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CORRIGENDUM TO: "SHARP L^p HARDY TYPE AND UNCERTAINTY PRINCIPLE INEQUALITIES ON THE SPHERE"

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Abstract. The aim of this note is to put straight the inaccuracy in Theorem 1 of the paper mentioned in the title (J. Math. Ineq. 13 (4), 1011–1022). The inaccuracy was due to an omission of a factor in the proof of the theorem.

This note is to put straight the inaccuracy in inequality (2) of [1, Theorem 1] (Sharp L^p Hardy type and uncertainty principle inequalities on the sphere, J. Math. Ineq. 13 (4), 1011–1022), which consequently affected inequality (4) of Proposition 1 in the said paper. The inaccuracy was due to an omission of a factor in the proof of the theorem. The authors hereby apologise for the inconviniences this might have caused the readers.

All the notations and assumptions of Theorem 1 in the paper [1] are still valid and therefore applied in this note. The theorem below and its proof show the correction required on [1, Thorem 1].

THEOREM 1. *Let $N \geqslant 3$, $2 \leqslant p < N$ and $q \in \mathbb{S}^N$, then there exists a positive constant $A(N,p)$ such that for all $f \in C^\infty(\mathbb{S}^N)$,*

$$A(N,p)\int_{\mathbb{S}^N}\frac{|f|^p}{|\tan d(q,x)|^{p-2}}dV+\int_{\mathbb{S}^N}|\nabla f|^p dV \geqslant \left(\frac{N-p}{p}\right)^p\int_{\mathbb{S}^N}\frac{|f|^p}{|\tan d(x,q)|^p}dV, \tag{1}$$

where $A(N,p)=(p-1)\left(\frac{N-p}{p}\right)^{p-1}$ and $d(x,q)$ is the geodesic distance from x to a fixed point q on $\mathbb{S}^N$. Moreover, the constant $\left(\frac{N-p}{p}\right)^p$ is sharp.

REMARK 1. Instead of the first term on the left hand side of inequality (1) above we wrote $\left(\frac{N-p}{p}\right)^{p-1}\int_{\mathbb{S}^N}\frac{|f|^p}{\sin^{p-2}d(q,x)}$ in [1] (cf. inequality (2) of [1]). However, the inequality (2) of [1] may hold true but cannot be derived by the method used in the paper.

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Proof. As in [1], let $\gamma = -\frac{N-p}{p}$. $f = \rho^\gamma \phi \in C^\infty(\mathbb{S}^N)$, $\rho = \sin r_q$, $q \in \mathbb{S}^N$. A straightforward computation gives rise to

$$\begin{aligned}|\nabla f|^p &= |\gamma \rho^{\gamma-1}\nabla\rho\phi + \rho^\gamma\nabla\phi|^p \\ &\geqslant |\gamma|^p\rho^{\gamma p-p}|\nabla\rho|^p|\phi|^p + p|\gamma|^{p-2}\gamma\rho^{\gamma p-p+1}|\phi|^{p-2}\phi|\nabla\rho|^{p-2}\langle\nabla\rho,\nabla\phi\rangle. \qquad (2)\end{aligned}$$

The factor $|\nabla\rho|^{p-2}$ (appearing in the second term on right hand side) in line 2 of inequality (2) above was erroneously not considered in [1] and we wrote

$$|\nabla f|^p \geqslant |\gamma|^p\rho^{\gamma p-p}|\nabla\rho|^p|\phi|^p + \frac{|\gamma|^{p-2}\gamma}{\gamma p-p+2}\langle\nabla\rho^{\gamma p-p+2},\nabla|\phi|^p\rangle$$

instead of a correct inequality whose last term should have read

$$\frac{|\gamma|^{p-2}\gamma}{\gamma p-p+2}|\nabla\rho|^{p-2}\langle\nabla\rho^{\gamma p-p+2},\nabla|\phi|^p\rangle.$$

So, application of divergence theorem on the integral $\int_{\mathbb{S}^N}|\nabla\rho|^{p-2}\langle\nabla\rho^{\gamma p-p+2},\nabla|\phi|^p\rangle dv$ may become complicated and still not yield the desired quantities. Then instead of divergence theorem we will use a more direct approach via integration by parts.

Considering the following two identities

$$p|\phi|^{p-2}\phi\nabla\phi = \nabla|\phi|^p$$

and

$$\rho^{(\gamma p-p+1)}|\nabla\rho|^{p-2}\nabla\rho = \frac{\rho^{-(p-2)(\gamma p-p+1)}}{(\gamma p-p+2)^{p-1}}|\nabla\rho^{\gamma p-p+2}|^{p-1},$$

we arrive at (by using the above two identities in inequality (2))

$$\begin{aligned}|\nabla f|^p \geqslant\ & |\gamma|^p\rho^{\gamma p-p}|\nabla\rho|^p|\phi|^p \\ & + \frac{|\gamma|^{p-2}\gamma}{(\gamma p-p+2)^{p-1}}\rho^{-(p-2)(\gamma p-p+1)}\langle|\nabla\rho^{\gamma p-p+2}|^{p-1},\nabla|\phi|^p\rangle. \qquad (3)\end{aligned}$$

Now substituting back the values γ, f and ϕ into (3) and using the fact that $|\nabla r_q| = 1$ gives

$$\begin{aligned}|\nabla f|^p \geqslant\ & \Big(\frac{N-p}{p}\Big)^p\frac{|f|^p}{|\sin r_q|^p}(\cos r_q)^p - \frac{1}{(2-N)^{p-1}}\Big(\frac{N-p}{p}\Big)^{p-1} \\ & \times(\sin r_q)^{(p-2)(N-1)}\langle|\nabla(\sin r_q)^{2-N}|^{p-1},\nabla(|f|^p(\sin r_q)^{N-p})\rangle. \qquad (4)\end{aligned}$$

By a way of simplification

$$\begin{aligned}&(\sin r_q)^{(p-2)(N-1)}\langle|\nabla(\sin r_q)^{2-N}|^{p-1},\nabla(|f|^p(\sin r_q)^{N-p})\rangle \\ &\qquad = |(2-N)|^{p-1}(\sin r_q)^{1-N}(\cos r_q)^{p-1}|\nabla r_q|^{p-1}\nabla(|f|^p(\sin r_q)^{N-p}).\end{aligned}$$

Substituting the last expression into (4) and integrating over $\mathbb{S}^N$ (using integration by parts and $|\nabla r_q| = 1$) yields

$$\begin{aligned}\int_{\mathbb{S}^N} |\nabla f|^p dV \geqslant \Big(\frac{N-p}{p}\Big)^p \int_{\mathbb{S}^N} \frac{|f|^p}{|\tan r_q|^p} dV + \Big(\frac{N-p}{p}\Big)^{p-1} \\ \times \int_{\mathbb{S}^N} |f|^p (\sin r_q)^{N-p} \operatorname{div}\big((\sin r_q)^{1-N} (\cos r_q)^{p-1} \nabla r_q\big)\, dV \\ = \Big(\frac{N-p}{p}\Big)^p \int_{\mathbb{S}^N} \frac{|f|^p}{|\tan r_q|^p} dV - (p-1)\Big(\frac{N-p}{p}\Big)^{p-1} \\ \times \int_{\mathbb{S}^N} |f|^p (\sin r_q)^{2-p} (\cos r_q)^{p-2} dV\end{aligned}$$

for the fact that the term $\operatorname{div}\big((\sin r_q)^{1-N} \nabla r_q\big)$ vanishes since $\Delta r_q = \frac{\cos r_q}{\sin r_q}$ and

$$\langle \nabla (\sin r_q)^{1-N}, \nabla r_q \rangle = -\Delta r_q / (\sin r_q)^{N-1}.$$

The last inequality therefore recovers (1) for $2 \leqslant p < N$. Note that this also holds for $1 < p < 2$.

In what follows, the constant $\Big(\frac{N-p}{p}\Big)^p$ can be shown to be sharp in the sense that

$$\Big(\frac{N-p}{p}\Big)^p \geqslant \inf_{f \in C^\infty(\mathbb{S}^N)\setminus\{0\}} \frac{\int_{\mathbb{S}^N} |\nabla f|^p dV + A(N,p) \int_{\mathbb{S}^N} \frac{|f|^p}{|\tan r_q|^{p-2}} dV}{\int_{\mathbb{S}^N} \frac{|f|^p}{|\tan r_q|^p} dV}$$

by following the same argument in paper [1]. The proof is complete. □

Consequently, the uncertainty principle inequality in [1, Proposition 1] (that is inequality (4) on page 1019 of the paper) now reads as follows: For all functions $f \in C^\infty(\mathbb{S}^N)$

$$\begin{aligned}\Big(\int_{\mathbb{S}^N} |f|^p |\tan d(x,\xi)^q| dV\Big)^{p/q} \Big(\int_{\mathbb{S}^N} |\nabla f|^p dV + A(N,p) \int_{\mathbb{S}^N} \frac{|f|^p}{(|\tan d|^{p-2} d(x,\xi))} dV\Big) \\ \geqslant \Big(\frac{N-p}{p}\Big)^p \Big(\int_{\mathbb{S}^N} |f|^p dV\Big)^p,\end{aligned} \tag{5}$$

where $A(N,p) = (p-1)((N-p)/p)^{p-1}$ and $1/p + 1/q = 1$. The proof of this inequality is still correct as stated in [1, Proposition 1].

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