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COMPLETE MOMENT CONVERGENCE FOR WEIGHTED SUMS OF m–ASYMPTOTIC NEGATIVELY ASSOCIATED RANDOM VARIABLES

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Abstract. In the paper, we establish the complete moment convergence and complete convergence for weighted sums of arrays of rowwise m-asymptotic negatively associated random variables. As an application, the strong law of large numbers for weighted sums of m-ANA random variables is presented. The obtained results generalize the corresponding ones for NA random variables and ρ^*-mixing random variables.

1. Introduction

In classical probability space $(\Omega, \mathscr{F}, P)$, the limit properties for independent random variables have been quite mature. As data in finance, economics, telecommunications and other fields became more complex, statisticians proposed various dependent variables. Recently, Wu et al. (2021) introduced a new dependent structure: m-asymptotic negatively association (m-ANA, for short).

Now, let us recall the concepts of other related dependent structures before we present the definition of m-ANA. The first one is the concept of negatively associated (NA, for short) random variables, which was introduced by Joag-Dev and Proschan (1983) as follows.

DEFINITION 1.1. A finite family $\{X_i, 1 \leqslant i \leqslant n\}$ of random variables is defined as NA, if for every pair of disjoint subsets A and B of $\{1, 2, \cdots, n\}$,

$$\mathrm{Cov}\left(f_1(X_i, i \in A), f_2(X_j, j \in B)\right) \leqslant 0,$$

whenever f_1 and f_2 are coordinate-wise increasing functions such that the covariance above exists. An infinite family of random variables is NA, if every finite subfamily is NA.

Another important concept of dependent random variables is ρ^*-mixing, which was introduced by Bradley (1992) as follows.

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DEFINITION 1.2. A sequence $\{X_n, n \geqslant 1\}$ of random variables is called ρ^*-mixing, if the mixing coefficient

$$\rho^*(s) = \sup\{\rho(S,T) : S,T \subset \mathbb{N}, \text{dist}(S,T) \geqslant s\} \to 0, \text{ as } s \to \infty,$$

where

$$\rho(S,T) = \sup\left\{ \frac{|EXY - EXEY|}{\sqrt{\text{Var}(X) \cdot \text{Var}(Y)}} : X \in L_2(\sigma(S)), Y \in L_2(\sigma(T)) \right\},$$

$\text{dist}(S,T) = \min_{i \in S, j \in T} |j - i|$, $\sigma(S)$ and $\sigma(T)$ are the σ-fields generated by $\{X_i, i \in S\}$ and $\{X_j, j \in T\}$, respectively.

Since the concepts of NA and ρ^*-mixing random variables were introduced, a number of useful limit results have been established by many authors. For NA random variables, we refer to Matula (1992) for three series theorem and Kolmogorov type inequality, Shao and Su (1999) for the law of the iterated logarithm, Shao (2000) and Yang (2001) for Rosenthal-type moment inequality, Wang et al. (2011) for the strong limit theorems and so on. For ρ^*-mixing random variables, we refer the readers to Peligrad and Gut (1999), Utev and Peligrad (2003), Wu and Jiang (2008), Sung (2013), Wu et al. (2017) and among others.

Zhang and Wang (1999) introduced the following concept of asymptotically negatively associated (ANA, for short) random variables.

DEFINITION 1.3. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be ANA (or ρ^--mixing) if

$$\rho^-(s) = \sup\{\rho^-(S,T) : S,T \subset \mathbb{N}, \text{dist}(S,T) \geqslant s\} \to 0, \text{ as } s \to \infty,$$

where

$$\rho^-(S,T) = 0 \vee \left\{ \frac{\text{Cov}(f_1(X_i, i \in S), f_2(X_j, j \in T))}{\sqrt{\text{Var}(f_1(X_i, i \in S)) \cdot \text{Var}(f_2(X_j, j \in T))}} : f_1, f_2 \in \mathscr{C} \right\},$$

$\mathscr{C}$ is the set of nondecreasing functions.

It is obvious that $\rho^-(s) \leqslant \rho^*(s)$ and a sequence of ANA random variables is NA if and only if $\rho^-(1) = 0$. Thus, the class of ANA random variables includes NA random variables and ρ^*-mixing random variables as special cases. Therefore, it has attracted more and more attention from many statisticians, and many meaningful results have been established. For example, Zhang and Wang (1999) established some Rosenthal type inequalities and discussed the convergence rates in the strong law of large numbers; Zhang (2000a) showed a functional central limit theorem; Zhang (2000b) investigated the central limit theorems under lower moment conditions or the Lindeberg condition; Wang and Lu (2006) established some inequalities for the maximum of partial sums and weak convergence; Wang and Zhang (2007) investigated the law of the iterated logarithm; Liu and Liu (2009) showed moments of the maximum of normed partial sums; Yuan and Wu (2010) obtained the limiting behavior of the maximum of partial sums; Ko (2014) established the Hájek-Rényi inequality and the strong law of large numbers; Tang et al. (2018) studied the asymptotic normality of the wavelet estimator in the nonparametric regression model, and so forth. It is worth noting that NA and ρ^*-mixing are both ANA. However, the converse is not always true, and Zhang and Wang (1999) gave an example of ANA random variables which is neither NA nor ρ^*-mixing.

The following concept of m-ANA random variables, which was introduced by Wu et al. (2021), is a natural extension of ANA random variables.

DEFINITION 1.4. Let $m \geqslant 1$ be a fixed integer. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be m-ANA if for any $n \geqslant 2$ and any $i_1, i_2, \ldots, i_n$ such that $|i_k - i_j| \geqslant m$ for all $1 \leqslant k \neq j \leqslant n$, we have that $X_{i_1}, X_{i_2}, \ldots, X_{i_n}$ are ANA.

An array $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ of random variables is said to be rowwise m-ANA if for every $n \geqslant 1$, $\{X_{ni}, 1 \leqslant i \leqslant n\}$ are m-ANA random variables.

It is easy to see that if $m = 1$, then a sequence of m-ANA random variables is ANA. Moreover, since NA implies ANA, then m-NA, the concept of which was first proposed by Hu et al. (2009), implies m-ANA. Consequently, studying the limit properties of m-ANA random variables is of great interest. Wu et al. (2021) also established a general result on complete moment convergence for weighted sums of m-ANA random variables, and provided some applications in nonparametric models and conditional Value-at-risk estimator. To the best of our knowledge, there are few results for m-ANA random variables. In the paper, we will further investigate the complete moment convergence for arrays of rowwise m-ANA random variables. Recently, Sung (2011) established the following complete convergence for weighted sums of NA random variables.

THEOREM A. *Let $\{X_n, n \geqslant 1\}$ be a sequence of identically distributed NA random variables, and let $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of real constants satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n) \tag{1.1}$$

for some $0 < \alpha \leqslant 2$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Furthermore, suppose that $EX_1 = 0$ for $1 < \alpha \leqslant 2$. If

$$\begin{cases} E|X_1|^{\alpha} < \infty & \text{for } \alpha > \gamma, \\ E|X_1|^{\alpha} \log(1 + |X_1|) < \infty & \text{for } \alpha = \gamma, \\ E|X_1|^{\gamma} < \infty & \text{for } \alpha < \gamma, \end{cases}$$

then

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} a_{ni} X_i \right| > \varepsilon b_n \right) < \infty \text{ for all } \varepsilon > 0.$$

When $\alpha \neq \gamma$, Zhou et al. (2011) extended Theorem A for NA random variables to the case of ρ^*-mixing random variables satisfying

$$\sum_{i=1}^{n} |a_{ni}|^{\max\{\alpha, \gamma\}} = O(n)$$

by using a different method. Sung (2013) obtained that the case $\alpha = \gamma$ of Theorem A holds true for ρ^*-mixing random variables. This is an open problem left by Zhou et al. (2011).

In this paper, under the same conditions of Theorem A, we will further investigate the stronger limit properties: complete moment convergence. The concept of complete moment convergence was introduced by Chow (1988). Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables and a_n, b_n, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|X_n| - \varepsilon\}_+^q < \infty \text{ for all } \varepsilon > 0,$$

then $\{X_n, n \geqslant 1\}$ is said to be complete moment convergent. It is easy to show that it is a more general concept than complete convergence, which was introduced by Hsu and Robbins (1947).

Our purpose of this paper is to establish the complete moment convergence for weighted sums of arrays of rowwise m-ANA random variables . The result obtained in the paper extends Theorem A from complete convergence for NA random variables and ρ^*-mixing random variables to complete moment convergence for m-ANA random variables. As a corollary, we present the strong law of large numbers for weighted sums of m-ANA random variables.

Throughout the paper, C denotes a positive constant which may be different in various places. Let $\log x = \ln\max(x,e)$ and $I(A)$ be the indicator function of the set A. Denote $x^+ = xI(x \geqslant 0)$, and $x^- = xI(x \leqslant 0)$. $a_n = O(b_n)$ stands for $a_n \leqslant Cb_n$.

2. Main results

To prove the main results of the paper, we need the following important lemmas. The first one is a basic property for m-ANA random varibales, which can be referred to Remark 1.2 of Wu et al. (2021).

LEMMA 2.1. *Let random variables $X_1, X_2, \ldots, X_n$ be m-ANA, and $f_1, f_2, \ldots, f_n$ be all nondecreasing (or nonincreasing) functions, then random variables $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ are m-ANA.*

The next lemma is the Rosenthal-type maximum inequality for m-ANA random varibales, which can be obatined in Lemma 3.2 of Wu et al. (2021).

LEMMA 2.2. *Suppose that $\{X_i, i \geqslant 1\}$ is a sequence of m-ANA random variables with $EX_i = 0$, $E|X_i|^p < \infty$ for some $p \geqslant 2$. Then there exists a positive constant C depending only on m, p, and $\rho^-(\cdot)$ such that for all $n \geqslant 1$,*

$$E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} X_i\right|\right)^p \leqslant C\left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}.$$

Now, we state the main results of this paper.

THEOREM 2.1. *Let* $\{X_{ni}, X, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *be an array of rowwise* m*-ANA random variables with identical distribution, and let* $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *be an array of constants satisfying* (1.1) *for some* $0 < \alpha \leqslant 2$. *Assume that* $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ *for some* $\gamma > 0$, *and* $EX = 0$ *for* $1 < \alpha \leqslant 2$. *If*

$$\begin{cases} E|X|^\alpha < \infty & \text{for } \alpha > \gamma, \\ E|X|^\alpha \log(1+|X|) < \infty & \text{for } \alpha = \gamma, \\ E|X|^\gamma < \infty & \text{for } \alpha < \gamma, \end{cases} \tag{2.1}$$

then for $0 < q \leqslant \alpha$,

$$\sum_{n=1}^{\infty} \frac{1}{n} E\left(\frac{1}{b_n} \max_{1 \leqslant j \leqslant n} \left|\sum_{i=1}^{j} a_{ni} X_{ni}\right| - \varepsilon\right)_+^q < \infty \text{ for all } \varepsilon > 0. \tag{2.2}$$

From the proof of Theorem 2.1, we can obtain the following result of complete convergence.

THEOREM 2.2. *Let* $\{X_{ni}, X, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *be an array of rowwise* m*-ANA random variables with identical distribution, and let* $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *be an array of constants satisfying* (1.1) *for some* $0 < \alpha \leqslant 2$. *Assume that* $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ *for some* $\gamma > 0$, *and* $EX = 0$ *for* $1 < \alpha \leqslant 2$. *If* (2.1) *holds, then*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1 \leqslant j \leqslant n} \left|\sum_{i=1}^{j} a_{ni} X_{ni}\right| > \varepsilon b_n\right) < \infty \text{ for all } \varepsilon > 0.$$

REMARK 2.1. Under the conditions of Theorem 2.1, we can obtain that for any $\varepsilon > 0$

$$\begin{aligned} \infty &> \sum_{n=1}^{\infty} \frac{1}{n} E\left(\frac{1}{b_n} \max_{1 \leqslant j \leqslant n} \left|\sum_{i=1}^{j} a_{ni} X_{ni}\right| - \varepsilon\right)_+^q \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} P\left(\frac{1}{b_n} \max_{1 \leqslant j \leqslant n} \left|\sum_{i=1}^{j} a_{ni} X_{ni}\right| - \varepsilon > t^{1/q}\right) dt \\ &\geqslant \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\varepsilon^q} P\left(\max_{1 \leqslant j \leqslant n} \left|\sum_{i=1}^{j} a_{ni} X_{ni}\right| > 2b_n \varepsilon\right) dt \\ &= \varepsilon^q \sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1 \leqslant j \leqslant n} \left|\sum_{i=1}^{j} a_{ni} X_{ni}\right| > 2b_n \varepsilon\right). \end{aligned}$$

Hence, we get that complete moment convergence implies complete convergence.

REMARK 2.2. In Theorems 2.1 and 2.2, the assumption of identical distribution can be weakened to stochastic domination, i.e., there exists a positive constant C such that

$$P(|X_{ni}| > x) \leqslant CP(|X| > x)$$

for all $x \geqslant 0$, $1 \leqslant i \leqslant n$ and $n \geqslant 1$.

COROLLARY 2.1. *Let* $\{X_n, X, n \geqslant 1\}$ *be a sequence of identically distributed m-ANA random variables and* $\{a_n, n \geqslant 1\}$ *be a sequence of constants satisfying* (1.1) *for some* $0 < \alpha \leqslant 2$. *Assume that* $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ *for some* $\gamma > 0$, *and* $EX = 0$ *for* $1 < \alpha \leqslant 2$. *If* (2.1) *holds, then*

$$\frac{1}{b_n}\sum_{i=1}^{n} a_i X_i \to 0 \ a.s., \text{ as } n \to \infty.$$

3. Proofs of main results

In this section, we give the detailed proofs of our main results.

Proof of Theorem 2.1. For all $\varepsilon > 0$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n}E\left(\frac{1}{b_n}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}X_{ni}\right|-\varepsilon\right)_+^q \\
&=\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\infty}P\left(\frac{1}{b_n}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}X_{ni}\right|-\varepsilon>t^{1/q}\right)dt \\
&=\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{1}P\left(\frac{1}{b_n}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}X_{ni}\right|-\varepsilon>t^{1/q}\right)dt \\
&\quad+\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}P\left(\frac{1}{b_n}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}X_{ni}\right|-\varepsilon>t^{1/q}\right)dt \\
&\leqslant\sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}X_{ni}\right|>\varepsilon b_n\right)+\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}a_{ni}X_{ni}\right|>b_nt^{1/q}\right)dt \\
&\doteq I_1+I_2.
\end{aligned} \tag{3.1}$$

To prove (2.2), it is sufficient to show that $I_1 < \infty$ and $I_2 < \infty$.

Next, we give the proof of $I_1 < \infty$. Without loss of generality, assume that $\sum_{i=1}^{n}|a_{ni}|^\alpha \leqslant n$, and $a_{ni} \geqslant 0$ (otherwise we shall use a_{ni}^+ and a_{ni}^- instead of a_{ni}, and note that $a_{ni} = a_{ni}^+ - a_{ni}^-$). For fixed $n \geqslant 1$, denote for $1 \leqslant i \leqslant n$ that

$$Y_{ni} = -b_nI(a_{ni}X_{ni} < -b_n) + a_{ni}X_{ni}I(|a_{ni}X_{ni}| \leqslant b_n) + b_nI(a_{ni}X_{ni} > b_n),$$

$$Z_{ni} = (a_{ni}X_{ni} + b_n)I(a_{ni}X_{ni} < -b_n) + (a_{ni}X_{ni} - b_n)I(a_{ni}X_{ni} > b_n).$$

First, we will show that

$$\frac{1}{b_n}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}EY_{ni}\right| \to 0, \text{ as } n \to \infty. \tag{3.2}$$

For $0<\alpha\leqslant 1$, we have by Markov's inequality and $E|X|^\alpha<\infty$ that

$$\begin{aligned}\frac{1}{b_n}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}EY_{ni}\right| &\leqslant \frac{1}{b_n}\sum_{i=1}^{n}E|Y_{ni}|\\ &\leqslant \frac{1}{b_n}\sum_{i=1}^{n}E|a_{ni}X|I(|a_{ni}X|\leqslant b_n)+\sum_{i=1}^{n}P(|a_{ni}X|>b_n)\\ &\leqslant \frac{1}{b_n^\alpha}\sum_{i=1}^{n}a_{ni}^\alpha E|X|^\alpha I(|a_{ni}X|\leqslant b_n)+\frac{1}{b_n^\alpha}\sum_{i=1}^{n}a_{ni}^\alpha E|X|^\alpha\\ &\leqslant C(\log n)^{-\alpha/\gamma}E|X|^\alpha\to 0,\ \text{as } n\to\infty.\end{aligned}$$

For $1<\alpha\leqslant 2$, we can also obtain by $EX=0$ and $E|X|^\alpha<\infty$ that

$$\begin{aligned}\frac{1}{b_n}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}EY_{ni}\right| &= \frac{1}{b_n}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}EZ_{ni}\right|\\ &\leqslant \frac{1}{b_n}\sum_{i=1}^{n}E|Z_{ni}|\\ &\leqslant \frac{1}{b_n}\sum_{i=1}^{n}E|a_{ni}X|I(|a_{ni}X|>b_n)\\ &\leqslant \frac{1}{b_n^\alpha}\sum_{i=1}^{n}a_{ni}^\alpha E|X|^\alpha I(|a_{ni}X|>b_n)\\ &\leqslant C(\log n)^{-\alpha/\gamma}E|X|^\alpha\to 0,\ \text{as } n\to\infty.\end{aligned}$$

Thus, (3.2) holds true, which implies that for any $\varepsilon>0$ and all n large enough,

$$\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}EY_{ni}\right|\leqslant \varepsilon b_n/2.$$

Noting that

$$\begin{aligned}I_1 &\leqslant \sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}Y_{ni}\right|>\varepsilon b_n\right)+\sum_{n=1}^{\infty}\frac{1}{n}P\left(\bigcup_{i=1}^{n}(|a_{ni}X_{ni}|>b_n)\right)\\ &\leqslant C\sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\right|>\varepsilon b_n/2\right)+C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}P(|a_{ni}X|>b_n)\\ &\doteq J_1+J_2.\end{aligned}\tag{3.3}$$

For J_1, let $p>\max\{2,\gamma,2\gamma/\alpha\}$. It follows from Markov's inequality, Lemma 2.1, Lemma 2.2, C_r inequality and Jensen's inequality that

$$\begin{aligned}J_1 &\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\right|^p\right)\\ &\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\sum_{i=1}^{n}E|Y_{ni}-EY_{ni}|^p+C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\left(\sum_{i=1}^{n}E(Y_{ni}-EY_{ni})^2\right)^{p/2}\end{aligned}$$

$$\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\sum_{i=1}^{n}E|Y_{ni}|^p+C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\left(\sum_{i=1}^{n}EY_{ni}^2\right)^{p/2}$$
$$\doteq J_{11}+J_{12}. \tag{3.4}$$

Note that

$$\begin{aligned}J_{11} &\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\sum_{i=1}^{n}E|a_{ni}X|^pI(|a_{ni}X|\leqslant b_n)+C\sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}P(|a_{ni}X|>b_n)\\ &\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\sum_{i=1}^{n}E|a_{ni}X|^pI(|a_{ni}X|\leqslant b_n)+C\sum_{n=1}^{\infty}\frac{1}{nb_n^\alpha}\sum_{i=1}^{n}E|a_{ni}X|^\alpha I(|a_{ni}X|>b_n)\\ &\doteq J_{111}+J_{112}.\end{aligned} \tag{3.5}$$

Actually, it is easy to show by (1.1) that

$$\begin{aligned}J_{112} &= C\sum_{n=1}^{\infty}n^{-2}(\log n)^{-\alpha/\gamma}\sum_{i=1}^{n}E|a_{ni}X|^\alpha I(|a_{ni}X|^\alpha>n(\log n)^{\alpha/\gamma})\\ &\leqslant C\sum_{n=1}^{\infty}n^{-2}(\log n)^{-\alpha/\gamma}\sum_{i=1}^{n}E|a_{ni}X|^\alpha I\left(\sum_{i=1}^{n}|a_{ni}|^\alpha|X|^\alpha>n(\log n)^{\alpha/\gamma}\right)\\ &\leqslant C\sum_{n=1}^{\infty}n^{-2}(\log n)^{-\alpha/\gamma}\sum_{i=1}^{n}E|a_{ni}X|^\alpha I\left(|X|>(\log n)^{1/\gamma}\right)\\ &\leqslant C\sum_{n=1}^{\infty}n^{-1}(\log n)^{-\alpha/\gamma}E|X|^\alpha I\left(|X|>(\log n)^{1/\gamma}\right)\\ &= C\sum_{n=1}^{\infty}n^{-1}(\log n)^{-\alpha/\gamma}\sum_{k=n}^{\infty}E|X|^\alpha I(\log k<|X|^\gamma\leqslant\log(k+1))\\ &= C\sum_{k=1}^{\infty}E|X|^\alpha I(\log k<|X|^\gamma\leqslant\log(k+1))\sum_{n=1}^{k}n^{-1}(\log n)^{-\alpha/\gamma}\\ &\leqslant\begin{cases}C\sum_{k=1}^{\infty}E|X|^\alpha I(\log k<|X|^\gamma\leqslant\log(k+1)) & \text{for } \alpha>\gamma,\\ C\sum_{k=1}^{\infty}(\log\log k)E|X|^\alpha I(\log k<|X|^\gamma\leqslant\log(k+1)) & \text{for } \alpha=\gamma,\\ C\sum_{k=1}^{\infty}(\log k)^{1-\alpha/\gamma}E|X|^\alpha I(\log k<|X|^\gamma\leqslant\log(k+1)) & \text{for } \alpha<\gamma,\end{cases}\\ &\leqslant\begin{cases}CE|X|^\alpha<\infty & \text{for } \alpha>\gamma,\\ CE|X|^\alpha\log(1+|X|)<\infty & \text{for } \alpha=\gamma,\\ CE|X|^\gamma<\infty & \text{for } \alpha<\gamma.\end{cases}\end{aligned} \tag{3.6}$$

For $j\geqslant 1$ and $n\geqslant 2$, let

$$I_{nj}=\{1\leqslant i\leqslant n: n^{1/\alpha}(j+1)^{-1/\alpha}<|a_{ni}|\leqslant n^{1/\alpha}j^{-1/\alpha}\}.$$

Thus, $\{I_{nj}, j\geqslant 1\}$ are disjoint, and $\bigcup_{j\geqslant 1}I_{nj}=\{1\leqslant i\leqslant n: 0<|a_{ni}|\leqslant n^{1/\alpha}\}$. It follows by (1.1) that

$$\begin{aligned} n &\geqslant \sum_{i=1}^{n}|a_{ni}|^{\alpha}=\sum_{j=1}^{\infty}\sum_{i\in I_{nj}}|a_{ni}|^{\alpha}\geqslant\sum_{j=1}^{\infty}\sharp I_{nj}n(j+1)^{-1} \\ &\geqslant \sum_{j=k}^{\infty}\sharp I_{nj}n(j+1)^{-1}=\sum_{j=k}^{\infty}\sharp I_{nj}n(j+1)^{-p/\alpha}(j+1)^{p/\alpha-1} \\ &\geqslant \sum_{j=k}^{\infty}\sharp I_{nj}n(j+1)^{-p/\alpha}(k+1)^{p/\alpha-1}, \end{aligned}$$

which implies that for $k\geqslant 1$,

$$\sum_{j=k}^{\infty}\sharp I_{nj}j^{-p/\alpha}\leqslant C(k+1)^{1-p/\alpha}. \tag{3.7}$$

It is easily checked that

$$\begin{aligned} J_{111} &\leqslant C\sum_{n=2}^{\infty}n^{-1-p/\alpha}(\log n)^{-p/\gamma}\sum_{i=1}^{n}E|a_{ni}X|^{p}I(|a_{ni}X|\leqslant n^{1/\alpha}(\log n)^{1/\gamma}) \\ &= C\sum_{n=2}^{\infty}n^{-1-p/\alpha}(\log n)^{-p/\gamma}\sum_{j=1}^{\infty}\sum_{i\in I_{nj}}|a_{ni}|^{p}E|X|^{p}I(|a_{ni}X|\leqslant n^{1/\alpha}(\log n)^{1/\gamma}) \\ &\leqslant C\sum_{n=2}^{\infty}n^{-1-p/\alpha}(\log n)^{-p/\gamma}\sum_{j=1}^{\infty}\sharp I_{nj}n^{p/\alpha}j^{-p/\alpha}E|X|^{p}I(|X|\leqslant (j+1)^{1/\alpha}(\log n)^{1/\gamma}) \\ &= C\sum_{n=2}^{\infty}n^{-1}(\log n)^{-p/\gamma}\sum_{j=1}^{\infty}\sharp I_{nj}j^{-p/\alpha}E|X|^{p}I(|X|\leqslant(\log n)^{1/\gamma}) \\ &\quad +C\sum_{n=2}^{\infty}n^{-1}(\log n)^{-p/\gamma}\sum_{j=1}^{\infty}\sharp I_{nj}j^{-p/\alpha}E|X|^{p} \\ &\quad \times I((\log n)^{1/\gamma}<|X|\leqslant (j+1)^{1/\alpha}(\log n)^{1/\gamma}) \\ &\doteq J_{1111}+J_{1112}. \end{aligned} \tag{3.8}$$

If $\alpha>\gamma$, by (3.7) and $p>\alpha$, we have

$$\begin{aligned} J_{1111} &\leqslant C\sum_{n=2}^{\infty}n^{-1}(\log n)^{-p/\gamma}E|X|^{p}I(|X|\leqslant(\log n)^{1/\gamma}) \\ &\leqslant C\sum_{n=2}^{\infty}n^{-1}(\log n)^{-\alpha/\gamma}E|X|^{\alpha}I(|X|\leqslant(\log n)^{1/\gamma}) \\ &\leqslant CE|X|^{\alpha}\sum_{n=2}^{\infty}n^{-1}(\log n)^{-\alpha/\gamma}\leqslant CE|X|^{\alpha}<\infty. \end{aligned} \tag{3.9}$$

If $\alpha \leqslant \gamma$, by (3.7) and $p > \gamma$, we have

$$\begin{aligned} J_{1111} &\leqslant C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-p/\gamma} E|X|^p I(|X| \leqslant (\log n)^{1/\gamma}) \\ &\leqslant C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-p/\gamma} \sum_{m=2}^{n} E|X|^p I(\log(m-1) < |X|^\gamma \leqslant \log m) \\ &\leqslant C\sum_{m=2}^{\infty} E|X|^p I(\log(m-1) < |X|^\gamma \leqslant \log m) \sum_{n=m}^{\infty} n^{-1}(\log n)^{-p/\gamma} \\ &\leqslant C\sum_{m=2}^{\infty} (\log m)^{1-p/\gamma} E|X|^p I(\log(m-1) < |X|^\gamma \leqslant \log m) \\ &\leqslant CE|X|^\gamma < \infty. \end{aligned} \tag{3.10}$$

Next, we consider J_{1112}. By (3.7) and (3.6), we can obtain

$$\begin{aligned} J_{1112} &\leqslant C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-p/\gamma} \\ &\quad \times \sum_{j=1}^{\infty} \sharp I_{nj} j^{-p/\alpha} \sum_{k=1}^{j} E|X|^p I(k^{1/\alpha}(\log n)^{1/\gamma} < |X| \leqslant (k+1)^{1/\alpha}(\log n)^{1/\gamma}) \\ &= C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-p/\gamma} \\ &\quad \times \sum_{k=1}^{\infty} E|X|^p I(k^{1/\alpha}(\log n)^{1/\gamma} < |X| \leqslant (k+1)^{1/\alpha}(\log n)^{1/\gamma}) \sum_{j=k}^{\infty} \sharp I_{nj} j^{-p/\alpha} \\ &\leqslant C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-p/\gamma} \\ &\quad \times \sum_{k=1}^{\infty} (k+1)^{1-p/\alpha} E|X|^p I(k^{1/\alpha}(\log n)^{1/\gamma} < |X| \leqslant (k+1)^{1/\alpha}(\log n)^{1/\gamma}) \\ &\leqslant C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-\alpha/\gamma} \\ &\quad \times \sum_{k=1}^{\infty} E|X|^\alpha I(k^{1/\alpha}(\log n)^{1/\gamma} < |X| \leqslant (k+1)^{1/\alpha}(\log n)^{1/\gamma}) \\ &= C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-\alpha/\gamma} E|X|^\alpha I(|X| > (\log n)^{1/\gamma}) \\ &\leqslant \begin{cases} CE|X|^\alpha < \infty & \text{for } \alpha > \gamma, \\ CE|X|^\alpha \log(1+|X|) < \infty & \text{for } \alpha = \gamma, \\ CE|X|^\gamma < \infty & \text{for } \alpha < \gamma. \end{cases} \end{aligned} \tag{3.11}$$

Combined with (3.5), (3.6), (3.8)–(3.11), $J_{11} < \infty$ can be obtained immediately.

In view of Markov's inequality and $p > 2\gamma/\alpha$, we have

$$\begin{aligned} J_{12} &= C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\left(\sum_{i=1}^{n}\left[E|a_{ni}X|^2 I(|a_{ni}X|\leqslant b_n)+b_n^2P(|a_{ni}X|>b_n)\right]\right)^{p/2} \\ &\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\left(\sum_{i=1}^{n}b_n^{2-\alpha}E|a_{ni}X|^{\alpha}I(|a_{ni}X|\leqslant b_n)+\sum_{i=1}^{n}b_n^{2-\alpha}E|a_{ni}X|^{\alpha}\right)^{p/2} \\ &\leqslant C\sum_{n=1}^{\infty}n^{-1}(\log n)^{-\alpha p/(2\gamma)}(E|X|^{\alpha})^{p/2}<\infty. \end{aligned} \tag{3.12}$$

Therefore, according to (3.4) and (3.12), we have $J_1 < \infty$. In addition, similar to the proof of J_{112}, we can get that $J_2 < \infty$. This completes the proof of $I_1 < \infty$ by (3.3).

Applying the similar method of $I_1 < \infty$, we prove $I_2 < \infty$. For fixed $n \geqslant 1$ and any $t \geqslant 1$, denote for $1 \leqslant i \leqslant n$ that

$$Y_{ni}^{'} = -b_nt^{1/q}I(a_{ni}X_{ni}<-b_nt^{1/q})+a_{ni}X_{ni}I(|a_{ni}X_{ni}|\leqslant b_nt^{1/q})+b_nt^{1/q}I(a_{ni}X_{ni}>b_nt^{1/q}),$$

$$Z_{ni}^{'} = (a_{ni}X_{ni}+b_nt^{1/q})I(a_{ni}X_{ni}<-b_nt^{1/q})+(a_{ni}X_{ni}-b_nt^{1/q})I(a_{ni}X_{ni}>b_nt^{1/q}).$$

First, we shall show that

$$\max_{t\geqslant 1}\frac{1}{b_nt^{1/q}}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}EY_{ni}^{'}\right|\to 0,\ \text{as } n\to\infty.$$

For $0 < \alpha \leqslant 1$, by Markov's inequality and $E|X|^{\alpha} < \infty$, we can obatin that

$$\begin{aligned} \max_{t\geqslant 1}\frac{1}{b_nt^{1/q}}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}EY_{ni}^{'}\right| &\leqslant \max_{t\geqslant 1}\frac{1}{b_nt^{1/q}}\sum_{i=1}^{n}E|Y_{ni}^{'}| \\ &\leqslant \max_{t\geqslant 1}\frac{1}{b_nt^{1/q}}\sum_{i=1}^{n}E|a_{ni}X|I(|a_{ni}X|\leqslant b_nt^{1/q}) \\ &\quad +\max_{t\geqslant 1}\sum_{i=1}^{n}P(|a_{ni}X|>b_nt^{1/q}) \\ &\leqslant \max_{t\geqslant 1}\frac{1}{b_n^{\alpha}t^{\alpha/q}}\sum_{i=1}^{n}a_{ni}^{\alpha}E|X|^{\alpha}I(|a_{ni}X|\leqslant b_nt^{1/q}) \\ &\quad +\max_{t\geqslant 1}\frac{1}{b_n^{\alpha}t^{\alpha/q}}\sum_{i=1}^{n}a_{ni}^{\alpha}E|X|^{\alpha} \\ &\leqslant C(\log n)^{-\alpha/\gamma}E|X|^{\alpha}\to 0,\ \text{as } n\to\infty. \end{aligned}$$

For $1 < \alpha \leqslant 2$, we can also obtain by $EX = 0$ and $E|X|^{\alpha} < \infty$ that

$$\begin{aligned} \max_{t\geqslant 1}\frac{1}{b_nt^{1/q}}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}EY_{ni}^{'}\right| &= \max_{t\geqslant 1}\frac{1}{b_nt^{1/q}}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}EZ_{ni}^{'}\right| \\ &\leqslant \max_{t\geqslant 1}\frac{1}{b_nt^{1/q}}\sum_{i=1}^{n}E|Z_{ni}^{'}| \end{aligned}$$

$$\begin{aligned}&\leqslant \max_{t\geqslant 1}\frac{1}{b_n t^{1/q}}\sum_{i=1}^{n}E|a_{ni}X|I(|a_{ni}X|>b_n t^{1/q})\\&\leqslant \max_{t\geqslant 1}\frac{1}{b_n^{\alpha} t^{\alpha/q}}\sum_{i=1}^{n}a_{ni}^{\alpha}E|X|^{\alpha}I(|a_{ni}X|>b_n t^{1/q})\\&\leqslant C(\log n)^{-\alpha/\gamma}E|X|^{\alpha}\to 0,\ \text{as } n\to\infty.\end{aligned}$$

Thus, for any $t\geqslant 1$ and all n large enough,

$$\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}EY_{ni}^{'}\right|\leqslant b_n t^{1/q}/2,$$

which implies that

$$\begin{aligned}I_2 &\leqslant \sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}Y_{ni}^{'}\right|>b_n t^{1/q}\right)dt\\&\quad+\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}P\left(\bigcup_{i=1}^{n}(|a_{ni}X_{ni}|>b_n t^{1/q})\right)dt\\&\leqslant C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}(Y_{ni}^{'}-EY_{ni}^{'})\right|>b_n t^{1/q}/2\right)dt\\&\quad+C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\sum_{i=1}^{n}P(|a_{ni}X|>b_n t^{1/q})dt\\&\doteq J_3+J_4.\end{aligned}\tag{3.13}$$

On the basis of $q\leqslant\alpha$ and $J_{112}<\infty$, we have

$$\begin{aligned}J_4 &= C\sum_{n=1}^{\infty}\frac{1}{nb_n^{q}}\sum_{i=1}^{n}E|a_{ni}X|^{q}I(|a_{ni}X|>b_n)\\&\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^{\alpha}}\sum_{i=1}^{n}E|a_{ni}X|^{\alpha}I(|a_{ni}X|>b_n)<\infty.\end{aligned}\tag{3.14}$$

Taking $p>\max\{2,\gamma,2\gamma/\alpha\}$, it follows from Markov's inequality, Lemma 2.1, Lemma 2.2, C_r inequality and Jensen's inequality that

$$\begin{aligned}J_3 &\leqslant C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{1}{b_n^{p}t^{p/q}}E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}(Y_{ni}^{'}-EY_{ni}^{'})\right|^{p}\right)dt\\&\leqslant C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{1}{b_n^{p}t^{p/q}}\sum_{i=1}^{n}E|Y_{ni}^{'}|^{p}dt+C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{1}{b_n^{p}t^{p/q}}\left(\sum_{i=1}^{n}E(Y_{ni}^{'})^{2}\right)^{p/2}dt\\&\doteq J_{31}+J_{32}.\end{aligned}\tag{3.15}$$

To prove $J_3<\infty$, it is sufficient to show that $J_{31}<\infty$ and $J_{32}<\infty$.

Note that

$$\begin{aligned} J_{31} &\leqslant C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\sum_{i=1}^{n}P(|a_{ni}X_{ni}|>b_nt^{1/q})dt \\ &\quad +C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\int_1^{\infty}\frac{1}{t^{p/q}}\sum_{i=1}^{n}E|a_{ni}X_{ni}|^pI(|a_{ni}X_{ni}|\leqslant b_nt^{1/q})dt \\ &= C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\sum_{i=1}^{n}P(|a_{ni}X|>b_nt^{1/q})dt \\ &\quad +C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\int_1^{\infty}\frac{1}{t^{p/q}}\sum_{i=1}^{n}E|a_{ni}X|^pI(|a_{ni}X|\leqslant b_n)dt \\ &\quad +C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\int_1^{\infty}\frac{1}{t^{p/q}}\sum_{i=1}^{n}E|a_{ni}X|^pI(b_n<|a_{ni}X|\leqslant b_nt^{1/q})dt \\ &\doteq J_{311}+J_{312}+J_{313}. \end{aligned}$$

It is obvious that $J_{311}<\infty$ from $J_4<\infty$. By $p>q$ and $J_{111}<\infty$, we have

$$J_{312}\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\sum_{i=1}^{n}E|a_{ni}X|^pI(|a_{ni}X|\leqslant b_n)<\infty.$$

Taking $t=x^q$, by $p>\alpha\geqslant q$ and $J_{112}<\infty$, we have

$$\begin{aligned} J_{313} &= C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\int_1^{\infty}x^{q-1-p}\sum_{i=1}^{n}E|a_{ni}X|^pI(b_n<|a_{ni}X|\leqslant b_nx)dx \\ &= C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\sum_{m=1}^{\infty}\int_m^{m+1}x^{q-1-p}\sum_{i=1}^{n}E|a_{ni}X|^pI(b_n<|a_{ni}X|\leqslant b_nx)dx \\ &\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\sum_{m=1}^{\infty}m^{q-1-p}\sum_{i=1}^{n}E|a_{ni}X|^pI(b_n<|a_{ni}X|\leqslant b_n(m+1)) \\ &= C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\sum_{i=1}^{n}\sum_{m=1}^{\infty}\sum_{k=1}^{m}m^{q-1-p}E|a_{ni}X|^pI(b_nk<|a_{ni}X|\leqslant b_n(k+1)) \\ &= C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\sum_{i=1}^{n}\sum_{k=1}^{\infty}E|a_{ni}X|^pI(b_nk<|a_{ni}X|\leqslant b_n(k+1))\sum_{m=k}^{\infty}m^{q-1-p} \\ &\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^p}\sum_{i=1}^{n}\sum_{k=1}^{\infty}E|a_{ni}X|^pI(b_nk<|a_{ni}X|\leqslant b_n(k+1))k^{q-p} \\ &\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^q}\sum_{i=1}^{n}E|a_{ni}X|^qI(|a_{ni}X|>b_n) \\ &\leqslant C\sum_{n=1}^{\infty}\frac{1}{nb_n^\alpha}\sum_{i=1}^{n}E|a_{ni}X|^\alpha I(|a_{ni}X|>b_n)<\infty. \end{aligned}$$

Hence, $J_{31} < \infty$. Since $p > 2\gamma/\alpha$, we have

$$\begin{aligned} J_{32} &= C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{1}{b_n^p t^{p/q}}\left(\sum_{i=1}^{n}E|a_{ni}X_{ni}|^2 I(|a_{ni}X_{ni}|\leqslant b_n t^{1/q})\right.\\ &\quad\left.+\sum_{i=1}^{n} b_n^2 t^{2/q} P(|a_{ni}X_{ni}| > b_n t^{1/q})\right)^{p/2} dt\\ &\leqslant C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{1}{b_n^p t^{p/q}}\left(\sum_{i=1}^{n} b_n^{2-\alpha} t^{(2-\alpha)/q} E|a_{ni}X_{ni}|^{\alpha} I(|a_{ni}X_{ni}|\leqslant b_n t^{1/q})\right.\\ &\quad\left.+\sum_{i=1}^{n} b_n^{2-\alpha} t^{(2-\alpha)/q} E|a_{ni}X_{ni}|^{\alpha} I(|a_{ni}X_{ni}| > b_n t^{1/q})\right)^{p/2} dt\\ &= C\sum_{n=1}^{\infty}\frac{1}{n}\int_1^{\infty}\frac{1}{b_n^p t^{p/q}}\left(\sum_{i=1}^{n} b_n^{2-\alpha} t^{(2-\alpha)/q} E|a_{ni}X|^{\alpha}\right)^{p/2} dt\\ &\leqslant C\sum_{n=1}^{\infty} n^{-1+p/2} b_n^{-\alpha p/2}\int_1^{\infty} t^{-\alpha p/2q} dt\cdot (E|X|^{\alpha})^{p/2}\\ &= C(E|X|^{\alpha})^{p/2}\sum_{n=1}^{\infty} n^{-1}(\log n)^{-\alpha p/(2\gamma)} < \infty. \end{aligned}$$

Combining with (3.13)–(3.15), we can obtain $I_2 < \infty$. Therefore, the desired result (2.2) is established by (3.1).

The proof is completed. □

Proof of Corollary 2.1. Let $a_{ni} = a_i$ and $X_{ni} = X_i$ for each $1 \leqslant i \leqslant n$, $n \geqslant 1$ in Theorem 2.2, we have that for any $\varepsilon > 0$,

$$\begin{aligned} \infty &> \sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_i X_i\right| > \varepsilon b_n\right)\\ &= \sum_{k=0}^{\infty}\sum_{n=2^k}^{2^{k+1}-1}\frac{1}{n}P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_i X_i\right| > \varepsilon b_n\right)\\ &\geqslant \frac{1}{2}\sum_{k=0}^{\infty}P\left(\max_{1\leqslant j\leqslant 2^k}\left|\sum_{i=1}^{j} a_i X_i\right| > \varepsilon (2^{k+1})^{1/\alpha}(\log 2^{k+1})^{1/\gamma}\right), \end{aligned}$$

which together with Borel-Cantelli lemma yields that as $k \to \infty$,

$$\frac{1}{(2^k)^{1/\alpha}(\log 2^k)^{1/\gamma}}\max_{1\leqslant j\leqslant 2^{k+1}}\left|\sum_{i=1}^{j} a_i X_i\right| \to 0\ a.s..$$

On the other hand, for any fixed n, there always exists k such that $2^k \leqslant n < 2^{k+1}$. Thus we can obtain that

$$\frac{1}{n^{1/\alpha}(\log n)^{1/\gamma}}\left|\sum_{i=1}^{n} a_i X_i\right| \leqslant \frac{1}{(2^k)^{1/\alpha}(\log 2^k)^{1/\gamma}}\max_{1\leqslant j\leqslant 2^{k+1}}\left|\sum_{i=1}^{j} a_i X_i\right| \to 0\ a.s.,\ \text{as } k\to\infty.$$

The proof is completed. □

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