Journal of
Mathematical
Inequalities
Volume 15, Number 4 (2021), 1581–1596 doi:10.7153/jmi-2021-15-108

SELF–ADAPTIVE ALGORITHMS FOR AN EQUILIBRIUM SPLIT PROBLEM IN HILBERT SPACES

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(*Communicated by J. Kyu Kim*)

Abstract. In this paper, we propose and study iterative algorithms for solving the split problem: find a common element $x^\dagger \in C$ satisfying

$$\Theta(x^\dagger, y) + \langle Fx^\dagger, y - x^\dagger \rangle + \psi(x^\dagger, y) - \psi(x^\dagger, x^\dagger) \geqslant 0, \ \ \forall y \in C$$

and

$$Au \in Fix(S),$$

where S be an L-Lipschitzian quasi-pseudo-contractive operator. Weak and strong convergence theorems are given under some mild assumptions.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. Let $\Theta : C \times C \to R$ and $\psi : H \times H \to R$ be nonlinear bifunctions.

Recall that equilibrium problems aim to find an element $x^\dagger \in C$ such that

$$\Theta(x^\dagger, x) \geqslant 0, \ \ \forall x \in C, \tag{1.1}$$

which have been initially introduced by Blum and Oettli [7]. Equilibrium problems have proved very interest and useful, for they provide a novel and unified method to deal with various problems arising in pure and applied sciences, such as image reconstruction, network, economics, finance, ecology, optimization, elasticity and transportation. A large number of important problems can be regarded as special cases, for instance, fixed point problem, variational inequalities, game theory and Nash equilibrium. The iterative methods has been studied for the equilibrium problem (1.1) by many authors, see for instance ([9], [14]–[17]).

More generally, we consider the following mixed equilibrium problem: find $x^\dagger \in C$ such that

$$\Theta(x^\dagger, x) + \langle Fx^\dagger, y - x^\dagger \rangle + \psi(x^\dagger, x) - \psi(x^\dagger, x^\dagger) \geqslant 0, \ \ \forall x \in C. \tag{1.2}$$

Mathematics subject classification (2020): Primary 47H06, 47H09, 49J05, 47J25.
Keywords and phrases: Equilibrium split problem, quasi-pseudo-contractive operator, self-adaptive algorithm.

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whose solutions set is denoted by $EP(F)$. If $F \equiv 0$, then the mixed equilibrium problem (1.2) becomes the following mixed equilibrium problem: find $x^{\dagger} \in C$ such that

$$\Theta(x^{\dagger},x) + \psi(x^{\dagger},x) - \psi(x^{\dagger},x^{\dagger}) \geqslant 0, \ \ \forall x \in C \tag{1.3}$$

whose solutions set is denoted by $EP(1)$. These problems have been studied by many authors, see for instance [1, 5, 11, 12, 13, 18].

Recall that, the split common fixed point problem is to find an element $u \in H_1$ such that

$$u \in Fix(T) \quad \text{and} \quad Au \in Fix(S). \tag{1.4}$$

The split feasibility problem is to find an element satisfying

$$u \in C \quad \text{and} \quad Au \in Q. \tag{1.5}$$

The split common fixed point problem can be regarded as a generation of the split feasibility problem which characterizes various inverse problems arising in many real-world application problems, such as medical image reconstruction and intensity-modulated radiation therapy. The original split feasibility problem was introduced firstly by Censor and Elfving [24] in finite-dimensional Hilbert spaces, and has got many attention ever since and many iterative algorithms have been presented to solve these problems, see for example ([21], [23]–[26], [28], [30], [33]) and references therein.

Problem (1.4) was firstly introduced by Censor and Segal [27]. Note that solving (1.4) can be translated to solve the fixed point equation

$$u = S(u - \tau A^{*}(I - T)Au), \ \tau > 0.$$

Whereafter, Censor and Segal proposed an algorithm for directed operators. Since then, there has been growing interest in the split common fixed point problem ([20], [22], [29], [31], [32], [34], [38]).

Very recently, Yao et al. [10] present a new iterative algorithm for finding a common element of the set of solutions of a mixed equilibrium problem and the set of fixed points of a nonexpansive mapping and the set of a variational inclusion in a real Hilbert space. Yao et al. [37] proposed a new self-adaptive iterative algorithms for the split common fixed point problem of demicontractive operators. Onjai-Uea et al. ([6]) proposed the iterative algorithm to solve the problems for finding a common elements of the set of solution of the split equilibrium problem and the fixed point of hybrid-type multivalued mappings in Hilbert spaces.

Motivated and inspired by the above results and related literature, in the present paper, we consider the following problem: find a common element $x^{\dagger} \in C$ satisfying

$$\Theta(x^{\dagger},y) + \langle Fx^{\dagger}, y - x^{\dagger}\rangle + \psi(x^{\dagger},y) - \psi(x^{\dagger},x^{\dagger}) \geqslant 0, \ \ \forall y \in \mathscr{C}. \tag{1.6}$$

and

$$Au \in Fix(S).$$

In order to solve the problem, we construct an iterative algorithm. Under some mild assumptions, weak and strong convergence theorems are given.

2. Preliminaries

In this section, we collect some tools including some definitions, some useful inequalities and lemmas which will be used to derive our main results in the next section.

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Let $T: C \longrightarrow C$ be an operator. We use $Fix(T)$ to denote the set of fixed points of T, that is, $Fix(T) = \{x^\dagger | x^\dagger = Tx^\dagger, x^\dagger \in C\}$.

First, we give some definitions related to the involed operators.

DEFINITION 2.1. An operator T is called demicontractive if there exists a constant $k \in [0,1)$ such that

$$\|Tx - x^\dagger\|^2 \leqslant \|x - x^\dagger\|^2 + k\|Tx - x^\dagger\|^2,$$

or equivalently,

$$\langle x - Tx, x - x^\dagger\rangle \geqslant \frac{1-k}{2}\|x - Tx\|^2 \tag{2.1}$$

for all $x \in C$ and $x^\dagger \in Fix(T)$.

DEFINITION 2.2. An operator $T: C \longrightarrow C$ is said to be quasi-pseudo-contractive if $\|Tx - x^\dagger\|^2 \leqslant \|x - x^\dagger\|^2 + \|Tx - x\|^2$ for all $x \in C$ and $x^\dagger \in Fix(T)$.

REMARK 2.3. The class of quasi-pseudo-contractive operators [2] contains important operators such as the pseudocontractive operators, the demicontractive operators, the directed operators, the quasi-nonexpansive operators and the strictly pseudo-contractive operators with fixed points. Such a class of operators is fundamental because it includes many types of nonlinear operators arising in applied mathematics and optimization.

DEFINITION 2.4. An operator $T: C \longrightarrow C$ is said to be L-Lipschitzian if there exists $L > 0$ such that $\|Tx - Ty\| \leqslant L\|x - y\|$ for all $x, y \in C$.

Usually, the convergence of fixed point algorithms requires some additional smoothness properties of the mapping T such as demi-closedness.

DEFINITION 2.5. An operator T is said to be demiclosed if, for any sequence $\{x_n\}$ which weakly converges to $x^\dagger$, and if $Tx_n \longrightarrow w$, then $Tx^\dagger = w$.

DEFINITION 2.6. A sequence $\{x_n\}$ is called Fejér-monotone with respect to a given nonempty set Ω if for every $x^\dagger \in \Omega$,

$$\|x_{n+1} - x^\dagger\| \leqslant \|x_n - x^\dagger\|$$

for $\forall n \geqslant 0$.

Recall that the metric projection $P_C : H \to C$ is characterized by

$$\langle x - P_C x, z - P_C x\rangle \leqslant 0 \tag{2.2}$$

for all $x \in H, z \in C$, and possess the following properties

$$\begin{aligned} &\langle x - y, P_C x - P_C y\rangle \geqslant \|P_C x - P_C y\|^2 \\ &\text{or} \quad \|P_C x - P_C y\|^2 \leqslant \|x - y\|^2 - \|(I - P_C)x - (I - P_C)y\|^2 \end{aligned} \tag{2.3}$$

for all $x, y \in H$.

Next, we adopt the following notations:

- $x_n \rightharpoonup x^\dagger$ means that $\{x_n\}$ converges weakly to $x^\dagger$;
- $x_n \to x^\dagger$ means that $\{x_n\}$ converges strongly to $x^\dagger$;
- $\omega_w(x_n)$ stands for the set of cluster points in the weak topology, that is,

$$\omega_w(x_n) = \{x^\dagger : \exists x_{n_j} \rightharpoonup x^\dagger\}.$$

DEFINITION 2.7. A bifunction $\psi : H \times H \to \mathbb{R}$ is said to be skew-symmetric if

$$\psi(x,x) - \psi(x,y) - \psi(y,x) + \psi(y,y) \geqslant 0, \ \forall x, y \in H.$$

The skew symmetric bifunctions can be regarded as an analog of monotonicity of gradient and non negativity of second derivative for the convex function. Please refer to see ([19]).

Throughout this paper, we assume that $\Theta : C \times C \to \mathbb{R}$ satisfy the following conditions:

(H1) $\Theta(x,x) = 0$ for all $x \in C$;
(H2) Θ is monotone, i.e., $\Theta(x,y) + \Theta(y,x) \leqslant 0$ for all $x, y \in C$;
(H3) For $\forall y \in C$ fixed, the function $x \to \Theta(x,y)$ is upper-hemicontinuous, i.e.,

$$\limsup_{t \to 0} \Theta(tz + (1-t)x, y) \leqslant \Theta(x,y), \ \forall x, y, z \in C, \ t \in [0,1];$$

(H4) For each $x \in C$ fixed, the function $y \to \Theta(x,y)$ is convex and lower semicontinuous.

LEMMA 2.8. ([3], [4]) *Let $\Theta : C \times C \to \mathbb{R}$ and $\psi : H \times H \to \mathbb{R}$ be nonlinear bifunctions. For any $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : \Theta(z,y) + \psi(z,y) - \psi(z,z) + \frac{1}{r}\langle y - z, z - x\rangle \geqslant 0, \ \forall y \in \mathscr{C}\}. \tag{2.4}$$

Suppose that the following conditions are satisfied:
(i) Θ satisfies condition (H1)–(H4);
(ii) ψ is skew-symmetric, convex in second argument and continuous;
(iii) For $\forall x \in H$, there exists a compact subset $D_x \subset H$ and $y_0 \in C \bigcap D_x$ such that, for each $z \in C \backslash D_x$,

$$\Theta(z, y_0) + \psi(z, y_0) - \psi(z,z) + \frac{1}{r}\langle y_0 - z, z - x\rangle < 0.$$

Then we have the following results:
(1) *For any* $x \in H$, $T_r(x) \neq \emptyset$ *and* $T_r(x)$ *is single-valued;*
(2) T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leqslant \langle T_r x - T_r y, x - y\rangle;$$

(3) $Fix(T_r)$ = *EP(1);*
(4) *EP(1) is closed and convex.*

LEMMA 2.9. *If the sequence* $\{x_n\}$ *is Fejér monotone with respect to* Ω, *then we have the following conclusions:*
(i) $x_n \rightharpoonup x^\dagger \in \Omega$ *iff* $\omega_w(x_n) \subset \Omega$*;*
(ii) the sequence $\{P_\Omega x_n\}$ *converges strongly;*
(iii) if $x_n \rightharpoonup x^\dagger \in \Omega$, *then* $x^\dagger = \lim_{n\to\infty} P_\Omega x_n$.

For all $x, y \in H$, the following conclusions hold:

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \ \ t \in [0,1],$$

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$$

and

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, x+y\rangle.$$

LEMMA 2.10. ([36]) *If* $T : C \to C$ *be an L-Lipschitzian operator with* $L \geqslant 1$. *Then*

$$Fix(((1-\delta)I + \delta T)T) = Fix(T((1-\delta)I + \delta T)) = Fix(T),$$

where $\delta \in (0, \frac{1}{L})$.

LEMMA 2.11. ([36]) *If* $T : C \to C$ *be an L-Lipschitzian quasi-pseudocontractive operator. Then we have*

$$\|T((1-\zeta)I + \zeta T)x - x^\dagger\|^2 \leqslant \|x - x^\dagger\|^2 + (1-\zeta)\|T((1-\zeta) + \zeta T)x - x\|^2$$

for all $x \in C$ *and* $x^\dagger \in Fix(T)$ *when* $0 < \zeta < \frac{1}{\sqrt{1+L^2}+1}$.

LEMMA 2.12. ([8]) *Let* $T : C \to C$ *be a nonexpansive mapping. Then* $I - T$ *is demi-closed at* 0, *i.e. if* $x_n \rightharpoonup x^\dagger \in C$ *and* $x_n - Tx_n \to 0$, *then* $x^\dagger = Tx^\dagger$.

LEMMA 2.13. ([35]) *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leqslant (1-\gamma_n)\alpha_n + \delta_n, \ \ n \in N,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*
(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$*;*
(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leqslant 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} \alpha_n = 0$.

3. Main results

Throughout the present article, let H_1 and H_2 be two real Hilbert spaces. We use $\langle\cdot,\cdot\rangle$ to denote the inner product, and $\|\cdot\|$ stands for the corresponding norm. Let $S : H_2 \longrightarrow H_2$ be an L-Lipschitzian quasi-pseudo-contractive operator with $L > 1$. Denoted the fixed point sets of S by $Fix(S)$. Let $F : C \to H_1$ be an be a α-inverse strongly monotone. Let $A : H_1 \longrightarrow H_2$ be a bounded linear operator with its adjoint A^*. Let $B : H_1 \longrightarrow H_1$ is a strong positive linear bounded operator with coefficient μ.

Throughout, assume

$$\Omega = \{u|u \in EP(F) \text{ and } Au \in Fix(S)\} \neq \emptyset$$

which is the set of solutions of problem (1.6).

REMARK 3.1. In the light of Lemma 2.8, it can be seen easily that

$$EP(F) = Fix[T_r(I - rF)].$$

ALGORITHM 3.2. First, we select an initial element $x_1 \in \mathscr{C}$. Assume x_n has been given. Compute

$$\begin{aligned} &\Theta(u_n,y) + \langle Fx_n, y-u_n\rangle + \psi(u_n,y) - \psi(u_n,u_n) \\ &\qquad + \frac{1}{r}\langle y-u_n, u_n-x_n)\rangle \geqslant 0, \ \ \forall y \in \mathscr{C}, \\ &v_n = \{I - S[(1-\eta_n)I + \eta_n S]\}Ax_n, \ \ 0 < \eta_n < \frac{1}{\sqrt{1+L^2}+1}, \\ &w_n = x_n - u_n + A^* v_n, \end{aligned} \tag{3.1}$$

where $r > 0$ is a constant in $(0, 2\alpha)$. If

$$\|w_n\| = 0, \tag{3.2}$$

then stop; otherwise, compute $\{x_{n+1}\}$ by the following manner

$$x_{n+1} = P_C(x_n - \tau_n w_n), \tag{3.3}$$

where

$$\tau_n = \lambda_n \frac{\|x_n - u_n\|^2 + \|v_n\|^2}{\|w_n\|^2}, \ \lambda_n > 0. \tag{3.4}$$

PROPOSITION 3.3. $\|w_n\| = 0 \Leftrightarrow x_n \in \Omega$.

Proof. Assume $\|w_n\| = 0$. For any $x^\dagger \in \Omega$, we have

$$\begin{aligned} 0 &= \langle w_n, x_n - x^\dagger\rangle \\ &= \langle x_n - u_n + A^* v_n, x_n - x^\dagger\rangle \\ &= \langle x_n - T_r(x_n - rFx_n), x_n - x^\dagger\rangle \\ &\quad + \langle A^*[I - S((1-\eta_n)I + \eta_n S)]Ax_n, x_n - x^\dagger\rangle \end{aligned} \tag{3.5}$$

Since T_r is firmly nonexpansive and F is α-inverse strongly monotone, we have

$$\begin{aligned}
&\|T_r(x-rFx)-T_r(y-rFy)\|^2 \\
&\quad\leqslant \|x-rFx-(y-rFy)\|^2 \\
&\quad\leqslant \|x-y\|^2+r(r-2\alpha)\|Fx-Fy\|^2 \\
&\quad\leqslant \|x-y\|^2.
\end{aligned} \tag{3.6}$$

In particular, $T_r(I-rF)$ is nonexpansive on C. Since $x^\dagger = T_r(I-rF)x^\dagger$, we have

$$\langle x_n - T_r(x_n - rFx_n), x_n - x^\dagger\rangle \geqslant \frac{1}{2}\|x_n - T_r(x_n - rFx_n)\|^2. \tag{3.7}$$

By lemma 2.11, we obtain that $S((1-\eta_n)I+\eta_n S)$ is demicontractive, from (2.1), we deduce

$$\begin{aligned}
&\langle A^*[I-S((1-\eta_n)I+\eta_n S)]Ax_n, x_n - x^\dagger\rangle \\
&\quad= \langle [I-S((1-\eta_n)I+\eta_n S)]Ax_n, Ax_n - Ax^\dagger\rangle \\
&\quad\geqslant \frac{\eta_n}{2}\|Ax_n - S((1-\eta_n)I+\eta_n S)Ax_n\|^2.
\end{aligned} \tag{3.8}$$

By(3.5), (3.7) and (3.8) we get

$$\begin{aligned}
0 &= \langle w_n, x_n - x^\dagger\rangle \\
&\geqslant \frac{1}{2}\|x_n - T_r(x_n - rFx_n)\|^2 \\
&\quad + \frac{\eta_n}{2}\|(I-S((1-\eta_n)I+\eta_n S))Ax_n\|^2
\end{aligned} \tag{3.9}$$

which implies

$$\|x_n - T_r(x_n - rFx_n)\| = 0 \tag{3.10}$$

and

$$\|(I-S((1-\eta_n)I+\eta_n S))Ax_n\| = 0. \tag{3.11}$$

Equivalently,

$$x_n \in Fix(T_r(I-rF)) = EP(F)$$

and

$$Ax_n \in Fix(S((1-\eta_n)I+\eta_n S)) = Fix(S).$$

Therefore, $x_n \in \Omega$. □

If Algorithm 3.2 does not terminate in a finite number of iterations. We have the following theorem.

THEOREM 3.4. *Suppose that $I-S$ is demiclosed at zero. If $\Omega \neq \emptyset$ and the following conditions are satisfied:*

(C_1) $0<\eta \leqslant \eta_n < \frac{1}{\sqrt{1+L^2}+1}$;

(C_2) $0<\liminf_{n\to\infty}\lambda_n \leqslant \limsup_{n\to\infty}\lambda_n < \min\{1,\eta\}$.

Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges weakly to a solution $z^*(=\lim_{n\to\infty}P_\Omega(x_n))$.

Proof. Firstly, we prove that the sequence $\{x_n\}$ is Fejér-monotone with respect to Ω. From (3.9), for $z^\natural \in \Omega$, we have

$$\begin{aligned}
0 &= \langle w_n, x_n - z^\natural\rangle \\
&\geqslant \frac{1}{2}\|x_n - T_r(x_n - rFx_n)\|^2 + \frac{\eta_n}{2}\|(I-S((1-\eta_n)I+\eta_n S))Ax_n\|^2 \\
&\geqslant \frac{1}{2}\min\{1,\eta\}(\|x_n - T_r(x_n - rFx_n)\|^2 + \|(I-S((1-\eta_n)I+\eta_n S))Ax_n\|^2) \\
&= \frac{1}{2}\min\{1,\eta\}(\|x_n-u_n\|^2+\|v_n\|^2).
\end{aligned} \tag{3.12}$$

According to (3.2), (3.3), (3.4) and (3.12), we derive

$$\begin{aligned}
&\|x_{n+1} - z^\natural\|^2 \\
&= \|P_C(x_n - \tau_n w_n) - z^\natural\|^2 \\
&\leqslant \|x_n - z^\natural - \tau_n w_n\|^2 \\
&= \|x_n - z^\natural\|^2 - 2\tau_n\langle w_n, x_n - z^\natural\rangle + \tau_n^2\|w_n\|^2 \\
&\leqslant \|x_n - z^\natural\|^2 - \min\{1,\eta\}\frac{\lambda_n(\|x_n-u_n\|^2+\|v_n\|^2)^2}{\|w_n\|^2} \\
&\quad + \frac{\lambda_n^2(\|x_n-u_n\|^2+\|v_n\|^2)^2}{\|w_n\|^2} \\
&= \|x_n - z^\natural\|^2 - \lambda_n[\min\{1,\eta\}-\lambda_n]\frac{(\|x_n-u_n\|^2+\|v_n\|^2)^2}{\|w_n\|^2}.
\end{aligned} \tag{3.13}$$

Consequently, the sequences $\{x_n\}$ is Fejér-monotone and the sequence $\{x_n\}$ and $\{Ax_n\}$ are bounded.

Next, we show $\omega_w(x_n) \subset \Omega$. Further, from (3.13), we obtain

$$\lambda_n[\min\{1,\eta\}-\lambda_n]\frac{(\|x_n-u_n\|^2+\|v_n\|^2)^2}{\|w_n\|^2} \leqslant \|x_n - z^\natural\|^2 - \|x_{n+1} - z^\natural\|^2 \tag{3.14}$$

which implies that

$$\lim_{n\to\infty}\frac{(\|x_n-u_n\|^2+\|v_n\|^2)^2}{\|w_n\|^2} = 0. \tag{3.15}$$

This together with the boundedness of the sequence $\{u_n\}$ implies that

$$\lim_{n\to\infty}\|x_n-u_n\| = 0 \Leftrightarrow \lim_{n\to\infty}\|x_n - T_r(I-rF)x_n\| = 0 \tag{3.16}$$

and

$$\lim_{n\to\infty}\|v_n\| = 0 \Leftrightarrow \lim_{n\to\infty}\|Ax_n - S((1-\eta_n)I+\eta_n S)Ax_n\| = 0. \tag{3.17}$$

Then,

$$\begin{aligned}&\|Ax_n - SAx_n\|\\&\quad\leqslant \|Ax_n - S((1-\eta_n)I+\eta_n S)Ax_n\|\\&\qquad + \|S((1-\eta_n)I+\eta_n S)Ax_n - SAx_n\|\\&\quad\leqslant \|Ax_n - S((1-\eta_n)I+\eta_n S)Ax_n\|\\&\qquad + \eta_n L\|Ax_n - SAx_n\|.\end{aligned} \tag{3.18}$$

From (3.17) and (3.18), we obtain

$$\begin{aligned}\|Ax_n - SAx_n\| &\leqslant \frac{1}{1-\eta_n L}\|Ax_n - S((1-\eta_n)I+\eta_n S))Ax_n\|\\ (\text{by}(C_1)) &\leqslant \frac{1}{\eta}\|Ax_n - S((1-\eta_n)I+\eta_n S))Ax_n\| \to 0.\end{aligned} \tag{3.19}$$

Owing to (3.16), Lemma 2.12 and the demiclosedness (at zero) $I-S$, we deduce immediately $\omega_w(x_n) \subset \Omega$. To this end, the conditions of Lemma 2.9 are all satisfied. Consequently, $x_n \rightharpoonup z^* = \lim_{n\to\infty} P_\Omega x_n$. The proof is completed. □

Algorithm 3.2 has only weak convergence. Now, we present a new algorithm with strong convergence.

ALGORITHM 3.5. First, we select an initial element $x_1 \in \mathscr{C}$. Assume x_n has been given. Compute

$$\begin{aligned}&\Theta(u_n,y) + \langle Tu_n + Fx_n, y-u_n\rangle + \psi(u_n,y) - \psi(u_n,u_n)\\&\qquad + \frac{1}{r}\langle y-u_n, u_n - x_n)\rangle \geqslant 0, \quad \forall y \in \mathscr{C},\\&v_n = \{I - S[(1-\eta_n)I+\eta_n S]\}Ax_n, \quad 0<\eta_n<\frac{1}{\sqrt{1+L^2}+1},\\&w_n = x_n - u_n + A^* v_n,\end{aligned} \tag{3.20}$$

where $r>0$ is a constant in $(0,2\alpha)$. If

$$\|w_n\| = 0, \tag{3.21}$$

then stop; otherwise, compute $\{x_{n+1}\}$ by the following manner

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1-\alpha_n B)P_C(x_n - \tau_n w_n), \tag{3.22}$$

where $\alpha_n \in (0,1)$ and

$$\tau_n = \lambda_n \frac{\|x_n-u_n\|^2 + \|v_n\|^2}{\|w_n\|^2}, \ \lambda_n > 0. \tag{3.23}$$

The following proposition is well-known. It is very useful for our main theorem. For sake of completeness, we give the proof.

PROPOSITION 3.6. *Let $G: C \longrightarrow H$ be an L-Lipschitzian α-strongly monotone operator. Then the following variational inequality*

$$x^\dagger \in C, \ \langle Gx^\dagger, x - x^\dagger \rangle \geqslant 0, \ \forall x \in C. \tag{3.24}$$

has a unique solution $x^\dagger$.

Proof. Recall that an operator is called to be α-strongly monotone operator if

$$\langle Gx - Gy, x - y \rangle \geqslant \alpha \|x - y\|^2$$

for some constant $\alpha > 0$ and all $x, y \in C$. Letting $0 < \rho < \frac{2\alpha}{L^2}$, we can deduce

$$\begin{aligned} &\|P_C(x - \rho Gx)) - P_C(y - \rho Gy)\|^2 \\ &\quad \leqslant \|x - \rho Gx - (y - \rho Gy)\|^2 \\ &\quad = \|x - y\|^2 + \rho^2 \|Gx - Gy\|^2 - 2\rho \langle x - y, Gx - Gy \rangle \\ &\quad \leqslant \|x - y\|^2 + \rho^2 L^2 \|x - y\|^2 - 2\rho\alpha \|x - y\|^2 \\ &\quad = (1 - \rho(2\alpha - \rho L^2))\|x - y\|^2. \end{aligned} \tag{3.25}$$

Hence, $P_C(I - \rho G)$ is a contraction on C. Hence, there exists a unique fixed point $x^\dagger \in C$ satisfying $P_C(x^\dagger - \rho Gx^\dagger) = x^\dagger$ which is equivalent to the following variational inequality

$$\langle Gx^\dagger, x - x^\dagger \rangle \geqslant 0, \ \forall x \in C. \quad \square$$

If Algorithm 3.5 does not terminate in a finite number of iterations, then we show the following theorem.

THEOREM 3.7. *Suppose that $I - S$ is demiclosed at zero. If $\Omega \neq \emptyset$ and the following conditions are satisfied:*

(C_1) $0 < \eta \leqslant \eta_n < \frac{1}{\sqrt{1+L^2}+1}$, $\rho < \frac{1}{2}$, $\gamma < \frac{\mu}{2\rho}$;

(C_2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = +\infty$;

(C_3) $0 < \liminf_{n\to\infty} \lambda_n \leqslant \limsup_{n\to\infty} \lambda_n < \theta$, *where* $\theta = \min\{1, \eta\}$.

Then the sequence $\{x_n\}$ generated by Algorithm 3.5 converges strongly to the unique solution $z^\sharp$ of the following variational inequality VI(3.26)

$$z^\sharp \in \Omega, \ \langle (B - \gamma f)z^\dagger, z^\sharp - z^\dagger \rangle \geqslant 0, \ \forall z^\dagger \in \Omega. \tag{3.26}$$

Proof. First, note that

$$\begin{aligned} &\langle Bx - \gamma f(x) - (By - \gamma f(y)), x - y \rangle \\ &\quad = \langle Bx - By, x - y \rangle - \gamma \langle f(x) - f(y), x - y \rangle \\ &\quad \geqslant \mu \|x - y\|^2 - \gamma\rho \|x - y\|^2 \\ &\quad = (\mu - \gamma\rho)\|x - y\|^2 \end{aligned} \tag{3.27}$$

and

$$\begin{aligned}\|Bx &-\gamma f(x)-(By-\gamma f(y))\| \\ &=\|Bx-By\|+\gamma\|f(x)-f(y)\| \\ &\leqslant(\|B\|+\gamma\rho)\|x-y\|.\end{aligned} \tag{3.28}$$

Consequently, in view of (C_1), we deduce that $B-\gamma f$ is $(\|B\|+\gamma\rho)$-Lipschitzian $(\mu-\gamma\rho)$-strongly monotone. In virtue of Proposition 3.6, the variational inequality $VI(3.26)$ has an unique solution denoted by $z^\sharp$.

Next, let $h_n=x_n-\tau_n w_n$. By (3.13), we have

$$\|h_n-z^\sharp\|^2\leqslant\|x_n-z^\sharp\|^2-\lambda_n[\min\{1,\eta\}-\lambda_n]\frac{(\|x_n-u_n\|^2+\|v_n\|^2)^2}{\|w_n\|^2} \tag{3.29}$$

and therefore $\|h_n-z^\sharp\|\leqslant\|x_n-z^\sharp\|$. Thus, from (3.22), we obtain

$$\begin{aligned}\|x_{n+1}-z^\sharp\|&=\|\alpha_n\gamma f(x_n)+(1-\alpha_nB)P_Ch_n-z^\sharp\| \\ &\leqslant\alpha_n\|\gamma f(x_n)-Bz^\sharp\|+\|(1-\alpha_nB)(P_Ch_n-z^\sharp)\| \\ &\leqslant\alpha_n\|\gamma f(x_n)-\gamma f(z^\sharp)\|+\alpha_n\|\gamma f(z^\sharp)-Bz^\sharp\|+(1-\alpha_n\mu)\|x_n-z^\sharp\| \\ &\leqslant\alpha_n\gamma\rho\|x_n-z^\sharp\|+\alpha_n\|\gamma f(z^\sharp)-Bz^\sharp\|+(1-\alpha_n\mu)\|x_n-z^\sharp\| \\ &\leqslant\alpha_n(\mu-\gamma\rho)\frac{\|\gamma f(z^\sharp)-Bz^\sharp\|}{\mu-\gamma\rho}+[1-\alpha_n(\mu-\gamma\rho)]\|x_n-z^\sharp\| \\ &\leqslant\max\left\{\frac{\|\gamma f(z^\sharp)-Bz^\sharp\|}{\mu-\gamma\rho},\|x_n-z^\sharp\|\right\}.\end{aligned} \tag{3.30}$$

By induction, we derive

$$\|x_{n+1}-z^\sharp\|\leqslant\max\left\{\frac{\|\gamma f(z^\sharp)-Bz^\sharp\|}{\mu-\gamma\rho},\|x_1-z^\sharp\|\right\}.$$

Hence, $\{x_n\}$ is bounded and so is $\{Ax_n\}$.

From (3.22), we have

$$\begin{aligned}\|x_{n+1}&-z^\sharp\|^2 \\ &=\|\alpha_n(\gamma f(x_n)-Bz^\sharp)+(1-\alpha_nB)(P_Ch_n-z^\sharp)\|^2 \\ &\leqslant(1-\alpha_n\mu)\|h_n-z^\sharp\|^2+2\alpha_n\langle\gamma f(x_n)-Bz^\sharp,x_{n+1}-z^\sharp\rangle \\ &\leqslant(1-\alpha_n\mu)\|h_n-z^\sharp\|^2+2\alpha_n\langle\gamma f(x_n)-\gamma f(z^\sharp),x_{n+1}-z^\sharp\rangle \\ &\quad+2\alpha_n\langle\gamma f(z^\sharp)-Bz^\sharp,x_{n+1}-z^\sharp\rangle \\ &\leqslant(1-\alpha_n\mu)\|h_n-z^\sharp\|^2+2\alpha_n\gamma\rho\|x_n-z^\sharp\|\cdot\|x_{n+1}-z^\sharp\| \\ &\quad+2\alpha_n\langle\gamma f(z^\sharp)-Bz^\sharp,x_{n+1}-z^\sharp\rangle \\ &\leqslant(1-\alpha_n\mu)\|h_n-z^\sharp\|^2+\alpha_n\gamma\rho(\|x_n-z^\sharp\|^2+\|x_{n+1}-z^\sharp\|^2) \\ &\quad+2\alpha_n\langle\gamma f(z^\sharp)-Bz^\sharp,x_{n+1}-z^\sharp\rangle\end{aligned} \tag{3.31}$$

$$\text{(by (3.29))} \leqslant (1-\alpha_n\mu)\Big(\|x_n - z^\sharp\|^2 - \lambda_n[\min\{1,\eta\} - \lambda_n]\frac{(\|x_n-u_n\|^2+\|v_n\|^2)^2}{\|w_n\|^2}\Big)$$
$$+\alpha_n\gamma\rho(\|x_n-z^\sharp\|^2+\|x_{n+1}-z^\sharp\|^2)+2\alpha_n\langle\gamma f(z^\sharp)-Bz^\sharp, x_{n+1}-z^\sharp\rangle.$$

Hence, we obtain

$$\begin{aligned}
&\|x_{n+1} - z^\sharp\|^2 \\
&\leqslant \frac{1}{1-\alpha_n\gamma\rho}\Big\{[1-\alpha_n(\mu-\gamma\rho)]\|x_n-z^\sharp\|^2+2\alpha_n\langle\gamma f(z^\sharp)-Bz^\sharp,x_{n+1}-z^\sharp\rangle \\
&\quad -(1-\alpha_n\mu)\lambda_n[\min\{1,\eta\}-\lambda_n]\frac{(\|x_n-u_n\|^2+\|v_n\|^2)^2}{\|w_n\|^2}\Big\} \\
&\leqslant \Big[1-\frac{\alpha_n(\mu-2\gamma\rho)}{1-\alpha_n\gamma\rho}\Big]\|x_n-z^\sharp\|^2 \\
&\quad +\frac{\alpha_n(\mu-2\gamma\rho)}{1-\alpha_n\gamma\rho}\Big[\frac{2\langle\gamma f(z^\sharp)-Bz^\sharp,x_{n+1}-z^\sharp\rangle}{\mu-2\gamma\rho} \\
&\quad -\frac{(1-\alpha_n\mu)\lambda_n(\theta-\lambda_n)}{\alpha_n(\mu-2\gamma\rho)}\frac{(\|x_n-u_n\|^2+\|v_n\|^2)^2}{\|w_n\|^2}\Big]
\end{aligned} \tag{3.32}$$

Set $\phi_n = \|x_n - z^\sharp\|^2$ and

$$\zeta_n = \frac{2\langle\gamma f(z^\sharp)-Bz^\sharp,x_{n+1}-z^\sharp\rangle}{\mu-2\gamma\rho} - \frac{(1-\alpha_n\mu)\lambda_n(\theta-\lambda_n)}{\alpha_n(\mu-2\gamma\rho)}\frac{(\|x_n-u_n\|^2+\|v_n\|^2)^2}{\|w_n\|^2}$$

for all $n \geqslant 1$. Let

$$\hat{\alpha}_n = \frac{\alpha_n(\mu-2\gamma\rho)}{1-\alpha_n\gamma\rho}.$$

It can be readily seen that

$$\alpha_n(\mu-2\gamma\rho) < \hat{\alpha}_n < \frac{\alpha_n(\mu-2\gamma\rho)}{1-\rho}$$

for large enough $n\in\mathbb{Z}^+$. Hence, by (C_2), we get that $\lim_{n\to\infty}\hat{\alpha}_n = 0$ and $\sum_{n=1}^{\infty}\hat{\alpha}_n = +\infty$. Then, from (3.32), we have

$$0 \leqslant \phi_{n+1} \leqslant (1-\hat{\alpha}_n)\phi_n + \hat{\alpha}_n\zeta_n, \quad n \geqslant 1. \tag{3.33}$$

Evidently,

$$\zeta_n \leqslant \frac{2\langle\gamma f(z^\sharp)-Bz^\sharp,x_{n+1}-z^\sharp\rangle}{\mu-2\gamma\rho} \leqslant \frac{2}{\mu-2\gamma\rho}\|\gamma f(z^\sharp)-Bz^\sharp\|\cdot\|x_{n+1}-z^\sharp\|.$$

Hence, $\limsup_{n\to\infty}\zeta_n < +\infty$.

Next, we show that $\limsup_{n\to\infty}\zeta_n \geqslant 0$. By contradiction, assume that $\limsup_{n\to\infty}\zeta_n = -\varsigma < 0$, then, $\exists N$ such that $\zeta_n < -\frac{\varsigma}{2}$ for $\forall n > N$. It follows from (3.33) that

$$\phi_{n+1} \leqslant (1-\hat{\alpha}_n)\phi_n + \hat{\alpha}_n\zeta_n = \phi_n + \hat{\alpha}_n(\zeta_n - \phi_n) < \phi_n - \frac{\hat{\alpha}_n\varsigma}{2} \tag{3.34}$$

for all $n \geqslant N$. Thus,

$$\phi_{n+1} \leqslant \phi_N - \frac{\varsigma}{2}\sum_{i=N}^{n}\hat{\alpha}_i.$$

Taking $\limsup_{n\to\infty}$ in the last inequality, we obtain

$$0 \leqslant \limsup_{n\to\infty} \phi_{n+1} \leqslant \phi_N - \frac{\varsigma}{2}\sum_{i=N}^{\infty}\hat{\alpha}_i = -\infty,$$

which is a contradiction. Therefore, $0 \leqslant \limsup_{n\to\infty}\zeta_n < +\infty$. Consequently, we can take a subsequence $\{n_i\}$ such that

$$\begin{aligned}\limsup_{n\to\infty}\zeta_n &= \lim_{i\to\infty}\zeta_{n_i}\\ &= \lim_{i\to\infty}\Big[-\frac{(1-\alpha_{n_i}\mu)\lambda_{n_i}(\theta-\lambda_{n_i})}{\alpha_{n_i}(\mu-2\gamma\rho)}\frac{(\|x_{n_i}-u_{n_i}\|^2+\|v_{n_i}\|^2)^2}{\|w_{n_i}\|^2}\\ &\quad+\frac{2\langle\gamma f(z^\sharp)-Bz^\sharp, x_{n_i+1}-z^\sharp\rangle}{\mu-2\gamma\rho}\Big].\end{aligned} \tag{3.35}$$

By the boundedness of the real sequence

$$\Big\{\frac{2\langle\gamma f(z^\sharp)-Bz^\sharp, x_{n_i+1}-z^\sharp\rangle}{\mu-2\gamma\rho}\Big\},$$

without loss of generality, we may assume the limit

$$\lim_{i\to\infty}\frac{2\langle\gamma f(z^\sharp)-Bz^\sharp, x_{n_i+1}-z^\sharp\rangle}{\mu-2\gamma\rho}$$

exists. Consequently, from (3.35), the following limit also exists

$$\lim_{i\to\infty}-\frac{(1-\alpha_{n_i}\mu)\lambda_{n_i}(\theta-\lambda_{n_i})}{\alpha_{n_i}(\mu-2\gamma\rho)}\frac{(\|x_{n_i}-u_{n_i}\|^2+\|v_{n_i}\|^2)^2}{\|w_{n_i}\|^2}.$$

This together with conditions (C_1), (C_2) and (C_3) implies that

$$\lim_{i\to\infty}\frac{(\|x_{n_i}-u_{n_i}\|^2+\|v_{n_i}\|^2)^2}{\|w_{n_i}\|^2} = 0, \tag{3.36}$$

which yields $\lim_{i\to\infty}\|x_{n_i}-u_{n_i}\| = 0$ and $\lim_{i\to\infty}\|v_{n_i}\| = 0$. By a similar proof as in Theorem 3.4, we conclude that any weak cluster point of $\{x_{n_i}\}$ belongs to Ω. Note that

$$\begin{aligned}&\|x_{n_i+1}-x_{n_i}\|\\ &\quad= \|\alpha_{n_i}\gamma f(x_{n_i})+(1-\alpha_{n_i}B)P_C h_{n_i}-x_{n_i}\|\\ (\text{by } x_{n_i}\in C)&\quad\leqslant \alpha_{n_i}\|\gamma f(x_{n_i})-Bx_{n_i}\|+(1-\alpha_{n_i}\mu)\|P_C h_{n_i}-P_C x_{n_i}\|\\ &\quad\leqslant \alpha_{n_i}\|\gamma f(x_{n_i})-Bx_{n_i}\|+(1-\alpha_{n_i}\mu)\|h_{n_i}-x_{n_i}\|\\ &\quad\leqslant \alpha_{n_i}\|\gamma f(x_{n_i})-Bx_{n_i}\|+\tau_{n_i}\|w_{n_i}\|\\ &\quad\leqslant \alpha_{n_i}\|\gamma f(x_{n_i})-Bx_{n_i}\|+\lambda_{n_i}\frac{\|x_{n_i}-u_{n_i}\|^2+\|v_{n_i}\|^2}{\|w_{n_i}\|}\\ &\quad\to 0\ (\text{by}(3.36)),\end{aligned} \tag{3.37}$$

this indicates that $\omega_w(x_{n_i+1}) \subset \Omega$. Without losing generality, we assume that

$$x_{n_i+1} \rightharpoonup z^{\dagger} \in \Omega.$$

Now, by (3.35), we infer that

$$\begin{aligned}
\limsup_{n\to\infty} \zeta_n &= \lim_{i\to\infty} \zeta_{n_i} \\
&= \lim_{i\to\infty} \Big[-\frac{(1-\alpha_{n_i}\mu)\lambda_{n_i}(\theta-\lambda_{n_i})}{\alpha_{n_i}(\mu-2\gamma\rho)} \frac{(\|x_{n_i}-u_{n_i}\|^2+\|v_{n_i}\|^2)^2}{\|w_{n_i}\|^2} \\
&\quad + \frac{2\langle \gamma f(z^{\sharp})-Bz^{\sharp}, x_{n_i+1}-z^{\sharp}\rangle}{\mu-2\gamma\rho} \Big] \\
&\leqslant \lim_{i\to\infty} \frac{2\langle \gamma f(z^{\sharp})-Bz^{\sharp}, x_{n_i+1}-z^{\sharp}\rangle}{\mu-2\gamma\rho} \\
&= \frac{2\langle \gamma f(z^{\sharp})-Bz^{\sharp}, z^{\dagger}-z^{\sharp}\rangle}{\mu-2\gamma\rho} \\
&\leqslant 0
\end{aligned} \tag{3.38}$$

owing to the assumption that $z^{\sharp}$ is the unique solution of $VI(3.26)$. Applying Lemma 2.13 and (3.38) to (3.33), we conclude that $x_n \to z^{\sharp}$.

This completes the proof. □

Declarations

Availablity of data and materials. Not applicable.

Competing interests. The authors declare that they have no competing interests.

Fundings. This work was supported by National Natural Science Foundation of China (No. 11761074), Project of Jilin Science and Technology Development for Leading Talent of Science and Technology Innovation in Middle and Young and Team Project (No. 20200301053RQ), and Natural Science Foundation of Jilin Province (No. 2020122336JC).

Authors' contributions. The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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(Received November 6, 2020)

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