|}{24}\right)\left\|\psi''\right\|_\infty\leqslant\frac{C}{n}. \tag{4.6}$$

Now, collecting the inequalities (4.2), (4.5) and (4.6) , we have

$$\begin{aligned}
&\left\|n\left\{K_{n,r,s}\left(fg\right)-K_{n,r,s}f\cdot K_{n,r,s}g\right\}-Xf'g'\right\|_\infty \\
&\qquad\leqslant C\left\{Q\left(fg,n\right)+\|f\|_\infty Q\left(g,n\right)+\|g\|_\infty Q\left(f,n\right)+1/n\right\}.
\end{aligned}$$

This completes the proof. □

We shall need the following Lemma in order to give a new Grüss-Voronovskaya type theorem.

LEMMA 4.2. [12] *Let* $I=[0,1]$ *and* $f\in C^\lambda\left(I\right),\lambda\in\mathbb{N}_0$. *For any* $h\in(0,1]$ *and* $\nu\in\mathbb{N}$ *there exists a function* $f_{h,\lambda+\nu}\in C^{2\lambda+\nu}(I)$ *with*

(*i*) $\left\|f^{(j)}-f^{(j)}_{h,\lambda+\nu}\right\|\leqslant C\cdot\omega_{\lambda+\nu}\left(f^{(j)},h\right)$, *for* $0\leqslant j\leqslant\lambda$,

(*ii*) $\left\|f^{(j)}_{h,\lambda+\nu}\right\|\leqslant C\cdot h^{-j}\cdot\omega_j\left(f,h\right)$ *for* $0\leqslant j\leqslant\lambda+\nu$,

(*iii*) $\left\|f^{(j)}_{h,\lambda+\nu}\right\|\leqslant C\cdot h^{-(\lambda+\nu)}\cdot\omega_{\lambda+\nu}\left(f^{(j-\lambda-\nu)},h\right)$ *for* $\lambda+\nu\leqslant j\leqslant 2\lambda+\nu$,

where the constant C *depends only on* λ *and* ν.

THEOREM 4.3. *Let $f,g \in C^1[0,1]$ and $n \geqslant 1$, then there is a constant C independent of n, f and g, such that*

$$\begin{aligned}&\left\|\{n(K_{n,r,s}fg - K_{n,r,s}f \cdot K_{n,r,s}g)\} - Xf'g'\right\|_\infty \\ &\leqslant C\Big\{\omega_3\left(f',n^{-\frac{1}{6}}\right)\omega_3\left(g',n^{-\frac{1}{6}}\right) + \|f'\|_\infty\omega_3\left(g',n^{-\frac{1}{6}}\right) + \|g'\|_\infty\omega_3\left(f',n^{-\frac{1}{6}}\right) \\ &\qquad + \max\left(\frac{\|f'\|_\infty}{n^{\frac{1}{2}}},\omega_3\left(f',n^{-\frac{1}{6}}\right)\right)\cdot\max\left(\frac{\|g'\|_\infty}{n^{\frac{1}{2}}},\omega_3\left(g',n^{-\frac{1}{6}}\right)\right)\Big\}.\end{aligned}$$

Proof. Let

$$J_{n,r,s}(f,g;x) = K_{n,r,s}(fg;x) - K_{n,r,s}(f;x)K_{n,r,s}(g;x) - \frac{x(1-x)}{n}f'(x)g'(x). \tag{4.7}$$

For $f,g \in C^1[0,1]$ fixed and $u,v \in C^4[0,1]$, we have

$$\begin{aligned}|J_{n,r,s}(f,g;x)| &= |J_{n,r,s}(f-u+u,g-v+v;x)| \\ &\leqslant |J_{n,r,s}(f-u,g-v;x)| + |J_{n,r,s}(u,g-v;x)| + |J_{n,r,s}(f-u,v;x)| + |J_{n,r,s}(u,v;x)|.\end{aligned} \tag{4.8}$$

Let $h(x) = x$, $x \in [0,1]$. Using [1, Theorem 4.1], there exists $\eta,\theta \in [0,1]$ such that

$$\begin{aligned}K_{n,r,s}(fg;x) - K_{n,r,s}(f;x)K_{n,r,s}(g;x) &= f'(\eta)g'(\theta)\left[K_{n,r,s}\left(h^2;x\right) - (K_{n,r,s}(h;x))^2\right] \\ &= f'(\eta)g'(\theta)\left\{\frac{nx(1-x)}{(n+1)^2} + \frac{1}{12(n+1)^2} + \frac{rsx(1-x)(r-1)}{(n+1)^2}\right\}.\end{aligned} \tag{4.9}$$

From (4.7) and (4.9), we get

$$\begin{aligned}|nJ_{n,r,s}(f,g;x)| &\leqslant \left[x(1-x)\frac{n^2}{(n+1)^2} + \frac{n}{12(n+1)^2} + x(1-x) + \frac{rsx(1-x)|r-1|n}{(n+1)^2}\right] \\ &\quad \times \|f'\|_\infty\|g'\|_\infty \\ &\leqslant 2\left[x(1-x) + \frac{n}{24(n+1)^2} + \frac{rsx(1-x)|r-1|n}{2(n+1)^2}\right]\|f'\|_\infty\|g'\|_\infty.\end{aligned} \tag{4.10}$$

Using (4.5), for any $\psi \in C^4[0,1]$, we get

$$\left|n[K_{n,r,s}(\psi;x) - \psi(x)] - \frac{1}{2}(X\psi')'(x)\right| \leqslant \frac{C}{n}\left(\|\psi'\|_\infty + \|\psi''\|_\infty + \|\psi'''\|_\infty + \left\|\psi^{(4)}\right\|_\infty\right).$$

But, for $\psi \in C^n[a,b], n \in \mathbb{N}$ one has (see [11], Remark 2.15)

$$\max_{0\leqslant k\leqslant n}\left\{\left\|\psi^{(k)}\right\|\right\} \leqslant C\max\left\{\|\psi\|_\infty, \left\|\psi^{(n)}\right\|_\infty\right\}.$$

Therefore,

$$\left|n[K_{n,r,s}(\psi;x) - \psi(x)] - \frac{1}{2}(X\psi')'(x)\right| \leqslant \frac{C}{n}\max\left\{\|\psi'\|_\infty, \left\|\psi^{(4)}\right\|_\infty\right\}. \tag{4.11}$$

For $u,v\in C^4[0,1]$, using the same decomposition as in the proof of Theorem 4.1, the relation (4.11) and Theorem 2.10, we get

$$\begin{aligned}
|J_{n,r,s}(u,v;x)| &\leqslant \left|K_{n,r,s}(uv;x)-(uv)(x)-\frac{1}{2n}\left(X(uv)'\right)'\right| \\
&\quad +|u(x)|\left|K_{n,r,s}(v;x)-v(x)-\frac{1}{2n}\left(Xv'\right)'\right| \\
&\quad +|v(x)|\left|K_{n,r,s}(u;x)-u(x)-\frac{1}{2n}\left(Xu'\right)'\right| \\
&\quad +|v(x)-K_{n,r,s}(v;x)|\,|K_{n,r,s}(u;x)-u(x)| \\
&\leqslant \frac{C}{n^2}\max\left\{\|u'\|_\infty,\left\|u^{(4)}\right\|_\infty\right\}\max\left\{\|v'\|_\infty,\left\|v^{(4)}\right\|_\infty\right\}. \qquad (4.12)
\end{aligned}$$

From (4.8), (4.10) and (4.12), we get

$$\begin{aligned}
|J_{n,r,s}(u,v;x)| &\leqslant \frac{2}{n}\left\{x(1-x)+\frac{n}{24(n+1)^2}+\frac{rsx(1-x)|r-1|n}{2(n+1)^2}\right\} \\
&\quad \times\{\|(f-u)'\|_\infty\|(g-v)'\|_\infty+\|u'\|_\infty\|(g-v)'\|_\infty+\|(f-u)'\|_\infty\|v'\|_\infty\} \\
&\quad +\frac{C}{n^2}\max\left\{\|u'\|_\infty,\left\|u^{(4)}\right\|_\infty\right\}\max\left\{\|v'\|_\infty,\left\|v^{(4)}\right\|_\infty\right\}.
\end{aligned}$$

Using Lemma 4.2, for $\lambda=1,\nu=2, f_{h,3}=u$ and $g_{h,3}=v$, then for all $h\in(0,1]$ and $n\in\mathbb{N}$, it follows that

$$\begin{aligned}
|J_{n,r,s}(f,g;x)| &\leqslant \frac{C}{n}\left\{\omega_3(f',h)\,\omega_3(g',h)+\frac{1}{h}\omega_1(f,h)\,\omega_3(g',h)+\frac{1}{h}\omega_1(g,h)\,\omega_3(f',h)\right\} \\
&\quad +\frac{C}{n^2}\max\left\{\frac{1}{h}\omega_1(f,h),\frac{1}{h^3}\omega_3(f',h)\right\}.\max\left\{\frac{1}{h}\omega_1(g,h),\frac{1}{h^3}\omega_3(g',h)\right\} \\
&\leqslant \frac{C}{n}\{\omega_3(f',h)\,\omega_3(g',h)+\|f'\|_\infty\omega_3(g',h)+\|g'\|_\infty\omega_3(f',h)\} \\
&\quad +\frac{C}{n^2}\max\left\{\|f'\|_\infty,\frac{1}{h^3}\omega_3(f',h)\right\}.\max\left\{\|g'\|_\infty,\frac{1}{h^3}\omega_3(g',h)\right\}.
\end{aligned}$$

Choosing $h=n^{-\frac{1}{6}}$, we obtain

$$\begin{aligned}
|J_{n,r,s}(f,g;x)| &\leqslant \frac{C}{n}\Bigg\{\omega_3\left(f',n^{-\frac{1}{6}}\right)\omega_3\left(g',n^{-\frac{1}{6}}\right)+\|f'\|_\infty\omega_3\left(g',n^{-\frac{1}{6}}\right) \\
&\quad +\|g'\|_\infty\omega_3\left(f',n^{-\frac{1}{6}}\right) \\
&\quad +\max\left(\frac{\|f'\|_\infty}{n^{\frac{1}{2}}},\omega_3\left(f',n^{-\frac{1}{6}}\right)\right)\cdot\max\left(\frac{\|g'\|_\infty}{n^{\frac{1}{2}}},\omega_3\left(g',n^{-\frac{1}{6}}\right)\right)\Bigg\}. \qquad \square
\end{aligned}$$

Acknowledgement. The authors are extremely thankful to Science and Engineering Research Board (SERB), Govt. of India, for providing the financial support under Teachers Associateship for Research Excellence (TARE) Award (TAR/2018/000356).

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(Received October 10, 2020)

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