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PERTURBATION STRATEGY FOR SPLITTING OPERATOR METHOD TO SOLVE THE SET–VALUED VARIATIONAL INEQUALITIES

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Abstract. In this paper, we suggest a new perturbation strategy for a splitting operator method for solving the set-valued variational inequalities with strongly monotone and compact mappings, under the mild condition, and prove the global convergence of the method. Also, we discuss the self-adaptive strategy and find the approximate solution of the set-valued variational inequality problems.

1. Introduction

Let Ω be a nonempty closed convex subset of n-dimensional Euclidean space $\mathbb{R}^n$ and $T:\mathbb{R}^n \longrightarrow 2^{\mathbb{R}^n}$ be a continuous mapping. $2^{\mathbb{R}^n}$ denotes the family of all nonempty subsets of $\mathbb{R}^n$. We consider the set-valued variational inequality problem, which is to find a vector $u^* \in \Omega$ such that

$$(T(u^*))^\top (u-u^*) \geqslant 0,\ \forall u \in \Omega. \tag{1.1}$$

We note that, if T is a single-valued mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, then (1.1) reduces to a classical variational inequality problem studied by G. Stampacchia [1].

The set-valued variational inequalities have important applications in mathematical programming, economics, transportation and structural analysis; *e.g., see* [2, 3, 4, 5, 6, 7, 8]. When T has explicit expression, there are various numerical methods that have been studied by many researchers; *e.g., see* [9, 10, 11, 12, 13], the required method is based on monotonicity properties of T. In this case, when T is only monotone and even non-monotone, one of the classical perturbation strategies is the Tikhonov regularization approach which solves the perturbed set-valued variational inequality problem

$$(T(u^*)+\varepsilon u^*)^\top (u-u^*) \geqslant 0,\ \forall u \in \Omega, \tag{1.2}$$

where ε is a positive parameter, *see* [14]. The solution of the original problem is obtained by letting $\varepsilon \longrightarrow 0$.

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However, in many applications, the mapping T can be split into two set-valued mappings and one part of T is known. That is,

$$T(u) \equiv F(u) + G(u),$$

where F is unknown, F and G are two set-valued mappings from $\mathbb{R}^n$ to $2^{\mathbb{R}^n}$. Fortunately, for any given $G(u^\ell)$, the solution $\bar{u}^\ell$ of the set-valued variational inequality problem

$$(F(\bar{u}^\ell) + G(u^\ell))^\top (u - \bar{u}^\ell) \geqslant 0, \ \forall u \in \Omega, \tag{1.3}$$

can be obtained. Under the certain assumption, we can solve the problem by using the information in (1.3) and the problem is explicit structure. Under the assumption F and G are strongly monotone, they proved the global convergence of the method. Utilize an operator splitting method, first getting the solution of (1.3) from the oracle and then solving the following system of nonlinear equations to obtain the next iteration $\{u^{\ell+1}\}$:

$$\vartheta_\ell(u) = 0, \tag{1.4}$$

$$\vartheta_\ell(u) = u + \lambda G(u) - u^\ell - \lambda G(u^\ell) + \alpha(u^\ell - \bar{u}^\ell), \tag{1.5}$$

λ and α are two parameters and $\{u^\ell\}$ is the current iterate. The problem (1.3) can also be regarded as a perturbation of the underlying mapping from $F+G$ to $F+G+(G(u^\ell)-G)$. From the given assumption, F is strongly monotone and G is monotone, they proved the global convergence of the method.

In this paper, we consider a new perturbation strategy where the underlying mapping from $F+G$ to $F+G+(G(u^\ell)-G)$, we also add the regularization term to a proximal point algorithm regularization term [15]. The resulted perturbation form of (1.3) is

$$\Big(F(\bar{u}^\ell) + G(u^\ell)\Big)^\top (u - \bar{u}^\ell) + \left(\frac{1}{\lambda}(\bar{u}^\ell - u^\ell)\right)^\top (u - \bar{u}^\ell) \geqslant 0, \ \forall u \in \Omega, \tag{1.6}$$

where λ is a positive parameter. Another perturbation technique for solving a variational inequality problem is the Tikhonov regularization [13, 16]. When applied to the variational inequality problem (1.3), it leads to solving the following problem

$$\Big(F(\bar{u}^\ell) + G(u^\ell)\Big)^\top (u - \bar{u}^\ell) + \Big(\varepsilon_\ell \bar{u}^\ell\Big)^\top (u - \bar{u}^\ell) \geqslant 0, \ \forall u \in \Omega, \tag{1.7}$$

where $\{\varepsilon_\ell\}$ is a sequence of positive parameters. By letting $\varepsilon_\ell \longrightarrow 0$, we can get a solution of the subproblem (1.3). Comparing (1.6) with (1.7), the coefficient $\frac{1}{\lambda}$ of our regularization term is not required to tend to 0, which is important from the numerical point of view, since it provides more flexibility in choosing the parameter. Hence, our algorithm is different from the proximal point algorithm and the Tikhonov regularization algorithm. To improve the efficiency of the algorithm, we also adopt the two strategy as in [17, 18].

The first strategy is that the system of nonlinear equations are approximately solve. Note that it is expansive or even impossible to find the exact solution of (1.4). Hence, we find an approximate solution which satisfies the condition

$$\|\vartheta_\ell(u^{\ell+1})\| \leqslant \eta_\ell \|u^\ell - \bar{u}^\ell\|, \tag{1.8}$$

where $\{\eta_\ell\}$ is a nonnegative sequence satisfying

$$\sum_{\ell=0}^{\infty} \eta_\ell^2 < +\infty.$$

The second strategy is to use a varying parameter λ_ℓ and choose it self-adaptively. The computational results in [19, 20, 21] indicated that the operator splitting algorithms with fixed parameters may converge very slowly, and He *et al.* [20], Han [21] suggested to choose the parameters self-adaptively.

Throughout this paper, we make the assumption that the solution of set-valued variational inequality, denoted by Ω^*, is nonempty, we know that the solution set Ω^* is nonempty when the underlying mapping T is strongly monotone and compact if Ω is compact.

The rest of this paper is organised as follows. In section 2, we summarize some fruitful concepts and definitions. In section 3, we first describe our exact version of operator splitting method with perturbation strategy, and then prove its global convergence under certain mild assumption. In section 4, we discuss the self-adaptive strategy and last section we find the approximate solution of set-valued variational inequality problems.

2. Preliminaries

For any vector $u,v \in \mathbb{R}^n$, $u^\top v$ is their inner product is define by Euclidean norm $\|u\| = \sqrt{u^\top u}$. $P_\Omega(\cdot)$ denotes the projection under the Euclidean norm of a point onto Ω, *i.e.*

$$P_\Omega(v) = \arg\min\{\|v-u\| \mid u \in \Omega\},$$

where Ω is a nonempty closed convex subsets of $\mathbb{R}^n$. The property of the projection mapping is

$$(w - P_\Omega(v))^\top (v - P_\Omega(v)) \leqslant 0,\ \forall v \in \mathbb{R}^n,\ w \in \Omega \tag{2.1}$$

and P_Ω is nonexpansive, *i.e.*

$$\|P_\Omega(u) - P_\Omega(v)\| \leqslant \|u - v\|,\ \forall u,v \in \mathbb{R}^n. \tag{2.2}$$

It is well known [22] that solving the set-valued variational inequality is equivalent to solving the projection equation

$$u = P_\Omega[u - \lambda T(u)],$$

where λ is an arbitrary positive constant. Hence, solving the set-valued variational inequality amounts to finding a zero point of the continuous non-smooth function

$$\varphi(u,\lambda) = u - P_\Omega[u - \lambda T(u)]. \tag{2.3}$$

LEMMA 2.1. [23] *For a given* $u \in \mathbb{R}^n$, *let* $\tilde{\lambda} \geqslant \lambda > 0$. *Then it holds that*

$$\|\varphi(u,\tilde{\lambda})\| \geqslant \|\varphi(u,\lambda)\| \tag{2.4}$$

and

$$\frac{\|\varphi(u,\tilde{\lambda})\|}{\tilde{\lambda}} \leqslant \frac{\|\varphi(u,\lambda)\|}{\lambda}. \tag{2.5}$$

DEFINITION 2.2. A set-valued operator $T : \Omega \longrightarrow 2^{\mathbb{R}^n}$ is said to be

(i) monotone, if
$$(u-v)^\top (T(u)-T(v)) \geqslant 0, \forall\, u,v \in \Omega;$$

(ii) strongly monotone with modulus $\mu > 0$, if
$$(u-v)^\top (T(u)-T(v)) \geqslant \mu \|u-v\|^2, \forall\, u,v \in \Omega;$$

(iii) inverse strongly monotone with modulus $\mu > 0$, if
$$(u-v)^\top (T(u)-T(v)) \geqslant \mu \|T(u)-T(v)\|^2, \forall\, u,v \in \Omega;$$

(iv) Lipschitz continuous with respect to constant $\zeta > 0$, if
$$\|T(u)-T(v)\| \leqslant \zeta \|u-v\|, \ \forall\, u,v \in \Omega.$$

DEFINITION 2.3. A set-valued operator $T : \Omega \longrightarrow 2^{\mathbb{R}^n}$ is said to be $\hat{\mathscr{H}}$-Lipschitz continuous if there exists a constant $\delta > 0$ such that

$$\hat{\mathscr{H}}(T(u),T(v)) \leqslant \delta \|u-v\|, \ \forall u,v \in \Omega,$$

where $\hat{\mathscr{H}} : 2^{\mathbb{R}^n} \times 2^{\mathbb{R}^n} \longrightarrow (-\infty,+\infty) \bigcup \{+\infty\}$ is the Hausdorff metric, *i.e.*

$$\hat{\mathscr{H}}(A,B) = \max\{\sup_{u\in A}\inf_{v\in B} \|u-v\|, \sup_{u\in B}\inf_{v\in A} \|u-v\|\}, \ \forall A,B \in 2^{\mathbb{R}^n}.$$

3. Convergence theory

In this section we describe an algorithm and prove its global convergence.

ALGORITHM 3.1.

Step 0. Choose an arbitrary initial point $u^0 \in \Omega$, $\varepsilon > 0$, $2 > \alpha_{\max} > \alpha_0 > 0$, $\lambda_0 > 0$, and a nonnegative sequence $\{\tau_\ell\}$ satisfying

$$\sum_{\ell=0}^{\infty} \tau_\ell < +\infty. \tag{3.1}$$

Set $\ell = 0$.

Step 1. For given $\{u^\ell\}$, observe the system at an equilibrium state to obtain a solution of the following set valued variational inequality problem

$$(F(\bar{u}^\ell)+G(u^\ell)+\frac{1}{\lambda_\ell}(\bar{u}^\ell-u^\ell))^\top(u-\bar{u}^\ell)\geqslant 0,\ \forall u\in\Omega. \tag{3.2}$$

Step 2. Choose $\alpha_{\max}\geqslant\alpha_\ell\geqslant\alpha_0$ and solve the following system of nonlinear equations to get the next iterate $\{u^{\ell+1}\}$

$$u^{\ell+1}+\lambda_\ell G(u^{\ell+1})=u^\ell+\lambda_\ell G(u^\ell)-\alpha_\ell(u^\ell-\bar{u}^\ell) \tag{3.3}$$

and

$$\hat{\mathscr{H}}(G(u^{\ell+1}),G(u^\ell))\leqslant\delta\|u^{\ell+1}-u^\ell\|.$$

Step 3. If

$$\|u^{\ell+1}-u^\ell\|\leqslant\varepsilon,\ \text{ stop,}$$

otherwise choose a new parameter

$$\lambda_{\ell+1}\in\left[\frac{1}{1+\tau_\ell}\lambda_\ell,(1+\tau_\ell)\lambda_\ell\right].$$

Set $\ell=\ell+1$ and go to Step 1.

Since F is assumed to be monotone, the mapping F_ℓ define by

$$F_\ell(u)\equiv F(u)+G(u^\ell)+\frac{1}{\lambda_\ell}(u-u^\ell)$$

is strongly monotone. Consequently, (3.2) possesses a unique solution and we can observe the system to get the solution. Moreover, since G is monotone, the operator $I+\lambda_\ell G$ is strongly monotone with modulus 1 for any parameter $\lambda_\ell>0$, and there is a unique solution of the system of equations (3.3).

REMARK 3.2. We note that, if $u^{\ell+1}=u^\ell$, then, we have

$$\bar{u}^\ell=u^\ell$$

from (3.3), and (3.2) can be rewritten as

$$(F(\bar{u}^\ell)+G(\bar{u}^\ell))^\top(u-\bar{u}^\ell)\geqslant 0,\ \forall u\in\Omega, \tag{3.4}$$

which means that $\bar{u}^\ell$ is a solution of the problem (1.1). Hence, it is reasonable to use

$$\|u^{\ell+1}-u^\ell\|\leqslant\varepsilon$$

as the stopping criterion. If

$$\|u^{\ell+1}-u^\ell\|\leqslant\varepsilon,$$

then $u^{\ell+1}$ can be regarded as an approximate solution (1.1).

LEMMA 3.3. *Assume that F and G are set-valued monotone mappings. Then the sequence $\{u^\ell\}$ generated by Algorithm 3.1 satisfies*

$$(u^\ell - \bar{u}^\ell)^\top (u^\ell - u^* + \lambda_\ell (G(u^\ell) - G(u))) \geqslant \|u^\ell - \bar{u}^\ell\|^2, \tag{3.5}$$

where $u \in \Omega$ is an arbitrary solution of problem (1.1) with $T \equiv F + G$.

Proof. Setting $u = u^\ell$ in (3.2), we have

$$(F(\bar{u}^\ell))^\top (u^* - \bar{u}^\ell) + (G(u^\ell))^\top (u^* - \bar{u}^\ell) + \frac{1}{\lambda_\ell}(\bar{u}^\ell - u^\ell)^\top (u^* - \bar{u}^\ell) \geqslant 0. \tag{3.6}$$

Then setting $u = \bar{u}^\ell$ in (1.1), we obtain

$$(F(u^*))^\top (\bar{u}^\ell - u^*) + (G(u^*))^\top (\bar{u}^\ell - u^*) \geqslant 0. \tag{3.7}$$

Adding (3.6), (3.7) and rearranging terms, we have

$$\begin{aligned}(G(u^\ell) - G(u^*))^\top (u^\ell - \bar{u}^\ell) &\geqslant (G(u^\ell) - G(u^*))^\top (u^\ell - u^*) + (F(\bar{u}^\ell) - F(u^*))^\top (\bar{u}^\ell - u^*) \\ &\quad + \frac{1}{\lambda_\ell}(u^\ell - \bar{u}^\ell)^\top (u^* - \bar{u}^\ell) \\ &\geqslant \frac{1}{\lambda_\ell}(u^\ell - \bar{u}^\ell)^\top (u^* - \bar{u}^\ell),\end{aligned} \tag{3.8}$$

where the second inequality follows from the monotonicity of F and G. Thus

$$\begin{aligned}&(u^\ell - \bar{u}^\ell)^\top (u^\ell - u^* + \lambda_\ell (G(u^\ell) - G(u^*))) \\ &= (u^\ell - \bar{u}^\ell)^\top (u^\ell - u^*) + \frac{1}{\lambda_\ell}(u^\ell - \bar{u}^\ell)^\top (G(u^\ell) - G(u^*)) \\ &\geqslant (u^\ell - \bar{u}^\ell)^\top (u^\ell - u^*) + (u^\ell - \bar{u}^\ell)^\top (u^* - \bar{u}^\ell) \\ &= \|u^\ell - \bar{u}^\ell\|^2.\end{aligned} \tag{3.9}$$

This completes the proof. □

LEMMA 3.4. *Assume that F and G are set-valued monotone mappings, and*

$$2 > \alpha_{\max} \geqslant \alpha_\ell \geqslant \alpha_0 > 0 \textit{ for all } \ell.$$

Then the sequence $\{u^\ell\}$ generated by Algorithm 3.1 satisfies

$$\begin{aligned}&\|u^{\ell+1} - u^* + \lambda_\ell (G(u^{\ell+1}) - G(u^*))\|^2 \\ &\quad \leqslant \|u^\ell - u^* + \lambda_\ell (G(u^\ell) - G(u^*))\|^2 - \alpha_\ell (2 - \alpha_\ell)\|u^\ell - \bar{u}^\ell\|^2.\end{aligned} \tag{3.10}$$

Proof. Using (3.3), we have

$$\begin{aligned}\|u^{\ell+1}-u^*+\lambda_\ell(G(u^{\ell+1})-G(u^*))\|^2 &= \|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))-\alpha_\ell(u^\ell-\bar{u}^\ell)\|^2\\ &\leqslant \|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\|^2+\alpha_\ell^2\|u^\ell-\bar{u}^\ell\|^2\\ &\quad -2\alpha_\ell(u^\ell-\bar{u}^\ell)^\top(u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*)))\\ &\leqslant \|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\|^2+\alpha_\ell^2\|u^\ell-\bar{u}^\ell\|^2\\ &\quad -2\alpha_\ell\|u^\ell-\bar{u}^\ell\|^2\\ &\leqslant \|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\|^2-\alpha_\ell(2-\alpha_\ell)\|u^\ell-\bar{u}^\ell\|^2,\end{aligned} \tag{3.11}$$

where the inequality follows from Lemma 3.3. Hence, proof is completed. □

REMARK 3.5. Since

$$0<\lambda_{\ell+1}\leqslant(1+\tau_\ell)\lambda_\ell,$$

it follows from the monotonicity of G that

$$\begin{aligned}\|u^{\ell+1}-u^*+\lambda_{\ell+1}(G(u^{\ell+1})-G(u^*))\|^2 &\leqslant (1+\tau_\ell)^2\|u^{\ell+1}-u^*+\lambda_\ell(G(u^{\ell+1})-G(u^*))\|^2\\ &\leqslant (1+\tau_\ell)^2\|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\|^2\\ &\quad -\alpha_\ell(2-\alpha_\ell)\|u^\ell-\bar{u}^\ell\|^2.\end{aligned} \tag{3.12}$$

Denote

$$\mathfrak{S}_p=\prod_{\ell=0}^{\infty}(1+\tau_\ell)^2.$$

Then (3.1) implies

$$\mathfrak{S}_p<+\infty,$$

and we have

$$\{\lambda_\ell\}\subset\left[\frac{1}{\sqrt{\mathfrak{S}_p}}\lambda_0,\sqrt{\mathfrak{S}_p}\lambda_0\right]=[\lambda_{\min},\lambda_{\max}],$$

i.e. the sequence $\{\lambda_\ell\}$ is bounded.

THEOREM 3.6. *Assume that F and G are set-valued monotone mappings, and*

$$2>\alpha_{\max}\geqslant\alpha_\ell\geqslant\alpha_0>0 \text{ for all } \ell.$$

Then the sequence $\{u^\ell\}$ generated by Algorithm 3.1 converges to a solution of (1.1).

Proof. Using (3.12) and $2>\alpha_\ell>0$, we have

$$\begin{aligned}\|u^{\ell+1}-u^*+\lambda_{\ell+1}(G(u^{\ell+1})-G(u^*))\|^2 &\leqslant (1+\tau_\ell)^2\|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\|^2\\ &\leqslant \prod_{\kappa=0}^{\ell}(1+\tau_\kappa)^2\|u^0-u^*+\lambda_0(G(u^0)-G(u^*))\|^2\\ &\leqslant \mathfrak{S}_p\|u^0-u^*+\lambda_0(G(u^0)-G(u^*))\|^2<+\infty.\end{aligned} \tag{3.13}$$

From the monotonicity of G, the sequence $\{u^\ell\}$ is bounded. Furthermore, it follows from (3.12) that

$$\begin{aligned}
&\alpha_\ell(2-\alpha_\ell)\|u^\ell-\bar{u}^\ell\|^2\\
&\leqslant (1+\tau_\ell)^2\|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\|^2-\|u^{\ell+1}-u^*+\lambda_{\ell+1}(G(u^{\ell+1})-G(u^*))\|^2\\
&=(\|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\|^2-\|u^{\ell+1}-u^*+\lambda_{\ell+1}(G(u^{\ell+1})-G(u^*))\|^2)\\
&\quad+(2\tau_\ell+\tau_\ell^2)\|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\|^2. \qquad (3.14)
\end{aligned}$$

Summing both sides for all ℓ, we have

$$\begin{aligned}
\sum_{\ell=0}^{\infty}\alpha_\ell(2-\alpha_\ell)\|u^\ell-\bar{u}^\ell\|^2 &\leqslant \sum_{\ell=0}^{\infty}(\|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\|^2\\
&\quad-\|u^{\ell+1}-u^*+\lambda_{\ell+1}(G(u^{\ell+1})-G(u^*))\|^2)\\
&\quad+\sum_{\ell=0}^{\infty}(2\tau_\ell+\tau_\ell^2)\|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\|^2\\
&\leqslant\left(\sum_{\ell=0}^{\infty}(2\tau_\ell+\tau_\ell^2)\mathfrak{S}_p+1\right)\|u^0-u^*+\lambda_0(G(u^0)-G(u^*))\|^2\\
&<+\infty, \qquad (3.15)
\end{aligned}$$

here the second inequality of (3.15) follows from (3.13) and the last inequality follows from (3.1). Since $\{\alpha_\ell\}$ is uniformly bounded away from 0 and 2, we have from (3.15) that

$$\lim_{\ell\longrightarrow\infty}\|u^\ell-\bar{u}^\ell\|=0. \qquad (3.16)$$

From (3.3),

$$\lim_{\ell\longrightarrow\infty}\|u^\ell-u^{\ell+1}\|=0.$$

Since $\{u^\ell\}$ is bounded, it has at least one cluster point. Let $\tilde{u}\in\Omega$ be a cluster point and $\{u^{\ell_j}\}$ be the corresponding subsequence converging to $\tilde{u}$. It follows from (3.16) that $\{\bar{u}^{\ell_j}\}$ also converges to $\tilde{u}$. Since

$$\hat{\mathscr{H}}(F(u^\ell),F(\tilde{u}))\leqslant\delta\|u^\ell-\tilde{u}\|=0 \text{ as } \ell\longrightarrow\infty$$

and

$$\hat{\mathscr{H}}(G(u^\ell),G(\tilde{u}))\leqslant\delta\|u^\ell-\tilde{u}\|=0 \text{ as } \ell\longrightarrow\infty.$$

Therefore

$$F(u^\ell)\longrightarrow F(\tilde{u})$$

and

$$G(u^\ell)\longrightarrow G(\tilde{u}).$$

On the other hand, (3.2) is equivalent to

$$\varphi(\bar{u}^\ell,\lambda_\ell)=\bar{u}^\ell-P_\Omega[u^\ell-\lambda_\ell(F(\bar{u}^\ell)+G(u^\ell))]=0. \qquad (3.17)$$

By using (2.4) and $\lambda_\ell \geqslant \lambda_{\min}$ for all ℓ, we have that (3.17) is equivalent to

$$\varphi(\bar{u}^\ell, \lambda_{\min}) = 0. \tag{3.18}$$

Hence, taking limit along this subsequence, it follows from the continuity of $\varphi(\cdot,\cdot)$,

$$\|\varphi(\tilde{u}, \lambda_{\min})\| = \lim_{j\longrightarrow\infty} \|\varphi(\bar{u}^{\ell_j}, \lambda_{\min})\| = 0,$$

which implies that $\tilde{u} \in \Omega$ is a solution of (1.1). The inequality (3.10) indicates that the whole sequence $\{u^\ell\}$ has just one cluster point. Hence, $\{u^\ell\}$ converges to $\tilde{u}$, a solution of (1.1). This completes the proof. □

4. Self-adaptive strategy

We showed in section 3 that if the parameter λ_ℓ satisfy

$$0 < \lambda_L \leqslant \lambda_\ell \leqslant \lambda_U < +\infty, \quad \forall \ell \geqslant 0. \tag{4.1}$$

Then the algorithm generates a sequence $\{u^\ell\}$ which converges to a solution of (1.1). This means that λ_ℓ can be selected randomly provided that the condition (4.1) is fulfilled, then the efficiency of the algorithm depends on a suitable λ_ℓ. Now, for a given constant $\tau > 0$,

$$\lambda_{\ell+1} = \begin{cases} \dfrac{1}{1+\tau_\ell}\lambda_\ell, & \text{if } \rho_\ell > 1+\tau, \\ (1+\tau_\ell)\lambda_\ell, & \text{if } \rho_\ell > \dfrac{1}{1+\tau}, \\ \lambda_\ell, & \text{otherwise,} \end{cases} \tag{4.2}$$

where $\tau_\ell \geqslant 0$, $\sum_{\ell=0}^{\infty} \tau_\ell < +\infty$, and

$$\rho_\ell = \frac{\|\lambda_\ell(T(u^{\ell+1}) - T(u^\ell))\|}{\|u^{\ell+1} - u^\ell\|}. \tag{4.3}$$

From (4.2) we have

$$\|\lambda_\ell(T(u^{\ell+1}) - T(u^\ell))\| \approx \|u^{\ell+1} - u^\ell\|.$$

Hence, if

$$\|\lambda_\ell(T(u^{\ell+1}) - T(u^\ell))\| > (1+\tau)\|u^{\ell+1} - u^\ell\|,$$

we should decrease λ in the next iteration, otherwise we should increase λ when

$$\|\lambda_\ell(T(u^{\ell+1}) - T(u^\ell))\| < \frac{1}{1+\tau}\|u^{\ell+1} - u^\ell\|.$$

Since the mapping T is not fully known and the system of nonlinear equation (3.3) only involved with the known part G. Therefore from the self-adaptive strategy (4.2), (4.3), using G instead of T, we have

$$\rho'_\ell = \frac{\|\lambda_\ell(G(u^{\ell+1}) - G(u^\ell))\|}{\|u^{\ell+1} - u^\ell\|} \tag{4.4}$$

then the set

$$\lambda_{\ell+1} = \begin{cases} \dfrac{1}{1+\tau_\ell}\lambda_\ell, & \text{if } \rho'_\ell > 1+\tau, \\ (1+\tau_\ell)\lambda_\ell, & \text{if } \rho'_\ell > \dfrac{1}{1+\tau}, \\ \lambda_\ell, & \text{otherwise.} \end{cases} \tag{4.5}$$

5. Approximate solution

Since $\bar{u}^\ell$ is a solution of (3.2). Therefore $\tilde{u}^\ell$ satisfies

$$\|\tilde{u}^\ell - \bar{u}^\ell\| \leqslant \eta_\ell \|u^\ell - \tilde{u}^\ell\|, \tag{5.1}$$

we regard it as an approximate solution, where $\eta_\ell \geqslant 0$ and

$$\sum_{\ell=0}^{\infty} \eta_\ell^2 < +\infty.$$

Now (3.3) can be rewritten as finding a zero point of ϑ_ℓ,

$$\vartheta_\ell(u) = u + \lambda_\ell G(u) - (u^\ell + \lambda_\ell G(u^\ell) - \alpha_\ell(u^\ell - \tilde{u}^\ell)). \tag{5.2}$$

We find an approximate solution of $\{u^{\ell+1}\}$ which satisfies

$$\|\vartheta_\ell(u^{\ell+1})\| \leqslant \eta_\ell \|u^\ell - \tilde{u}^\ell\|. \tag{5.3}$$

From above two strategies into Algorithm 3.1, we have

ALGORITHM 5.1.

Step 0. Choose an arbitrary initial point $u^0 \in \Omega$, $\varepsilon > 0$, $1 > \alpha_0 > 0$, $\lambda_0 > 0$, and a nonnegative sequence $\{\tau_\ell\}$ satisfying

$$\sum_{\ell=0}^{\infty} \tau_\ell < +\infty. \tag{5.4}$$

Set $\ell = 0$.

Step 1. For given u^ℓ, observe the system at an equilibrium state to obtain an approximate solution $\tilde{u}^\ell$ of (3.2) satisfying (5.1).

Step 2. Choose $\frac{1}{2} > \alpha_\ell \geqslant \alpha_0$ and find the next iteration $\{u^{\ell+1}\}$ according to (5.3).

Step 3. If

$$\|u^{\ell+1} - u^\ell\| \leqslant \varepsilon,$$

stop, otherwise choose a new parameter

$$\lambda_{\ell+1} \in \left[\frac{1}{1+\tau_\ell}\lambda_\ell, (1+\tau_\ell)\lambda_\ell\right],$$

according to the self-adaptive rule (4.4).

Set $\ell = \ell + 1$ and go to Step 1.

LEMMA 5.2. *Assume that F and G are set-valued monotone mappings. Then the sequence $\{u^\ell\}$ generated by Algorithm 5.1 satisfies*

$$\begin{aligned}
&\|u^{\ell+1}-u^*+\lambda_\ell(G(u^{\ell+1})-G(u^*))\|^2 \\
&\leqslant \left(1+\frac{8\eta_\ell^2}{\alpha_\ell}\right)\|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\|^2-\alpha_\ell\|u^\ell-\bar{u}^\ell\|^2 \\
&\quad -\alpha_\ell\|\bar{u}^\ell-\tilde{u}^\ell\|^2-\left(\frac{3\alpha_\ell}{4}-\frac{\alpha_\ell^3}{4}-\alpha_\ell^2-2\alpha_\ell\eta_\ell-\eta_\ell^2\right)\|u^\ell-\tilde{u}^\ell\|^2. \qquad (5.5)
\end{aligned}$$

Proof. Using (5.2), we have

$$\begin{aligned}
&\|u^{\ell+1}-u^*+\lambda_\ell(G(u^{\ell+1})-G(u^*))\|^2 \\
&\leqslant \|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))-\alpha_\ell(u^\ell-\tilde{u}^\ell)+\vartheta_\ell(u^{\ell+1})\|^2 \\
&\leqslant \|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\|^2+\alpha_\ell^2\|u^\ell-\tilde{u}^\ell\|^2+\|\vartheta_\ell(u^{\ell+1})\|^2 \\
&\quad -2\alpha_\ell\left(u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\right)^\top(u^\ell-\tilde{u}^\ell)-2\alpha_\ell(u^\ell-\tilde{u}^\ell)^\top\vartheta_\ell(u^{\ell+1}) \\
&\quad -2\left(u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\right)^\top\vartheta_\ell(u^{\ell+1}). \qquad (5.6)
\end{aligned}$$

For any two vectors a and b in $\mathbb{R}^n$, we have

$$2\|a\|\|b\|\leqslant p\|a\|^2+\frac{1}{p}\|b\|^2,\ \ \forall p>0. \qquad (5.7)$$

Therefore

$$\begin{aligned}
&2\left(u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\right)^\top\vartheta_\ell(u^{\ell+1}) \\
&\qquad\leqslant \frac{4\eta_\ell^2}{\alpha_\ell}\|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\|^2+\frac{\alpha_\ell}{4\eta_\ell^2}\|\vartheta_\ell(u^{\ell+1})\|^2 \\
&\qquad\leqslant \frac{4\eta_\ell^2}{\alpha_\ell}\|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\|^2+\frac{\alpha_\ell}{4}\|u^\ell-\tilde{u}^\ell\|^2, \qquad (5.8)
\end{aligned}$$

where the last inequality follows from (5.3).

Let $\bar{u}^\ell$ be the solution of (3.2). Then, we have

$$\begin{aligned}
&-2\alpha_\ell\left(u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\right)^\top(u^\ell-\tilde{u}^\ell) \\
&\leqslant -2\alpha_\ell\|u^\ell-\bar{u}^\ell\|^2-2\alpha_\ell\left(u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\right)^\top(\bar{u}^\ell-\tilde{u}^\ell) \\
&\leqslant -2\alpha_\ell\|u^\ell-\bar{u}^\ell\|^2+\alpha_\ell\left(\frac{4\eta_\ell^2}{\alpha_\ell^2}\|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\|^2+\frac{\alpha_\ell^2}{4\eta_\ell^2}\|\bar{u}^\ell-\tilde{u}^\ell\|^2\right) \\
&= -\alpha_\ell\|u^\ell-\bar{u}^\ell\|^2-\alpha_\ell\|u^\ell-\tilde{u}^\ell+\tilde{u}^\ell-\bar{u}^\ell\|^2 \\
&\quad +\frac{4\eta_\ell^2}{\alpha_\ell}\|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\|^2+\frac{\alpha_\ell^3}{4\eta_\ell^2}\|\bar{u}^\ell-\tilde{u}^\ell\|^2
\end{aligned}$$

$$\leqslant -\alpha_\ell\|u^\ell-\bar{u}^\ell\|^2-\alpha_\ell\|u^\ell-\tilde{u}^\ell\|^2-\alpha_\ell\|\bar{u}^\ell-\tilde{u}^\ell\|^2+2\alpha_\ell\eta_\ell\|u^\ell-\tilde{u}^\ell\|^2$$
$$+\frac{4\eta_\ell^2}{\alpha_\ell}\|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\|^2+\frac{\alpha_\ell^3}{4}\|u^\ell-\tilde{u}^\ell\|^2, \tag{5.9}$$

where the first inequality follows from Lemma 3.3, the second inequality follows from (5.7) and last inequality follows from (5.1) Substituting (5.8) and (5.9) into (5.6) and rearranging term, we get (5.5) immediately. This completes the proof. □

REMARK 5.3. Since

$$\eta_\ell \longrightarrow 0 \text{ as } \ell \longrightarrow \infty,$$

if we take

$$0<\alpha_0\leqslant\alpha_\ell<\frac{1}{2},$$

then there exists $\ell_1 \geqslant 0$ such that for all $\ell \geqslant \ell_1$,

$$\frac{3\alpha_\ell}{4}-\frac{\alpha_\ell^3}{4}-\alpha_\ell^2-2\alpha_\ell\eta_\ell-\eta_\ell^2>0. \tag{5.10}$$

Without loss of generality, we assume that (5.10) holds for all $\ell \geqslant 0$.

THEOREM 5.4. *Assume that F and G are set-valued monotone mappings and*

$$\frac{1}{2}>\alpha_\ell\geqslant\alpha_0>0 \textit{ for all } \ell.$$

Then the sequence $\{u^\ell\}$ generated by Algorithm 5.1 converges to a solution of (1.1).

Proof. Combining (3.12), (5.5) and the assumption $\frac{1}{2}>\alpha_\ell\geqslant\alpha_0>0$, we have

$$\begin{aligned}
&\|u^{\ell+1}-u^*+\lambda_{\ell+1}(G(u^{\ell+1})-G(u^*))\|^2\\
&\leqslant(1+\tau_\ell)^2\left(1+\frac{8\eta_\ell^2}{\alpha_\ell}\right)\|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\|^2-\alpha_\ell\|u^\ell-\bar{u}^\ell\|^2\\
&\leqslant\prod_{\kappa=0}^{\ell}(1+\tau_\kappa)^2\prod_{\kappa=0}^{\ell}\left(1+\frac{8\eta_\kappa^2}{\alpha_0}\right)\|u^0-u^*+\lambda_0(G(u^0)-G(u^*))\|^2\\
&\leqslant\mathfrak{S}_p\mathfrak{S}_\eta\|u^0-u^*+\lambda_0(G(u^0)-G(u^*))\|^2,
\end{aligned} \tag{5.11}$$

where

$$\mathfrak{S}_p=\prod_{\ell=0}^{\infty}(1+\tau_\ell)^2$$

and

$$\mathfrak{S}_\eta=\prod_{\ell=0}^{\infty}\left(1+\frac{8\eta_\ell^2}{\alpha_0}\right).$$

Since $\tau_\ell>0$, $\eta_\ell\geqslant0$, $\sum_{\ell=0}^{\infty}\tau_\ell<+\infty$, $\sum_{\ell=0}^{\infty}\eta_\ell^2<+\infty$, we have

$$\mathfrak{S}_p<+\infty \text{ and } \mathfrak{S}_\eta<+\infty.$$

From (5.11), we get that $\{u^\ell\}$ is bounded, and

$$\alpha_\ell\|u^\ell-\bar{u}^\ell\|^2 \leqslant (1+\tau_\ell)^2\left(1+\frac{8\eta_\ell^2}{\alpha_\ell}\right)\|u^\ell-u^*+\lambda_\ell(G(u^\ell)-G(u^*))\|^2 - \|u^{\ell+1}-u^*+\lambda_{\ell+1}(G(u^{\ell+1})-G(u^*))\|^2. \tag{5.12}$$

Summing the above sides for all $\ell \geqslant \ell_1$ and using a similar way as (3.15), we have

$$\sum_{\ell=\ell_1}^{\infty}\alpha_\ell\|u^\ell-\bar{u}^\ell\|^2 \leqslant \left(\sum_{\ell=0}^{\infty}\left((2\tau_\ell+\tau_\ell^2)\left(1+\frac{8\eta_\ell^2}{\alpha_\ell}\right)+\frac{8\eta_\ell^2}{\alpha_\ell}\right)\mathfrak{S}_p\mathfrak{S}_\eta+1\right) \times\|u^0-u^*+\lambda_0(G(u^0)-G(u^*))\|^2<+\infty. \tag{5.13}$$

Thus

$$\lim_{\ell\longrightarrow\infty}\|u^\ell-\bar{u}^\ell\|=0,$$

and

$$\hat{\mathscr{H}}(G(u^\ell),G(\bar{u}^\ell))\leqslant\delta\|u^\ell-\bar{u}^\ell\|=0,\ \text{as}\ \ell\longrightarrow\infty.$$

From the similar argument as those in Theorem 3.6, we assert that the sequence $\{u^\ell\}$ generated by Algorithm 5.1 converges to a solution of (1.1). This completes the proof. □

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