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A NEW ALGORITHM FOR FINDING A COMMON SOLUTION OF A SPLIT VARIATIONAL INEQUALITY PROBLEM, THE FIXED POINT PROBLEMS AND THE VARIATIONAL INCLUSION PROBLEMS

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Abstract. In this paper, we present a new iterative algorithm for finding a common element of the set of solutions of a split variational inequality problem, the set of fixed points of an infinite family of nonexpansive mappings and the set of solutions of a variational inclusion problem in Hilbert spaces. Under some mild conditions imposed on algorithm parameters, we prove that the proposed iterative algorithm have strong convergence.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H.

In this article, our study is related to a classical variational inequality problem (VIP) which aims to find an element $x^{\dagger} \in C$ such that

$$\langle Bx^{\dagger}, x - x^{\dagger} \rangle \geqslant 0, \ \ \forall x \in C, \tag{1.1}$$

where $B : H \to H$ is a given operator. It is well known that $x^{\sharp} \in VI(B,C)$ if and only if $x^{\sharp} = P_C(x^{\sharp} - \zeta Bx^{\sharp})$, where $\zeta > 0$, in other words, the VIP is equivalent to the fixed point problem (see [2]).

Variational inequality problem (VIP) was introduced by Stampacchia [14] and provide a useful tool for researching a large variety of interesting problems arising in physics, economics, finance, elasticity, optimization, network analysis, medical images, water resources, and structural analysis. For some related work, please refer to References ([12], [13], [15]–[22]).

The another motivation of this article is the split common fixed point problem which aims to find a point $u \in H_1$ such that

$$u \in Fix(T) \quad \text{and} \quad Au \in Fix(S). \tag{1.2}$$

The split common fixed point problem can be regarded as a generalization of the split feasibility problem. Recall that the split feasibility problem is to find a point satisfying

$$u \in C \quad \text{and} \quad Au \in Q, \tag{1.3}$$

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where C and Q are two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively and $A : H_1 \to H_2$ is a bounded linear operator. Inverse problems in various disciplines can be expressed as the split feasibility problems and the split common fixed point problem. Problem (1.2) was firstly introduced by Censor and Segal [26]. Note that solving (1.2) can be translated to solve the fixed point equation

$$u = S(u - \tau A^*(I - T)Au), \ \tau > 0.$$

Whereafter, Censor and Segal proposed an algorithm for directed operators. Since then, there has been growing interest in the split common fixed point problem ([23], [25], [27], [29]–[33]).

Now, we recall that the variational inclusion problem is to find an element $x^\dagger \in C$ such that

$$\theta \in D(x^\dagger) + R(x^\dagger), \tag{1.4}$$

where $D : C \to H$ is a single-valued mapping, $R : C \to 2^H$ is a set-valued mapping and θ is the zero vector in H.

The set of solutions of the problem (1.4) is denoted by $I(D,R)$. If $H = R^m$, then the problem (1.4) becomes the generalized equation introduced by Robinson ([3]). If $D \equiv 0$, then the problem (1.4) becomes the inclusion problem introduced by Rockafellar ([4]). It is known that the problem (1.4) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complementarity problems, variational inequalities, optimal control, mathematical economics, equilibria and game theory, etc. Also, various types of variational inclusions problems have been extended and generalized, for more details, refer to ([6]–[11]) and the references therein.

In this article, we will study the following split variational inequality problem of finding an element $x^\natural$ such that

$$x^\natural \in Fix(S) \bigcap VI(B,C) \bigcap I(D,R) \ \text{ and } \ Ax^\natural \in \bigcap_{n=1}^{\infty} Fix(T_n) \bigcap VI(F,Q). \tag{1.5}$$

Subsequently, we construct a new algorithm for solving the split variational inequality problem (1.5). Strong convergence theorems are established under some mild assumptions.

2. Preliminaries

In this section, we collect some tools including some definitions, useful inequalities and lemmas which will be used to derive our main results in the next section.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. Let $T : C \longrightarrow C$ be an operator. We use $Fix(T)$ to denote the set of fixed points of T, that is, $Fix(T) = \{u | u = Tu, u \in C\}$.

First, we give some definitions related to the involed operators.

DEFINITION 2.1. An operator $T : C \longrightarrow C$ is called to be nonexpansive if $\|Tu - Tv\| \leqslant \|u - v\|$ for all $u, v \in C$.

DEFINITION 2.2. An operator $T : C \longrightarrow C$ is called to be firmly nonexpansive if $\|Tu - Tv\|^2 \leqslant \|u - v\|^2 - \|(I - T)u - (I - T)v\|^2$ for all $u, v \in C$, or equivalently,

$$\langle Tu - Tv, u - v\rangle \geqslant \|Tu - Tv\|^2 \tag{2.1}$$

for all $u, v \in C$.

DEFINITION 2.3. An operator $T : C \longrightarrow C$ is called to be α-averaged, if there exists a nonexpansive operator U such that $T = (1 - \alpha)I + \alpha U$, where I is an identity mapping.

DEFINITION 2.4. An operator $T : C \longrightarrow C$ is said to be quasinonexpansive if $\|Tx - x^\dagger\| \leqslant \|x - x^\dagger\|$ for all $x \in C$ and $x^\dagger \in Fix(T)$, or equivalently,

$$\langle x - Tx, x - x^\dagger\rangle \geqslant \frac{1}{2}\|x - Tx\|^2 \tag{2.2}$$

for all $x \in C$ and $x^\dagger \in Fix(T)$.

REMARK 2.5. Obviously, if $Fix(T) \neq \emptyset$, then nonexpansive operators are quasinonexpansive operators.

DEFINITION 2.6. An operator $T : C \longrightarrow C$ is said to be strictly quasinonexpansive if $\|Tx - x^\dagger\| < \|x - x^\dagger\|$ for all $x \in C$ and $x^\dagger \in Fix(T)$.

REMARK 2.7. It is well known that an averaged operator T with $Fix(T) \neq \emptyset$ is strictly quasinonexpansive. For more details, Please refer to [1].

DEFINITION 2.8. An operator $F : C \longrightarrow H$ is said to be α-inverse strongly monotone if $\langle Fx - Fx^\dagger, x - x^\dagger\rangle \geqslant \alpha\|Fx - Fx^\dagger\|^2$ for some constant $\alpha > 0$ and all $x, x^\dagger \in C$.

Usually, some additional smoothness properties of the mapping are required in the study of the convergence of fixed point algorithms, such as demi-closedness.

DEFINITION 2.9. An operator T is said to be demiclosed at w if, for any sequence $\{u_n\}$ which weakly converges to u^*, and if $Tu_n \longrightarrow w$, then $Tu^* = w$.

Recall that the projection from H onto C, denoted P_C, assigns to each $u \in H$, the unique point $P_C u \in C$ satisfying

$$\|u - P_C u\| = inf\{\|u - v\| : v \in C\}.$$

P_C can be characterized by

$$\langle u - P_C u, v - P_C u\rangle \leqslant 0 \tag{2.3}$$

for all $u \in H, v \in C$, and $P_C : H \to C$ is firmly nonexpansive, that is,

$$\langle u - v, P_C u - P_C v\rangle \geqslant \|P_C u - P_C v\|^2$$
$$\text{or } \|P_C u - P_C v\|^2 \leqslant \|u-v\|^2 - \|(I-P_C)u - (I-P_C)v\|^2$$

for all $u, v \in H$.

For all $u, v \in H$, the following conclusions hold:

$$\|tu + (1-t)v\|^2 = t\|u\|^2 + (1-t)\|v\|^2 - t(1-t)\|u-v\|^2, \ \ t \in [0,1],$$

$$\|u+v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2$$

and

$$\|u+v\|^2 \leqslant \|u\|^2 + 2\langle v, u+v\rangle.$$

In the following text, we employ the following notations:

- $u_n \rightharpoonup u$ stands for that $\{u_n\}$ converges weakly to u;
- $u_n \to u$ stands for that $\{u_n\}$ converges strongly to u;
- $Fix(T)$ means the set of fixed points of T;
- $\omega_w(u_n)$ means the set of cluster points in the weak topology, that is,

$$\omega_w(u_n) = \{u : \exists u_{n_j} \rightharpoonup u\}.$$

LEMMA 2.10. ([24]) *Let H be a Hilbert space and $C(\neq \emptyset) \subset H$ be a closed convex set. If $F : C \to H$ is an α-inverse strongly monotone operator, then*

$$\|x - \gamma Fx - (y - \gamma Fy)\|^2 \leqslant \|x-y\|^2 + \gamma(\gamma - 2\alpha)\|Fx - Fy\|^2, \ \ \forall x, y \in C.$$

Especially, $I - \gamma F$ is nonexpansive provided $0 \leqslant \gamma \leqslant 2\alpha$.

LEMMA 2.11. ([9]) *Let C be a nonempty closed convex of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is, if $x_n \rightharpoonup x \in C$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be an infinite family of nonexpansive mappings and $\lambda_1, \lambda_2, \dots$ be real numbers such that $0 \leqslant \lambda_i \leqslant 1$ for each $i \in \mathbb{N}$. For any $n \in \mathbb{N}$, define a mapping W_n of C into C as follows:

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I,\\
&U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I,\\
&\dots\\
&U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I,\\
&U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I,\\
&\dots\\
&U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I,\\
&W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I.
\end{aligned} \tag{2.4}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \dots, T_1$ and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$. We have the following crucial lemma concerning W_n in ([5]):

LEMMA 2.12. *Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be an infinite family of nonexpansive mappings such that $\bigcap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$. Let $\lambda_1, \lambda_2, \dots$ be real numbers such that $0 \leqslant \lambda_i \leqslant b < 1$ for each $i \geqslant 1$. Then we have the following:*

(1) For any $x \in C$ and $k \geqslant 1$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists;

(2) $Fix(W) = \bigcap_{n=1}^{\infty} Fix(T_n)$, where $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$, $\forall x \in C$;

(3) For any bounded sequence $\{x_n\} \subset C$, $\lim_{n\to\infty} Wx_n = \lim_{n\to\infty} W_n x_n$.

LEMMA 2.13. ([28]) *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leqslant (1-\gamma_n)\alpha_n + \delta_n, \ \ n \in N,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leqslant 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$;

Then $\lim_{n\to\infty} \alpha_n = 0$.

3. Main results

Throughout the present article, let H_1 and H_2 be two real Hilbert spaces. Let $C(\neq \emptyset) \subset H_1$ and $Q(\neq \emptyset) \subset H_2$ be two closed convex sets. We use $\langle \cdot, \cdot \rangle$ to denote the inner product and $\|\cdot\|$ stands for the corresponding norm.

Next, we show two lemmas for quasinonexpansive operators. These lemmas will be very useful for our main theorem.

LEMMA 3.1. *Let $T_1 : C \to C$ be strictly quasinonexpansive operator, and let $T_2 : C \to C$ be quasinonexpansive operator. Suppose that $Fix(T_1) \bigcap Fix(T_2) \neq \emptyset$. Then $Fix(T_1) \bigcap Fix(T_2) = Fix(T_1 T_2)$.*

Proof. $Fix(T_1) \bigcap Fix(T_2) \subset Fix(T_1 T_2)$ is obvious. We only need to prove that

$$Fix(T_1 T_2) \subset Fix(T_1) \bigcap Fix(T_2).$$

Let $x^\sharp \in Fix(T_1 T_2)$ and $z \in Fix(T_1) \bigcap Fix(T_2)$. We consider the following two cases:

(i) $T_2 x^\sharp \in Fix(T_1)$. Then $T_2 x^\sharp = T_1 T_2 x^\sharp = x^\sharp$. Therefore, $x^\sharp \in Fix(T_2) \bigcap Fix(T_1)$.

(ii) $T_2 x^\sharp \notin Fix(T_1)$, and hence $x^\sharp \notin Fix(T_2)$. Since T_1 is strictly quasinonexpansive, we have that $\|x^\sharp - z\| = \|T_1 T_2 x^\sharp - z\| < \|T_2 x^\sharp - z\| \leqslant \|x^\sharp - z\|$, which yields a contradiction. $\square$

REMARK 3.2. As a matter of fact, in Lemma 3.1, assuming that $T_1 : C \to C$ is quasinonexpansive operator and $T_2 : C \to C$ is strictly quasinonexpansive operator, the conclusion still holds.

LEMMA 3.3. *Assume that* $\{\varpi_n\}$ *is a sequence of nonnegative real numbers such that*

$$\varpi_{n+1} \leqslant (1-\alpha_n)\varpi_n + \alpha_n\delta_n, \ \ n \geqslant 1,$$

where $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *such that* $\sum_{n=1}^{\infty}\alpha_n = \infty$. *Then* $\limsup_{n\to\infty}\delta_n \geqslant 0$.

Proof. By contradiction, we suppose that

$$\limsup_{n\to\infty}\delta_n = -\kappa < 0,$$

which implies that there exists large enough positive integer N such that $\delta_n < -\frac{\kappa}{2}$ for all $n > N$. Note that

$$\begin{aligned} \varpi_{n+1} &\leqslant (1-\alpha_n)\varpi_n + \alpha_n\delta_n \\ &= \varpi_n + \alpha_n(\delta_n - \varpi_n) \\ &< \varpi_n - \frac{\alpha_n\kappa}{2} \end{aligned} \tag{3.1}$$

for all $n > N$. Thus,

$$\varpi_{n+1} \leqslant \varpi_{N+1} - \frac{\kappa}{2}\sum_{i=N+1}^{n}\alpha_i.$$

Taking $\limsup_{n\to\infty}$ in the last inequality, we have

$$0 \leqslant \limsup_{n\to\infty}\varpi_{n+1} \leqslant \varpi_{N+1} - \frac{\kappa}{2}\sum_{i=N+1}^{\infty}\alpha_i = -\infty,$$

which yields a contradiction. Consequently, $\limsup_{n\to\infty}\delta_n \geqslant 0$. $\square$

In the sequel, we state several assumptions and symbols:

(H1): $A : H_1 \longrightarrow H_2$ is a bounded linear operator with its adjoint A^*.

(H2): $\{T_n\}_{n=1}^{\infty} : Q \to Q$ is an infinite family of nonexpansive mappings.

(H3): $S : C \longrightarrow C$ is a nonexpansive operator and $S_\delta = (1-\delta)I + \delta S$, where $\delta \in (0,1)$.

(H4): $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$.

(H5): $B : C \to H_1$ is a β-inverse strongly monotone operator.

(H6): $F : H_2 \to H_2$ is an α-inverse strongly monotone operator.

(H7): $D : C \to H_1$ is a κ-inverse strongly monotone operator.

(H8): R is a maximal monotone operator on H, such that the domain $D(R)$ of R is included in C.

(H9): $J_{R,\lambda} = (I+\lambda R)^{-1}$ is the resolvent of R for λ.

(H10): $\Omega = \{x^\natural | x^\natural \in Fix(S)\bigcap VI(B,C)\bigcap I(D,R) \ and \ Ax^\natural \in \bigcap_{n=1}^{\infty} Fix(T_n)\bigcap VI(F,Q)\}$.

Throughout this paper, we assume that $\Omega \neq \emptyset$.

REMARK 3.4. It is well known that the resolvent $J_{R,\lambda}$ is firmly nonexpansive and $R^{-1}0 = Fix(J_{R,\lambda})$ for all $\lambda > 0$(see [1]).

In the sequel, we present the following iterative algorithm to solve (1.5).

ALGORITHM 3.5. Let $x^\dagger \in C$ be a fixed point. Let $x_1 \in C$ be an initial value. Let $\{\alpha_n\}$ be a real number sequence in $(0,1)$. Let $\gamma \in (0,2\beta)$, $\varsigma \in (0,2)$, $\xi \in (0,2\alpha)$ and $\lambda \in (0,2\kappa)$ be four real constants. Assume that the sequence $\{x_n\}$ has been constructed. For the current iteration x_n, compute

$$\begin{aligned} y_n &= x_n - S_\delta P_C(I-\gamma B)x_n, \\ z_n &= Ax_n - W_n P_Q(I-\xi F)Ax_n. \end{aligned} \tag{3.2}$$

Case 1. If $\|y_n + A^* z_n\| \neq 0$, then continue and construct x_{n+1} via the following manner

$$\begin{aligned} u_n &= P_C(x_n - \varsigma\tau_n(y_n + A^* z_n)), \\ x_{n+1} &= \alpha_n x^\dagger + (1-\alpha_n)J_{R,\lambda}(u_n - \lambda D u_n) \end{aligned} \tag{3.3}$$

where

$$\tau_n = \frac{\|y_n\|^2 + \|z_n\|^2}{\|y_n + A^* z_n\|^2}. \tag{3.4}$$

Case 2. If $\|y_n + A^* z_n\| = 0$, then continue and construct x_{n+1} via the following manner

$$x_{n+1} = \alpha_n x^\dagger + (1-\alpha_n)J_{R,\lambda}(x_n - \lambda D x_n). \tag{3.5}$$

REMARK 3.6. Observing that S_δ, W_n and W are averaged operators and therefore strictly quasinonexpansive operators, in view of Lemma 3.1, we obtain that

$$Fix(S_\delta P_C(I-\gamma B)) = Fix(S_\delta)\bigcap VI(B,C) = Fix(S)\bigcap VI(B,C),$$

$$Fix(W_n P_Q(I-\xi F)) = Fix(W_n)\bigcap VI(F,Q)$$

and

$$Fix(W P_Q(I-\xi F)) = Fix(W)\bigcap VI(F,Q).$$

From now on, we will divide our main result into several propositions.

PROPOSITION 3.7. $\|y_n + A^* z_n\| = 0$ *if and only if* $x_n \in \Omega_n$, *where*

$$\Omega_n = \{x^\natural \mid x^\natural \in Fix(S)\bigcap VI(B,C) \ \text{and} \ Ax^\natural \in Fix(W_n)\bigcap VI(F,Q)\} \neq \emptyset.$$

Evidently, $\Omega \subset \Omega_n$.

Proof. Noting that $Fix(W_n) = \bigcap_{j=1}^n Fix(T_j)$ (see Lemma 3.1 in [5]), it is obvious that $x_n \in \Omega_n$ then $\|y_n + A^* z_n\| = 0$. To see the converse, assume that $\|y_n + A^* z_n\| = 0$.

In virtue of Lemma 2.10, we have that $S_\delta P_C(I-\gamma B)$ and $W_nP_Q(I-\xi F)$ is nonexpansive. Then, for any $x^\dagger \in \Omega_n$, in the light of Definition 2.4, we can derive

$$\begin{aligned}
0 &= \langle y_n + A^* z_n, x_n - x^\dagger\rangle \\
&= \langle x_n - S_\delta P_C(I-\gamma B)x_n, x_n - x^\dagger\rangle \\
&\quad + \langle A^*(I - W_nP_Q(I-\xi F))Ax_n, x_n - x^\dagger\rangle \\
&= \langle x_n - S_\delta P_C(I-\gamma B)x_n, x_n - x^\dagger\rangle \\
&\quad + \langle (I - W_nP_Q(I-\xi F))Ax_n, Ax_n - Ax^\dagger\rangle \\
&\geqslant \frac{1}{2}(\|x_n - S_\delta P_C(I-\gamma B)x_n\|^2 + \|(I-W_nP_Q(I-\xi F))Ax_n\|^2) \\
&= \frac{1}{2}(\|y_n\|^2 + \|z_n\|^2).
\end{aligned} \tag{3.6}$$

It follows that

$$\|x_n - S_\delta P_C(I-\gamma B)x_n\|^2 + \|(I-W_nP_Q(I-\xi F))Ax_n\|^2 = 0,$$

which implies that $x_n \in Fix(S_\delta P_C(I-\gamma B))$ and $Ax_n \in Fix(W_nP_Q(I-\xi F))$. In view of Remark 3.6, we have that $x_n \in Fix(S)\bigcap VI(B,C)$ and $Ax_n \in Fix(W_n)\bigcap VI(F,Q)$, that is, $x_n \in \Omega_n$. This completes the proof. $\square$

PROPOSITION 3.8. *The sequence $\{x_n\}$ generated by Algorithm 3.5 is bounded.*

Proof. Let $x^\flat \in \Omega$, that is, $P_C(x^\flat - \zeta Bx^\flat) = J_{R,\lambda}(x^\flat - \lambda Dx^\flat) = Sx^\flat = x^\flat$ for $\zeta,\lambda > 0$, and $P_Q(Ax^\flat - \xi FAx^\flat) = T_nAx^\flat = Ax^\flat$ for $\forall \xi > 0$ and all $n \geqslant 1$. First, in case 1, by virtue of (3.6), we can derive that

$$\begin{aligned}
\|u_n - x^\flat\| &= \|P_C(x_n - \varsigma\tau_n(y_n + A^*z_n)) - x^\flat\|^2 \\
&\leqslant \|x_n - x^\flat - \varsigma\tau_n(y_n + A^*z_n)\|^2 \\
&\leqslant \|x_n - x^\flat\|^2 + \varsigma^2\tau_n^2\|y_n + A^*z_n\|^2 - 2\langle \varsigma\tau_n(y_n + A^*z_n), x_n - x^\flat\rangle \\
&\leqslant \|x_n - x^\flat\|^2 + \varsigma^2\tau_n^2\|y_n + A^*z_n\|^2 - \varsigma\tau_n(\|y_n\|^2 + \|z_n\|^2) \\
&\leqslant \|x_n - x^\flat\|^2 - \varsigma(1-\varsigma)\frac{(\|y_n\|^2 + \|z_n\|^2)^2}{\|y_n + A^*z_n\|^2}
\end{aligned} \tag{3.7}$$

and consequently,

$$\begin{aligned}
\|x_{n+1} - x^\flat\| &= \|\alpha_n x^\dagger + (1-\alpha_n)J_{R,\lambda}(u_n - \lambda Du_n) - x^\flat\| \\
&= \|\alpha_n(x^\dagger - x^\flat) + (1-\alpha_n)(J_{R,\lambda}(u_n - \lambda Du_n) - x^\flat)\| \\
&\leqslant \alpha_n\|x^\dagger - x^\flat\| + (1-\alpha_n)\|J_{R,\lambda}(u_n - \lambda Du_n) - x^\flat\| \\
(by(3.10)) &\leqslant \alpha_n\|x^\dagger - x^\flat\| + (1-\alpha_n)\|u_n - x^\flat\| \\
(by(3.7)) &\leqslant \alpha_n\|x^\dagger - x^\flat\| + (1-\alpha_n)\|x_n - x^\flat\| \\
&\leqslant \max\{\|x^\dagger - x^\flat\|, \|x_n - x^\flat\|\}.
\end{aligned} \tag{3.8}$$

In case 2, we can obtain that

$$\begin{aligned}\|x_{n+1}-x^\flat\| &= \|\alpha_n x^\dagger + (1-\alpha_n)J_{R,\lambda}(x_n-\lambda Dx_n)-x^\flat\|\\ &= \|\alpha_n(x^\dagger-x^\flat)+(1-\alpha_n)(J_{R,\lambda}(x_n-\lambda Dx_n)-x^\flat)\|\\ &\leqslant \alpha_n\|x^\dagger-x^\flat\|+(1-\alpha_n)\|J_{R,\lambda}(x_n-\lambda Dx_n)-x^\flat\|\\ (by(3.12)) &\leqslant \alpha_n\|x^\dagger-x^\flat\|+(1-\alpha_n)\|x_n-x^\flat\|\\ &\leqslant \max\{\|x^\dagger-x^\flat\|,\|x_n-x^\flat\|\}.\end{aligned} \tag{3.9}$$

Hence, by induction, we deduce

$$\|x_{n+1}-x^\flat\| \leqslant \max\{\|x^\dagger-x^\flat\|,\|x_n-x^\flat\|\} \leqslant \cdots \leqslant \max\{\|x^\dagger-x^\flat\|,\|x_1-x^\flat\|\}.$$

Therefore, the sequence $\{x_n\}$ is bounded. □

THEOREM 3.9. *Under the assuptions $(H1)$–$(H10)$, the sequence $\{x_n\}$ generated by Algorithm 3.5 converges strongly to $P_\Omega x^\dagger$.*

Proof. Let $x^\flat = P_\Omega x^\dagger$. In case 1, observe that

$$\begin{aligned}&\|J_{R,\lambda}(u_n-\lambda Du_n)-x^\flat\|^2\\ &\quad= \|J_{R,\lambda}(u_n-\lambda Du_n)-J_{R,\lambda}(x^\flat-\lambda Dx^\flat)\|^2\\ &\quad\leqslant \|(u_n-\lambda Du_n)-(x^\flat-\lambda Dx^\flat)\|^2-\|(I-J_{R,\lambda})(u_n-\lambda Du_n)-(I-J_{R,\lambda})(x^\flat-\lambda Dx^\flat)\|^2\\ &\quad\leqslant \|u_n-x^\flat\|^2+\lambda(\lambda-2d)\|Du_n-Dx^\flat\|^2\\ &\qquad-\|u_n-J_{R,\lambda}(u_n-\lambda Du_n)-\lambda(Du_n-Dx^\flat)\|^2\\ &\quad\leqslant \|x_n-x^\flat\|^2-\varsigma(1-\varsigma)\frac{\|y_n\|^2+\|z_n\|^2}{\|y_n+A^*z_n\|^2}+\lambda(\lambda-2d)\|Du_n-Dx^\flat\|^2\\ &\qquad-\|u_n-J_{R,\lambda}(u_n-\lambda Du_n)-\lambda(Du_n-Dx^\flat)\|^2.\end{aligned} \tag{3.10}$$

Hence, we get

$$\begin{aligned}&\|x_{n+1}-x^\flat\|^2\\ &\quad= \|\alpha_n(x^\dagger-x^\flat)+(1-\alpha_n)(J_{R,\lambda}(u_n-\lambda Du_n)-x^\flat)\|^2\\ &\quad\leqslant (1-\alpha_n)\|J_{R,\lambda}(u_n-\lambda Du_n)-x^\flat\|^2+2\alpha_n\langle x^\dagger-x^\flat,x_{n+1}-x^\flat\rangle\\ &\quad\leqslant (1-\alpha_n)(\|x_n-x^\flat\|^2-\varsigma(1-\varsigma)\frac{\|y_n\|^2+\|z_n\|^2}{\|y_n+A^*z_n\|^2}+\lambda(\lambda-2d)\|Du_n-Dx^\flat\|^2\\ &\qquad-\|u_n-J_{R,\lambda}(u_n-\lambda Du_n)-\lambda(Du_n-Dx^\flat)\|^2)\\ &\qquad+2\alpha_n\langle x^\dagger-x^\flat,x_{n+1}-x^\flat\rangle\\ &\quad\leqslant (1-\alpha_n)\|x_n-x^\flat\|^2+\alpha_n[2\langle x^\dagger-x^\flat,x_{n+1}-x^\flat\rangle\\ &\qquad-\frac{1-\alpha_n}{\alpha_n}\varsigma(1-\varsigma)\frac{(\|y_n\|^2+\|z_n\|^2)^2}{\|y_n+A^*z_n\|^2}+\frac{1-\alpha_n}{\alpha_n}\lambda(\lambda-2d)\|Du_n-Dx^\flat\|^2\\ &\qquad-\frac{1-\alpha_n}{\alpha_n}\|u_n-J_{R,\lambda}(u_n-\lambda Du_n)-\lambda(Du_n-Dx^\flat)\|^2].\end{aligned} \tag{3.11}$$

In case 2, note that

$$\begin{aligned}
&\|J_{R,\lambda}(x_n-\lambda Dx_n)-x^\flat\|^2\\
&\quad=\|J_{R,\lambda}(x_n-\lambda Dx_n)-J_{R,\lambda}(x^\flat-\lambda Dx^\flat)\|^2\\
&\quad\leqslant\|(x_n-\lambda Dx_n)-(x^\flat-\lambda Dx^\flat)\|^2\\
&\qquad-\|(I-J_{R,\lambda})(x_n-\lambda Dx_n)-(I-J_{R,\lambda})(x^\flat-\lambda Dx^\flat)\|^2\\
&\quad\leqslant\|x_n-x^\flat\|^2+\lambda(\lambda-2d)\|Dx_n-Dx^\flat\|^2\\
&\qquad-\|x_n-J_{R,\lambda}(x_n-\lambda Dx_n)-\lambda(Dx_n-Dx^\flat)\|^2
\end{aligned}\tag{3.12}$$

and therefore,

$$\begin{aligned}
\|x_{n+1}-x^\flat\|^2&=\|\alpha_n(x^\dagger-x^\flat)+(1-\alpha_n)(J_{R,\lambda}(x_n-\lambda Dx_n)-x^\flat)\|^2\\
&\leqslant(1-\alpha_n)\|J_{R,\lambda}(x_n-\lambda Dx_n)-x^\flat\|^2+2\alpha_n\langle x^\dagger-x^\flat,x_{n+1}-x^\flat\rangle\\
&\leqslant(1-\alpha_n)(\|x_n-x^\flat\|^2+\lambda(\lambda-2d)\|Dx_n-Dx^\flat\|^2\\
&\quad-\|x_n-J_{R,\lambda}(x_n-\lambda Dx_n)-\lambda(Dx_n-Dx^\flat)\|^2)\\
&\quad+2\alpha_n\langle x^\dagger-x^\flat,x_{n+1}-x^\flat\rangle\\
&\leqslant(1-\alpha_n)\|x_n-x^\flat\|^2+\alpha_n[2\langle x^\dagger-x^\flat,x_{n+1}-x^\flat\rangle\\
&\quad-\frac{1-\alpha_n}{\alpha_n}\|x_n-J_{R,\lambda}(x_n-\lambda Dx_n)-\lambda(Dx_n-Dx^\flat)\|^2\\
&\quad+\frac{1-\alpha_n}{\alpha_n}\lambda(\lambda-2d)\|Dx_n-Dx^\flat\|^2].
\end{aligned}\tag{3.13}$$

Now, let $\varpi_n=\|x_n-x^\flat\|$ and

$$\pi_n=\begin{cases}
2\langle x^\dagger-x^\flat,x_{n+1}-x^\flat\rangle-\dfrac{1-\alpha_n}{\alpha_n}\varsigma(1-\varsigma)\dfrac{(\|y_n\|^2+\|z_n\|^2)^2}{\|y_n+A^*z_n\|^2}\\
-\dfrac{1-\alpha_n}{\alpha_n}\|u_n-J_{R,\lambda}(u_n-\lambda Du_n)-\lambda(Du_n-Dx^\flat)\|^2\\
+\dfrac{1-\alpha_n}{\alpha_n}\lambda(\lambda-2d)\|Du_n-Dx^\flat\|^2, & \text{case } 1,\\
2\langle x^\dagger-x^\flat,x_{n+1}-x^\flat\rangle+\dfrac{1-\alpha_n}{\alpha_n}\lambda(\lambda-2d)\|Dx_n-Dx^\flat\|^2\\
-\dfrac{1-\alpha_n}{\alpha_n}\|x_n-J_{R,\lambda}(x_n-\lambda Dx_n)-\lambda(Dx_n-Dx^\flat)\|^2, & \text{case } 2.
\end{cases}\tag{3.14}$$

According to (3.11), (3.13) and (3.14), we can obtain

$$\varpi_{n+1}\leqslant(1-\alpha_n)\varpi_n+\alpha_n\pi_n,\quad n\geqslant 1.\tag{3.15}$$

Evidently, $\pi_n\leqslant 2\langle x^\dagger-x^\flat,x_{n+1}-x^\flat\rangle\leqslant 2\|x^\dagger-x^\flat\|\times\|x_{n+1}-x^\flat\|$. Hence, the sequence $\{\pi_n\}$ is bounded due to the boundedness of $\{x_n\}$. Thanks to Lemma 3.3, we obtain

$0 \leqslant \limsup_{n\to\infty} \pi_n < +\infty$. Therefore, there exists a subsequence $\{\pi_{n_k}\}$ of $\{\pi_n\}$ such that

$$\limsup_{n\to\infty} \pi_n = \lim_{k\to\infty} \pi_{n_k}. \tag{3.16}$$

From now on, without loss of generality, we assume that $x_{n_k} \rightharpoonup \hat{x}$ owing to the boundedness of the sequence $\{x_n\}$. Suppose that there exists a subsequence $\{\pi_{n_{k_j}}\}$ of $\{\pi_{n_k}\}$ in case 1. Note that

$$\begin{aligned}
\limsup_{n\to\infty}\pi_n &= \lim_{k\to\infty} \pi_{n_k} = \lim_{j\to\infty} \pi_{n_{k_j}} \\
&= \lim_{j\to\infty} 2\langle x^\dagger - x^\flat, x_{n_{k_j}+1} - x^\flat\rangle - \frac{1-\alpha_{n_{k_j}}}{\alpha_{n_{k_j}}}\varsigma(1-\varsigma)\frac{(\|y_{n_{k_j}}\|^2 + \|z_{n_{k_j}}\|^2)^2}{\|y_{n_{k_j}} + A^* z_{n_{k_j}}\|^2} \\
&\quad - \frac{1-\alpha_{n_{k_j}}}{\alpha_{n_{k_j}}}\|u_{n_{k_j}} - J_{R,\lambda}(u_{n_{k_j}} - \lambda D u_{n_{k_j}}) - \lambda(D u_{n_{k_j}} - D x^\flat)\|^2 \\
&\quad + \frac{1-\alpha_{n_{k_j}}}{\alpha_{n_{k_j}}}\lambda(\lambda - 2d)\|D u_{n_{k_j}} - D x^\flat\|^2.
\end{aligned} \tag{3.17}$$

Due to the boundedness of real sequence $\{\langle x^\dagger - x^\flat, x_{n_{k_j}+1} - x^\flat\rangle\}$, without loss of generality, we may assume $\lim_{j\to\infty}\langle x^\dagger - x^\flat, x_{n_{k_j}+1} - x^\flat\rangle$ exists. Then we have that the following limits also exists

$$\lim_{j\to\infty} \frac{1-\alpha_{n_{k_j}}}{\alpha_{n_{k_j}}}\varsigma(1-\varsigma)\frac{(\|y_{n_{k_j}}\|^2 + \|z_{n_{k_j}}\|^2)^2}{\|y_{n_{k_j}} + A^* z_{n_{k_j}}\|^2}, \tag{3.18}$$

$$\lim_{j\to\infty} \frac{1-\alpha_{n_{k_j}}}{\alpha_{n_{k_j}}}\|u_{n_{k_j}} - J_{R,\lambda}(u_{n_{k_j}} - \lambda D u_{n_{k_j}}) - \lambda(D u_{n_{k_j}} - D x^\flat)\|^2 \tag{3.19}$$

and

$$\lim_{j\to\infty} \frac{1-\alpha_{n_{k_j}}}{\alpha_{n_{k_j}}}\lambda(\lambda - 2d)\|D u_{n_{k_j}} - D x^\flat\|^2. \tag{3.20}$$

It follows from (3.18) that

$$\lim_{j\to\infty} \frac{(\|y_{n_{k_j}}\|^2 + \|z_{n_{k_j}}\|^2)^2}{\|y_{n_{k_j}} + A^* z_{n_{k_j}}\|^2} = 0. \tag{3.21}$$

By the boundedness of the sequence $\{y_n + A^* z_n\}$, we have

$$\lim_{j\to\infty} \|y_{n_{k_j}}\|^2 + \|z_{n_{k_j}}\|^2 = 0, \tag{3.22}$$

which implies that

$$\lim_{j\to\infty}\|y_{n_{k_j}}\| = \lim_{j\to\infty}\|x_{n_{k_j}} - S_\delta P_C(I-\gamma B)x_{n_{k_j}}\| = 0 \tag{3.23}$$

and

$$\lim_{j\to\infty}\|z_{n_{k_j}}\| = \lim_{j\to\infty}\|Ax_{n_{k_j}} - W_{n_{k_j}}P_Q(I-\xi F)Ax_{n_{k_j}}\| = 0. \tag{3.24}$$

Evidently, $S_\delta P_C(I-\gamma B): C\to C$ is nonexpansive. Applying Lemma 2.11 to (3.23), we have

$$\hat{x} = S_\delta P_C(I-\gamma B)\hat{x}. \tag{3.25}$$

In virtue of Remark 3.6, we can derive that $\hat{x}\in Fix(S)\bigcap VI(B,C)$. It follows from (3.20) and (3.19) that

$$\lim_{j\to\infty}\|Du_{n_{k_j}} - Dx^\flat\| = 0 \tag{3.26}$$

and

$$\lim_{j\to\infty}\|u_{n_{k_j}} - J_{R,\lambda}(u_{n_{k_j}} - \lambda Du_{n_{k_j}}) - \lambda(Du_{n_{k_j}} - Dx^\flat)\|^2 = 0. \tag{3.27}$$

Hence, we have that

$$\lim_{j\to\infty}\|u_{n_{k_j}} - J_{R,\lambda}(u_{n_{k_j}} - \lambda Du_{n_{k_j}})\| = 0. \tag{3.28}$$

Note that

$$\begin{aligned}\lim_{j\to\infty}\|u_{n_{k_j}} - x_{n_{k_j}}\| &= \lim_{j\to\infty}\|P_C(x_{n_{k_j}} - \varsigma\tau_{n_{k_j}}(y_{n_{k_j}} + A^*z_{n_{k_j}})) - P_Cx_{n_{k_j}}\|\\ &\leqslant \lim_{j\to\infty}\varsigma\|\tau_{n_{k_j}}(y_{n_{k_j}} + A^*z_{n_{k_j}})\|\\ &= \lim_{j\to\infty}\varsigma\frac{\|y_{n_{k_j}}\|^2 + \|z_{n_{k_j}}\|^2}{\|y_{n_{k_j}} + A^*z_{n_{k_j}}\|}\\ &= 0,\end{aligned} \tag{3.29}$$

which implies that $u_{n_{k_j}} \rightharpoonup \hat{x}$. Since $J_{R,\lambda}(I-\lambda D)$ is nonexpansive, applying Lemma 2.12 to (3.28), we have

$$\hat{x} = J_{R,\lambda}(\hat{x} - \lambda D\hat{x}),$$

which implies that $\hat{x}\in I(D,R)$. In the light of the boundedness of the sequence $P_Q(I-\xi F)Ax_n$ and Lemma 2.12, we obtain that

$$\lim_{j\to\infty}\|WP_Q(I-\xi F)Ax_{n_{k_j}} - W_{n_{k_j}}P_Q(I-\xi F)Ax_{n_{k_j}}\| = 0.$$

This together with (3.24) implies that

$$\lim_{j\to\infty}\|Ax_{n_{k_j}} - WP_Q(I-\xi F)Ax_{n_{k_j}}\| = 0. \tag{3.30}$$

Since $WP_Q(I-\xi F)$ is nonexpansive, in view of Lemma 2.12, we have that

$$A\hat{x} = WP_Q(I-\xi F)A\hat{x}. \tag{3.31}$$

Thanks to Remark 3.6, we can also get that $A\hat{x} \in Fix(W)\bigcap VI(F,Q)$. Hence, we obtain that $\hat{x} \in \Omega$.

Next, we assume that there exists a subsequence $\{\pi_{n_{k_j}}\}$ of $\{\pi_{n_k}\}$ in case 2, by Proposition 3.7, we get that $x_{n_{k_j}} \in \Omega_{n_{k_j}}$, that is, $x_{n_{k_j}} \in Fix(S)\bigcap VI(B,C)$ and $Ax_{n_{k_j}} \in Fix(W_{n_{k_j}})\bigcap VI(F,Q)$. Hence, we have that $\hat{x} \in Fix(S)\bigcap VI(B,C)$, $A\hat{x} \in VI(F,Q)$ and $Ax_{n_{k_j}} = W_{n_{k_j}}Ax_{n_{k_j}}$. Therefore, by virtue of the boundedness of the sequence $\{Ax_n\}$, applying Lemma 2.12, we have that

$$\lim_{j\to\infty} \|WAx_{n_{k_j}} - W_{n_{k_j}}Ax_{n_{k_j}}\| = \|WAx_{n_{k_j}} - Ax_{n_{k_j}}\| = 0. \tag{3.32}$$

Since W is nonexpansive, by Lemma 2.12, we derive that $A\hat{x} = WA\hat{x}$, that is, $A\hat{x} \in Fix(W)$. Observe that

$$\begin{aligned}
\limsup_{n\to\infty}\pi_n &= \lim_{k\to\infty}\pi_{n_k} = \lim_{j\to\infty}\pi_{n_{k_j}} \\
&= \lim_{j\to\infty} 2\langle x^\dagger - x^\flat, x_{n_{k_j}+1} - x^\flat\rangle + \frac{1-\alpha_{n_{k_j}}}{\alpha_{n_{k_j}}}\lambda(\lambda-2d)\|Dx_{n_{k_j}} - Dx^\flat\|^2 \\
&\quad - \frac{1-\alpha_{n_{k_j}}}{\alpha_{n_{k_j}}}\|x_{n_{k_j}} - J_{R,\lambda}(x_{n_{k_j}} - \lambda Dx_{n_{k_j}}) - \lambda(Dx_{n_{k_j}} - Dx^\flat)\|^2.
\end{aligned} \tag{3.33}$$

Subsequently, we may assume that the limit $\lim_{n\to\infty}\langle x^\dagger - x^\flat, x_{n_{k_j}+1} - x^\flat\rangle$ exists. Then, the following two limits also exists:

$$\lim_{j\to\infty} -\frac{1-\alpha_{n_{k_j}}}{\alpha_{n_{k_j}}}\|x_{n_{k_j}} - J_{R,\lambda}(x_{n_{k_j}} - \lambda Dx_{n_{k_j}}) - \lambda(Dx_{n_{k_j}} - Dx^\flat)\|^2 \tag{3.34}$$

and

$$\lim_{j\to\infty} \frac{1-\alpha_{n_{k_j}}}{\alpha_{n_{k_j}}}\lambda(\lambda-2d)\|Dx_{n_{k_j}} - Dx^\flat\|^2. \tag{3.35}$$

By a similar argument as in case 1, we have

$$\lim_{j\to\infty} \|x_{n_{k_j}} - J_{R,\lambda}(x_{n_{k_j}} - \lambda Dx_{n_{k_j}})\| = 0 \tag{3.36}$$

and therefore $\hat{x} = J_{R,\lambda}(\hat{x} - \lambda D\hat{x})$ which implies that $\hat{x} \in I(D,R)$. Consequently, $\hat{x} \in \Omega$.

In case 1, from (3.3), we get

$$\begin{aligned}
\|x_{n_{k_j}+1} - x_{n_{k_j}}\| &= \|\alpha_{n_{k_j}}x^\dagger + (1-\alpha_{n_{k_j}})J_{R,\lambda}(u_{n_{k_j}} - \lambda Du_{n_{k_j}}) - x_{n_{k_j}}\| \\
&\leqslant \alpha_{n_{k_j}}\|x^\dagger - x_{n_{k_j}}\| + (1-\alpha_{n_{k_j}})\|u_{n_{k_j}} - x_{n_{k_j}}\| \\
&\quad + (1-\alpha_{n_{k_j}})\|J_{R,\lambda}(u_{n_{k_j}} - \lambda Du_{n_{k_j}}) - u_{n_{k_j}}\|.
\end{aligned} \tag{3.37}$$

It follows from (3.28) and (3.29) that

$$\lim_{j\to\infty}\|x_{n_{k_j}+1}-x_{n_{k_j}}\|=0. \tag{3.38}$$

In case 2, from (3.5), we have

$$\begin{aligned}\|x_{n_{k_j}+1}-x_{n_{k_j}}\|&=\|\alpha_{n_{k_j}}x^\dagger+(1-\alpha_{n_{k_j}})J_{R,\lambda}(x_{n_{k_j}}-\lambda Dx_{n_{k_j}})-x_{n_{k_j}}\|\\&\leqslant\alpha_{n_{k_j}}\|x^\dagger-x_{n_{k_j}}\|+(1-\alpha_{n_{k_j}})\|J_{R,\lambda}(x_{n_{k_j}}-\lambda Dx_{n_{k_j}})-x_{n_{k_j}}\|.\end{aligned} \tag{3.39}$$

By (3.36), we derive that

$$\lim_{j\to\infty}\|x_{n_{k_j}+1}-x_{n_{k_j}}\|=0. \tag{3.40}$$

Hence, in case 1 or case 2, $x_{n_{k_j}+1}\rightharpoonup\hat{x}\in\Omega$. Therefore,

$$\limsup_{n\to\infty}\pi_n\leqslant\lim_{j\to\infty}\langle x^\dagger-x^\flat,x_{n_{k_j}+1}-x^\flat\rangle=\langle x^\dagger-x^\flat,\hat{x}-x^\flat\rangle\leqslant 0. \tag{3.41}$$

Finally, applying Lemma 2.13 to (3.15), we get $x_n\to x^\flat=P_\Omega x^\dagger$ which ends the proof. □

ALGORITHM 3.10. Let $x^\dagger\in C$ be a given point. Let $x_1\in C$ be an initial value. Let $\{\alpha_n\}$ be a real number sequence in $(0,1)$. Let $\gamma\in(0,2\beta)$, $\varsigma\in(0,2)$ and $\xi\in(0,2\alpha)$ be three real constants. Assume that the sequence $\{x_n\}$ has been constructed. For the current iteration x_n, compute

$$\begin{aligned}y_n&=x_n-SP_C(I-\gamma B)x_n,\\z_n&=Ax_n-W_nP_Q(I-\xi F)Ax_n.\end{aligned} \tag{3.42}$$

Case 1. If $\|y_n+A^*z_n\|\neq 0$, then continue and construct x_{n+1} via the following manner

$$\begin{aligned}u_n&=P_C(x_n-\varsigma\tau_n(y_n+A^*z_n)),\\x_{n+1}&=\alpha_nx^\dagger+(1-\alpha_n)u_n\end{aligned} \tag{3.43}$$

where

$$\tau_n=\frac{\|y_n\|^2+\|z_n\|^2}{\|y_n+A^*z_n\|^2}. \tag{3.44}$$

Case 2. If $\|y_n+A^*z_n\|=0$, then continue and construct x_{n+1} via the following manner

$$x_{n+1}=\alpha_nx^\dagger+(1-\alpha_n)x_n. \tag{3.45}$$

COROLLARY 3.11. *Let*

$$\hat{\Omega} = \{x^{\natural} | x^{\natural} \in Fix(S) \bigcap VI(B,C) \ \ and \ \ Ax^{\natural} \in \bigcap_{n=1}^{\infty} Fix(T_n) \bigcap VI(F,Q)\}.$$

Suppose that $\hat{\Omega} \neq \emptyset$*. Under the assuptions* $(H1)-(H6)$*, the sequence* $\{x_n\}$ *generated by Algorithm 3.10 converges strongly to* $P_{\hat{\Omega}} x^{\dagger}$*.*

ALGORITHM 3.12. Let $x^{\dagger} \in C$ be a given point. Let $x_1 \in C$ be an initial value. Let $\{\alpha_n\}$ be a real number sequence in $(0,1)$. Let $\varsigma \in (0,2)$ be a real constants. Assume that the sequence $\{x_n\}$ has been constructed. For the current iteration x_n, compute

$$\begin{aligned} y_n &= x_n - Sx_n, \\ z_n &= Ax_n - W_n A x_n. \end{aligned} \tag{3.46}$$

Case 1. If $\|y_n + A^* z_n\| \neq 0$, then continue and construct x_{n+1} via the following manner

$$\begin{aligned} u_n &= P_C(x_n - \varsigma \tau_n (y_n + A^* z_n)), \\ x_{n+1} &= \alpha_n x^{\dagger} + (1-\alpha_n) u_n \end{aligned} \tag{3.47}$$

where

$$\tau_n = \frac{\|y_n\|^2 + \|z_n\|^2}{\|y_n + A^* z_n\|^2}. \tag{3.48}$$

Case 2. If $\|y_n + A^* z_n\| = 0$, then continue and construct x_{n+1} via the following manner

$$x_{n+1} = \alpha_n x^{\dagger} + (1-\alpha_n) x_n. \tag{3.49}$$

COROLLARY 3.13. *Let*

$$\hat{\Omega} = \{x^{\natural} | x^{\natural} \in Fix(S) \ \ and \ \ Ax^{\natural} \in \bigcap_{n=1}^{\infty} Fix(T_n)\}.$$

Suppose that $\hat{\Omega} \neq \emptyset$*. Under the assuptions* $(H1)-(H4)$*, the sequence* $\{x_n\}$ *generated by Algorithm 3.12 converges strongly to* $P_{\hat{\Omega}} x^{\dagger}$*.*

Declarations

Availablity of data and materials. Not applicable.

Competing interests. The authors declare that they have no competing interests.

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