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ORDERING EXTREMES OF SCALE RANDOM VARIABLES UNDER ARCHIMEDEAN COPULA

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Abstract. In this paper, we discuss the hazard rate order and reversed hazard rate order of parallel and series systems when the components follow general scale model under Archimedean copula for dependence. Several examples are presented for illustrations as well.

1. Introduction

For modelling lifetime data possessing varying hazard shapes, it is desirable that the assumed lifetime distribution has considerable flexibility. A general family of distributions which includes some well-known distributions such as normal, exponentiated Weibull and gamma as special cases is the scale family of distributions. Suppose $F(\cdot)$ is an absolutely continuous distribution function with corresponding probability density function $f(\cdot)$. Then, independent random variables $X_1,\cdots,X_n$ are said to belong to the scale family of distributions if $\lambda_1 X_1,\cdots,\lambda_n X_n$ are independent and identically distributed (i.i.d) with common distribution $F(\cdot)$, where $\lambda_i > 0$ for $i = 1,\cdots,n$. In other words, $X_1,\cdots,X_n$ are said to belong to the scale family of distributions if $X_i \sim F(\lambda_i x)$, $i = 1,\cdots,n$. In this case, $F(\cdot)$ is said to be the baseline distribution function and λ_i's are the scale parameters. Stochastic comparisons of order statistics in the scale models were first introduced by Pledger and Proschan (1971) and since then, many researchers have worked in this direction including recently by Hu (1995), Bon and Paltanea (2006), and Khaledi et al. (2011).

One of the most commonly used systems in reliability is an r-out-of-n system. This system comprising of n components, works iff at least r components work, and it includes parallel, fail-safe and series systems all as special cases corresponding to $r = 1$, $r = n-1$ and $r = n$, respectively. Let $X_1,\cdots,X_n$ denote the lifetimes of components of a system and $X_{1:n} \leqslant \cdots \leqslant X_{n:n}$ represent the corresponding order statistics. Then, $X_{n-r+1:n}$ corresponds to the lifetime of a r-out-of-n system. Due to this direct connection, the theory of order statistics becomes quite important in studying

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$(n-r+1)$-out-of-n systems and in characterizing their important properties. For comprehensive discussions on various properties of order statistics and their applications, one may refer to Balakrishnan and Rao (1998a,b) and David and Nagaraja (2003).

The comparison of important characteristics associated with lifetimes of technical systems is an interesting topic in reliability theory, since it usually enables us to approximate complex systems with simpler systems and subsequently obtaining various bounds for important ageing characteristics of the complex system. A convenient tool for this purpose is the theory of stochastic orderings. Stochastic comparisons of series and parallel systems with heterogeneous components have been discussed extensively for the various lifetimes. We refer the readers to Zhao and Balakrishnan (2011a, 2011b), Fang and Zhang (2013), Kochar and Xu (2014), Li and Li (2015), Li and Fang (2015), Fang et al. (2016), Amini-Seresht et al. (2016), Barmalzan et al. (2017), Ding et al. (2017), for detailed discussions on this topic. Several researchers made some progress in comparing order statistics of random variables with Archimedean copulas. See; for example, Rezapour and Alamatsaz (2014), Li and Fang (2015), Li and Li (2015), Li et al. (2015), Fang et al. (2016), Zhang et al. (2018) and Fang and Li (2019).

Many authors discuss on ordering different order statistics from one sample and studying the impact of sample size on a given order statistic. For $i.i.d.$ random variables, Boland et al. (1994) were among the first to prove that order statistics can be ordered in terms of the hazard rate order and even the likelihood ratio order, and Raqab and Amin (1996) further established the existence of the likelihood ratio order between order statistics from $i.i.d.$ samples of different sizes. In the context of $X_k \leqslant_{hr} X_{n+1}$, $k=1,\cdots,n$, Boland et al. (1994) firstly established the hazard rate order $X_{i-1:n} \leqslant_{hr} X_{i:n+1}$ and further proved that $X_{n+1} \leqslant_{hr} X_k$ implies $X_{i:n} \geqslant_{hr} X_{i:n+1}$. Later, for a sample with observations arrayed in the likelihood ratio order, Bapat and Kochar (1994) established the likelihood ratio order between two general order statistics. In the literature, some discussions can be found on stochastic order between order statistics from either one dependent sample or two dependent samples of different sizes.

The rest of this paper is organized as follows. Section 2 reviews some basic concepts that will be used in the sequel. In Section 3, we discuss the hazard rate order and reversed hazard rate order of parallel systems when the components follow scale model under Archimedean copula for dependence. The hazard rate order and reversed hazard rate order of series systems when the components follow scale model under Archimedean copula have been discussed in Section 4. Finally, some concluding remarks are made in in Section 5. Several examples are presented for illustrations as well.

2. Preliminaries

In this section, we first present the definitions of some well-known concepts relating to stochastic orders and copulas. Throughout, we use 'increasing' to mean 'non-decreasing' and similarly 'decreasing' to mean 'non-increasing'.

2.1. Stochastic orders

The following definition introduce some well-known orders that compare the magnitude of two random variables.

DEFINITION 1. Suppose X and Y are two non-negative random variables with density functions f and g, distribution functions F and G, survival functions $\bar{F} = 1-F$ and $\bar{G} = 1-G$, hazard rates $r_X = f/\bar{F}$ and $r_Y = g/\bar{G}$, and reversed hazard rates $\tilde{r}_X = f/F$ and $\tilde{r}_Y = g/G$, respectively. Then, X is said to be smaller than Y in the sense of

(i) usual stochastic order (denoted by $X \leqslant_{st} Y$) if $\bar{F}(x) \leqslant \bar{G}(x)$ for all $x \in \mathbb{R}^+$. For all increasing functions $\phi : \mathbb{R} \to \mathbb{R}$, $X \leqslant_{st} Y$ if and only if $E(\phi(X)) \leqslant E(\phi(Y))$, when the expectations exist;

(ii) hazard rate order (denoted by $X \leqslant_{hr} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in $x \in \mathbb{R}^+$. In fact, $X \leqslant_{hr} Y$ if and only if $r_Y(x) \leqslant r_X(x)$ for all $x \in \mathbb{R}^+$;

(iii) reversed hazard rate order (denoted by $X \leqslant_{rh} Y$) if $G(x)/F(x)$ is increasing in $x \in \mathbb{R}^+$. In fact, $X \leqslant_{rh} Y$ if and only if $\tilde{r}_X(x) \leqslant \tilde{r}_Y(x)$ for all $x \in \mathbb{R}^+$.

It is well-known that both the hazard rate and reversed hazard rate orders imply the usual stochastic order. For a comprehensive discussion on various stochastic orders and their applications, we refer the readers to Müller and Stoyan (2002) and Shaked and Shanthikumar (2007).

2.2. Archimedean copula

Many stochastic comparisons between univariate random variables have been defined and discussed in a variety of contexts; see, Müller and Stoyan (2002) and Shaked and Shanthikumar (2007) for pertinent details. Most of the univariate stochastic orders are based on comparisons of marginal distributions of the underlying variables, without taking dependence between them into account. In the present work, we discuss stochastic comparisons of series and parallel systems with dependent scale distributed components under an Archimedean copula.

Archimedean copulas have been used extensively due to their mathematical tractability as well as their ability to capture wide range of dependence. For a decreasing and continuous function $\psi : [0,\infty) \longrightarrow [0,1]$ such that $\psi(0) = 1$, $\psi(+\infty) = 0$ and $\psi = \phi^{-1}$ being the pseudo-inverse,

$$C_\psi(u_1,\cdots,u_n) = \psi(\phi(u_1)+\cdots+\phi(u_n)) \qquad \text{for all } u_i \in [0,1],\ i = 1,\cdots,n,$$

is said to be an Archimedean copula with generator ψ if $(-1)^k\psi^{[k]}(x) \geqslant 0$ for $k = 0,\cdots,n-2$ and $(-1)^{n-2}\psi^{[n-2]}(x)$ is decreasing and convex. The Archimedean copula family includes many well-known copulas such as independence (product) copula, Clayton copula, and Ali-Mikhail-Haq (AMH) copula. For detailed discussions on copulas and their properties, one may refer to Nelsen (2006) and McNeil and Nešlehová (2009).

3. Results for parallel systems

In this section, we discuss the reversed hazard rate order and the hazard rate order of parallel systems with dependent general scale components, respectively.

THEOREM 1. *Suppose $X_1, \cdots, X_{n+1}$ are random variables with $X_i \sim F(\lambda_i x)$ and associated Archimedean copula with generator ψ. Further,*

(i) Suppose that $-ln(\psi(e^x))$ is convex and $\phi(F(x))$ is log-concave on the $\mathbb{R}$. If $\lambda_{n+1} \geqslant max\{\lambda_1, \ldots, \lambda_n\}$, then we have $X_{n:n} \leqslant_{rh} X_{n+1:n+1}$.

(ii) Suppose that $-ln(\psi(e^x))$ is convex and $\phi(F(x))$ is log-convex on the $\mathbb{R}$. If $\lambda_{n+1} \leqslant min\{\lambda_1, \ldots, \lambda_n\}$, then we have $X_{n:n} \geqslant_{rh} X_{n+1:n+1}$.

Proof. *(i)* The distribution function of $X_{n:n}$ is given by

$$F_{X_{n:n}}(x) = \psi\left[\sum_{i=1}^{n} \phi\left(F(\lambda_i x)\right)\right], \qquad x \in \mathbb{R}^+.$$

Then, to prove the desired result, it is sufficient to show that $F_{X_{n+1:n+1}}(x)/F_{X_{n:n}}(x)$ is increasing in x. Taking the derivative of $F_{X_{n+1:n+1}}(x)/F_{X_{n:n}}(x)$ with respect to x, we have

$$\begin{aligned}
\left[\frac{F_{X_{n+1:n+1}}(x)}{F_{X_{n:n}}(x)}\right]' &= \left[\frac{\psi\left[\sum_{i=1}^{n+1} \phi\left(F(\lambda_i x)\right)\right]}{\psi\left[\sum_{i=1}^{n} \phi\left(F(\lambda_i x)\right)\right]}\right]' \\
&\overset{sgn}{=} \frac{\psi'\left[\sum_{i=1}^{n+1} \phi\left(F(\lambda_i x)\right)\right]}{\psi\left[\sum_{i=1}^{n+1} \phi\left(F(\lambda_i x)\right)\right]} \sum_{i=1}^{n+1} \frac{\lambda_i F'(\lambda_i x)}{\psi'\left(\phi\left(F(\lambda_i x)\right)\right)} \\
&\quad - \frac{\psi'\left[\sum_{i=1}^{n} \phi\left(F(\lambda_i x)\right)\right]}{\psi\left[\sum_{i=1}^{n} \phi\left(F(\lambda_i x)\right)\right]} \sum_{i=1}^{n} \frac{\lambda_i F'(\lambda_i x)}{\psi'\left(\phi\left(F(\lambda_i x)\right)\right)} \\
&= \frac{\psi'\left(\sum_{i=1}^{n+1} \phi\left(F(\lambda_i x)\right)\right)}{\psi\left(\sum_{i=1}^{n+1} \phi\left(F(\lambda_i x)\right)\right)} \times \frac{\sum_{i=1}^{n+1} \phi\left(F(\lambda_i x)\right) \times \sum_{i=1}^{n+1} \frac{\lambda_i F'(\lambda_i x)}{\psi'(\phi(F(\lambda_i x)))}}{\sum_{i=1}^{n+1} \phi\left(F(\lambda_i x)\right)} \\
&\quad - \frac{\psi'\left(\sum_{i=1}^{n} \phi\left(F(\lambda_i x)\right)\right)}{\psi\left(\sum_{i=1}^{n} \phi\left(F(\lambda_i x)\right)\right)} \times \frac{\sum_{i=1}^{n} \phi\left(F(\lambda_i x)\right) \times \sum_{i=1}^{n} \frac{\lambda_i F'(\lambda_i x)}{\psi'(\phi(F(\lambda_i x)))}}{\sum_{i=1}^{n} \phi\left(F(\lambda_i x)\right)},
\end{aligned}$$

where $\overset{sgn}{=}$ means both sides have the same sign. Since the convexity of $-ln(\psi(e^x))$ implies that $\frac{t\psi'(t)}{\psi(t)}$ is decreasing in $t \geqslant 0$, we have, for all x,

$$\frac{\psi'\left(\sum_{i=1}^{n+1} \phi\left(F(\lambda_i x)\right)\right) \sum_{i=1}^{n+1} \phi\left(F(\lambda_i x)\right)}{\psi\left(\sum_{i=1}^{n+1} \phi\left(F(\lambda_i x)\right)\right)} \leqslant \frac{\psi'\left(\sum_{i=1}^{n} \phi\left(F(\lambda_i x)\right)\right) \sum_{i=1}^{n} \phi\left(F(\lambda_i x)\right)}{\psi\left(\sum_{i=1}^{n} \phi\left(F(\lambda_i x)\right)\right)} \leqslant 0.$$

Since $\sum_{i=1}^{n} \frac{\lambda_i F'(\lambda_i x)}{\psi'(\phi(F(\lambda_i x)))} \leqslant 0$, to prove $\left[\frac{F_{X_{n+1:n+1}}(x)}{F_{X_{n:n}}(x)}\right]' \geqslant 0$ it is sufficient to show that

$$\frac{\sum_{i=1}^{n+1} \frac{\lambda_i F'(\lambda_i x)}{\psi'(\phi(F(\lambda_i x)))}}{\sum_{i=1}^{n+1} \phi\left(F(\lambda_i x)\right)} \leqslant \frac{\sum_{i=1}^{n} \frac{\lambda_i F'(\lambda_i x)}{\psi'(\phi(F(\lambda_i x)))}}{\sum_{i=1}^{n} \phi\left(F(\lambda_i x)\right)},$$

which is equivalent to that, for all x,

$$\sum_{i=1}^{n}\frac{\lambda_{n+1}F'(\lambda_{n+1}x)\phi\left(F(\lambda_i x)\right)}{\psi'\left(\phi\left(F(\lambda_{n+1}x)\right)\right)}\leqslant\sum_{i=1}^{n}\frac{\lambda_i F'(\lambda_i x)\phi\left(F(\lambda_{n+1}x)\right)}{\psi'\left(\phi\left(F(\lambda_i x)\right)\right)}$$

$$\iff 0\geqslant\sum_{i=1}^{n}\phi\left(F(\lambda_{n+1}(x))\,\phi\left(F(\lambda_i x)\right)\right.$$
$$\times\left[\frac{\lambda_{n+1}F'(\lambda_{n+1}x)}{\psi'\left(\phi\left(F(\lambda_{n+1}x)\right)\right)\phi\left(F(\lambda_{n+1}x)\right)}-\frac{\lambda_i F'(\lambda_i x)}{\psi'\left(\phi\left(F(\lambda_i x)\right)\right)\phi\left(F(\lambda_i x)\right)}\right]. \qquad (1)$$

For $i=1,\ldots,n$, with

$$\frac{\lambda_{n+1}F'(\lambda_{n+1}x)}{\psi'\left(\phi\left(F(\lambda_{n+1}x)\right)\right)\phi\left(F(\lambda_{n+1}x)\right)}-\frac{\lambda_i F'(\lambda_i x)}{\psi'\left(\phi\left(F(\lambda_i x)\right)\right)\phi\left(F(\lambda_i x)\right)}\leqslant 0, \qquad (2)$$

we observe taht the inequality (1) is satisfied. Further, (2) is equivalent to

$$\frac{d\ln\left(\phi(F(\lambda_{n+1}x)\right)}{dx}-\frac{d\ln\left(\phi(F(\lambda_i x)\right)}{dx}\leqslant 0.$$

From the assumption $\lambda_{n+1}\geqslant\lambda_i$, for all $i=1,\cdots,n$, and $F(x)$ is an increasing function, we have $F(\lambda_i x)\leqslant F(\lambda_{n+1}x)$, for all $i=1,\ldots,n$. Now, we conclude that $F_{X_{n+1:n+1}}(x)/F_{X_{n:n}}(x)$ is increasing in x if $\phi(F(x))$ be log-concave.

(ii) The proof is similar to that of Part (i) and is therefore omitted for the sake of brevity. □

EXAMPLE 1. It should be mentioned that the condition "$-\ln\psi(e^x)$ is convex and $\phi(F(x))$ is log-convex" in Part (ii) of Theorem 1 are not uncommon and we can verify them for some Archimedean copulas and some marginals. For example, consider $\psi(x)=(x+1)^{-1}$. Thus, $-\ln(\psi(e^x))$ is convex. Let us take $F(x)=1-e^{-\lambda x},(x>0,\lambda>0)$, then $\phi(F(x))=\frac{1}{F(x)}-1=\frac{1}{e^{\lambda x}-1}$. Now, it is easy to observe that $\phi(F(x))$ is log-convex.

THEOREM 2. *Suppose* $X_1,\cdots,X_{n+1}$ *are random variables with* $X_i\sim F(\lambda_i x)$ *and associated Archimedean copula with generator* ψ. *Further,*

(i) Suppose that $-ln(1-\psi(e^x))$ *is convex and* $\phi(F(x))$ *is log-convex on the* $\mathbb{R}$. *If* $\lambda_{n+1}\geqslant\max\{\lambda_1,\cdots,\lambda_n\}$, *then we have* $X_{n:n}\leqslant_{hr}X_{n+1:n+1}$.

(ii) Suppose that $-ln(1-\psi(e^x))$ *is concave and* $\phi(F(x))$ *is log-convex on the* $\mathbb{R}$. *If* $\lambda_{n+1}\leqslant\min\{\lambda_1,\cdots,\lambda_n\}$, *then we have* $X_{n:n}\geqslant_{hr}X_{n+1:n+1}$.

Proof. The survival function of $X_{n:n}$ is given by

$$\bar{F}_{X_{n:n}}(x)=1-\psi\left[\sum_{i=1}^{n}\phi\left(F(\lambda_i x)\right)\right], \qquad x\in\mathbb{R}^+.$$

Then, to prove the desired result, it is sufficient to show that $\bar{F}_{X_{n+1:n+1}}(x)/\bar{F}_{X_{n:n}}(x)$ is increasing in x. Taking the derivative of $\bar{F}_{X_{n+1:n+1}}(x)/\bar{F}_{X_{n:n}}(x)$ with respect to x, we obtain

$$\begin{aligned}\left[\frac{\bar{F}_{X_{n+1:n+1}}(x)}{\bar{F}_{X_{n:n}}(x)}\right]' &= \left[\frac{1-\psi\left[\sum_{i=1}^{n+1}\phi\left(F(\lambda_i x)\right)\right]}{1-\psi\left[\sum_{i=1}^{n}\phi\left(F(\lambda_i x)\right)\right]}\right]' \\ &\overset{sgn}{=} \frac{-\psi'\left[\sum_{i=1}^{n+1}\phi\left(F(\lambda_i x)\right)\right]}{1-\psi\left[\sum_{i=1}^{n+1}\phi\left(F(\lambda_i x)\right)\right]}\sum_{i=1}^{n+1}\frac{\lambda_i F'(\lambda_i x)}{\psi'\left(\phi\left(F(\lambda_i x)\right)\right)} \\ &\quad+\frac{\psi'\left[\sum_{i=1}^{n}\phi\left(F(\lambda_i x)\right)\right]}{1-\psi\left[\sum_{i=1}^{n}\phi\left(F(\lambda_i x)\right)\right]}\sum_{i=1}^{n}\frac{\lambda_i F'(\lambda_i x)}{\psi'\left(\phi\left(F(\lambda_i x)\right)\right)} \\ &= \frac{-\psi'\left(\sum_{i=1}^{n+1}\phi\left(F(\lambda_i x)\right)\right)}{1-\psi\left(\sum_{i=1}^{n+1}\phi\left(F(\lambda_i x)\right)\right)}\times\frac{\sum_{i=1}^{n+1}\phi\left(F(\lambda_i x)\right)\times\sum_{i=1}^{n+1}\frac{\lambda_i F'(\lambda_i x)}{\psi'(\phi(F(\lambda_i x)))}}{\sum_{i=1}^{n+1}\phi\left(F(\lambda_i x)\right)} \\ &\quad+\frac{\psi'\left(\sum_{i=1}^{n}\phi\left(F(\lambda_i x)\right)\right)}{1-\psi\left(\sum_{i=1}^{n}\phi\left(F(\lambda_i x)\right)\right)}\times\frac{\sum_{i=1}^{n}\phi\left(F(\lambda_i x)\right)\times\sum_{i=1}^{n}\frac{\lambda_i F'(\lambda_i x)}{\psi'(\phi(F(\lambda_i x)))}}{\sum_{i=1}^{n}\phi\left(F(\lambda_i x)\right)}.\end{aligned}$$

The convexity of $-ln(1-\psi(e^x))$ implies $\frac{-t\psi'(t)}{1-\psi(t)}$ is decreasing in $t \geqslant 0$ and then, we have, for all x,

$$0 \leqslant \frac{-\psi'\left(\sum_{i=1}^{n+1}\phi\left(F(\lambda_i x)\right)\right)\sum_{i=1}^{n+1}\phi\left(F(\lambda_i x)\right)}{1-\psi\left(\sum_{i=1}^{n+1}\phi\left(F(\lambda_i x)\right)\right)} \leqslant \frac{-\psi'\left(\sum_{i=1}^{n}\phi\left(F(\lambda_i x)\right)\right)\sum_{i=1}^{n}\phi\left(F(\lambda_i x)\right)}{1-\psi\left(\sum_{i=1}^{n}\phi\left(F(\lambda_i x)\right)\right)}.$$

Because $\sum_{i=1}^{n}\frac{\lambda_i F'(\lambda_i x)}{\psi'(\phi(F(\lambda_i x)))} \leqslant 0$, to prove $\left[\frac{1-F_{X_{n+1:n+1}}(x)}{1-F_{X_{n:n}}(x)}\right]' \geqslant 0$ it suffices to show, for all x,

$$\frac{\sum_{i=1}^{n+1}\frac{\lambda_i F'(\lambda_i x)}{\psi'(\phi(F(\lambda_i x)))}}{\sum_{i=1}^{n+1}\phi\left(F(\lambda_i x)\right)} \geqslant \frac{\sum_{i=1}^{n}\frac{\lambda_i F'(\lambda_i x)}{\psi'(\phi(F(\lambda_i x)))}}{\sum_{i=1}^{n}\phi\left(F(\lambda_i x)\right)},$$

which is equivalent to showing that, for all x,

$$\begin{aligned}&\sum_{i=1}^{n}\frac{\lambda_{n+1}F'(\lambda_{n+1}x)\phi\left(F(\lambda_i x)\right)}{\psi'\left(\phi\left(F(\lambda_{n+1}x)\right)\right)} \geqslant \sum_{i=1}^{n}\frac{\lambda_i F'(\lambda_i x)\phi\left(F(\lambda_{n+1}x)\right)}{\psi'\left(\phi\left(F(\lambda_i x)\right)\right)} \\ \Longleftrightarrow\ & 0 \leqslant \sum_{i=1}^{n}\phi\left(F(\lambda_{n+1}(x)\right)\phi\left(F(\lambda_i x)\right) \\ &\times\left[\frac{\lambda_{n+1}F'(\lambda_{n+1}x)}{\psi'\left(\phi\left(F(\lambda_{n+1}x)\right)\right)\phi\left(F(\lambda_{n+1}x)\right)} - \frac{\lambda_i F'(\lambda_i x)}{\psi'\left(\phi\left(F(\lambda_i x)\right)\right)\phi\left(F(\lambda_i x)\right)}\right] \qquad (3)\end{aligned}$$

For $i = 1, \ldots, n$, with

$$\frac{\lambda_{n+1}F'(\lambda_{n+1}x)}{\psi'\left(\phi\left(F(\lambda_{n+1}x)\right)\right)\phi\left(F(\lambda_{n+1}x)\right)} - \frac{\lambda_i F'(\lambda_i x)}{\psi'\left(\phi\left(F(\lambda_i x)\right)\right)\phi\left(F(\lambda_i x)\right)} \geqslant 0, \qquad (4)$$

we can see that the inequality (3) is satisfied. Further, (4) is equivalent to

$$\frac{d\ln(\phi(F(\lambda_{n+1}x))}{dx}-\frac{d\ln(\phi(F(\lambda_i x))}{dx}\geqslant 0.$$

Due to the condition $\lambda_i \leqslant \lambda_{n+1}$, for all $i=1,\ldots,n$, and increasing property of $F(x)$, we have $F(\lambda_i x)\leqslant F(\lambda_{n+1}x)$, for all $i=1,\ldots,n$. Thus, it suffices that $\phi(F(x))$ is log-convex.

(ii) The proof is similar to that of Part (i) and is therefore omitted for the sake of brevity. □

EXAMPLE 2. It should be mentioned that the condition "$-\ln(1-\psi(e^x))$ is convex and $\phi(F(x))$ is log-convex" in Part (i) of Theorem 2 are satisfied by some Archimedean copulas and some marginals. For example, consider $\psi(x)=(x+1)^{-1}$. Thus, $-\ln(1-\psi(e^x))$ is convex. Now, let us take $F(x)=1-e^{-\lambda x},(\lambda>0)$, then $\phi(F(x))=\frac{1}{F(x)}-1=\frac{1}{e^{\lambda x}-1}$ and so, $\phi(F(x))$ is log-convex.

4. Results for series systems

In this section, we discuss the reversed hazard rate order and the hazard rate order of series systems with dependent general scale components, respectively.

THEOREM 3. *Suppose $X_1,\cdots,X_{n+1}$ are random variables with $X_i\sim F(\lambda_i x)$ and associated survival Archimedean copula with generator ψ. Further,*

(i) Suppose that $-ln(1-\psi(e^x))$ is convex and $\phi(1-F(x))$ is log-concave on the $\mathbb{R}$. If $\lambda_{n+1}\geqslant \max\{\lambda_1,\cdots,\lambda_n\}$, then we have $X_{1:n+1}\leqslant_{rh} X_{1:n}$.

(ii) Suppose that $-ln(1-\psi(e^x))$ is concave and $\phi(1-F(x))$ is log-concave on the $\mathbb{R}$. If $\lambda_{n+1}\leqslant \min\{\lambda_1,\cdots,\lambda_n\}$, then we have $X_{1:n+1}\geqslant_{rh} X_{1:n}$.

Proof. The distribution function of $X_{1:n}$ is given by

$$F_{X_{1:n}}(x)=1-\psi\left[\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)\right].$$

Then, for obtaining the desired result, it is sufficient to show that $F_{X_{1:n}}(x)/F_{X_{1:n+1}}(x)$ is increasing in x. Taking the derivative of $F_{X_{1:n}}(x)/F_{X_{1:n+1}}(x)$ with respect to x, we obtain

$$\begin{aligned}\left[\frac{F_{X_{1:n}}(x)}{F_{X_{1:n+1}}(x)}\right]' &= \left[\frac{1-\psi\left[\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)\right]}{1-\psi\left[\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)\right]}\right]'\\ &\overset{sgn}{=} \frac{\psi'\left(\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)\right)}{1-\psi\left(\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)\right)}\times\sum_{i=1}^{n}\frac{\lambda_i F'(\lambda_i x)}{\psi'\left(\phi\left(1-F(\lambda_i x)\right)\right)}\\ &\quad -\frac{\psi'\left(\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)\right)}{1-\psi\left(\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)\right)}\times\sum_{i=1}^{n+1}\frac{\lambda_i F'(\lambda_i x)}{\psi'\left(\phi\left(1-F(\lambda_i x)\right)\right)}\end{aligned}$$

$$= \frac{\psi'\left(\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)\right)}{1-\psi\left(\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)\right)} \times \frac{\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)}{\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)}$$
$$\times \sum_{i=1}^{n} \frac{\lambda_i F'(\lambda_i x)}{\psi'\left(\phi\left(1-F(\lambda_i x)\right)\right)}$$
$$- \frac{\psi'\left(\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)\right)}{1-\psi\left(\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)\right)} \times \frac{\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)}{\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)}$$
$$\times \sum_{i=1}^{n+1} \frac{\lambda_i F'(\lambda_i x)}{\psi'\left(\phi\left(1-F(\lambda_i x)\right)\right)}.$$

As the convexity of $-ln(1-\psi(e^x))$ implies that $\frac{t\psi'(t)}{1-\psi(t)}$ is increasing in $t \geqslant 0$, we have, for all x,

$$0 \geqslant \frac{\psi'\left(\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)\right)\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)}{1-\psi\left(\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)\right))}$$
$$\geqslant \frac{\psi'\left(\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)\right)\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)}{1-\psi\left(\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)\right)}.$$

Because

$$\sum_{i=1}^{n} \frac{\lambda_i F'(\lambda_i x)}{\psi'\left[\phi\left(1-F(\lambda_i x)\right)\right]} \leqslant 0,$$

to prove $[F_{X_{1:n}}(x)/F_{X_{1:n+1}}(x)]' \geqslant 0$ it suffices to verify, for all x,

$$\frac{\sum_{i=1}^{n} \frac{\lambda_i F'(\lambda_i x)}{\psi'(\phi(1-F(\lambda_i x)))}}{\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)} \leqslant \frac{\sum_{i=1}^{n+1} \frac{\lambda_i F'(\lambda_i x)}{\psi'(\phi(1-F(\lambda_i x)))}}{\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)},$$

which is equivalent to showing that, for all x,

$$\sum_{i=1}^{n} \frac{\lambda_{n+1}F'(\lambda_{n+1}x)\phi\left(1-F(\lambda_i x)\right)}{\psi'\left(\phi\left(1-F(\lambda_{n+1}x)\right)\right)} \geqslant \sum_{i=1}^{n} \frac{\lambda_i F'(\lambda_i x)\phi\left(1-F(\lambda_{n+1}x)\right)}{\psi'\left(\phi\left(1-F(\lambda_i x)\right)\right)}$$
$$\Longleftrightarrow \; 0 \leqslant \sum_{i=1}^{n}\phi\left(1-F(\lambda_{n+1}x)\right)\phi\left(1-F(\lambda_i x)\right)$$
$$\times\left[\frac{\lambda_{n+1}F'(\lambda_{n+1}x)}{\psi'\left(\phi\left(1-F(\lambda_{n+1}x)\right)\right)\phi\left(1-F(\lambda_{n+1}x)\right)} - \frac{\lambda_i F'(\lambda_i x)}{\psi'\left(\phi\left(1-F(\lambda_i x)\right)\right)\phi\left(1-F(\lambda_i x)\right)}\right]$$
$$\Longleftrightarrow \; 0 \leqslant \sum_{i=1}^{n}\phi\left(1-F(\lambda_{n+1}x)\right)\phi\left(1-F(\lambda_i x)\right)$$
$$\times\left[-\frac{d\left(\ln(\phi(1-F(\lambda_{n+1}x)))\right)}{dx} + \frac{d\left(\ln(\phi(1-F(\lambda_i x)))\right)}{dx}\right]. \tag{5}$$

Since $\lambda_i \leqslant \lambda_{n+1}$, for all i, then (5) holds if

$$\frac{d^2\ln(\phi(1-F(x)))}{dx^2}$$

is negative. Thus, it suffices that $\phi(1-F(x))$ is log-concave.

(ii) The proof is similar to that of Part (i) and is therefore omitted for the sake of brevity. □

EXAMPLE 3. The condition "$-\ln(1-\psi(e^x))$ is convex and $\phi(1-F(x))$ is log-concave" in Part (i) of Theorem 3 are satisfied by some Archimedean copulas and some marginals. Consider $\psi(x)=(x+1)^{-1}$. Thus, $-\ln(1-\psi(e^x))$ is convex. On the other hand, if we set $F(x)=1-e^{-\lambda x}, (\lambda>0)$, then $\phi(1-F(x))=\frac{1}{F(x)}-1=e^{\lambda x}-1$. Now, it is easy to observe that $\phi(1-F(x))$ is log-concave.

THEOREM 4. *Suppose $X_1,\cdots,X_{n+1}$ are random variables with $X_i\sim F(\lambda_i x)$ and associated survival Archimedean copula with generator ψ. Further,*

(i) Suppose that $-ln(\psi(e^x))$ is convex and $\phi(1-F(x))$ is log-convex on the $\mathbb{R}$. If $\lambda_{n+1}\geqslant\max\{\lambda_1,\cdots,\lambda_n\}$, then we have $X_{1:n+1}\leqslant_{hr}X_{1:n}$.

(ii) Suppose that $-ln(\psi(e^x))$ is concave and $\phi(1-F(x))$ is log-convex on the $\mathbb{R}$. If $\lambda_{n+1}\leqslant\min\{\lambda_1,\cdots,\lambda_n\}$, then we have $X_{1:n+1}\leqslant_{hr}X_{1:n}$.

Proof. The survival function of $X_{1:n}$ is given by

$$\bar{F}_{X_{1:n}}(x)=\psi\left(\sum_{i=1}^{n}\phi(1-F(\lambda_i x))\right).$$

Then, to prove the desired result, it is sufficient to show that $\bar{F}_{X_{1:n}}(x)/\bar{F}_{X_{1:n+1}}(x)$ is increasing in x. Taking the derivative of $\bar{F}_{X_{1:n}}(x)/\bar{F}_{X_{1:n+1}}(x)$ with respect to x, we obtain

$$\begin{aligned}
&\left[\frac{\bar{F}_{X_{1:n}}(x)}{\bar{F}_{X_{1:n+1}}(x)}\right]'\\
&=\left(\frac{\psi\left[\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)\right]}{\psi\left[\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)\right]}\right)'\\
&\overset{sgn}{=}\frac{\psi'\left(\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)\right)}{\psi\left(\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)\right)}\times\sum_{i=1}^{n}\frac{-\lambda_i F'(\lambda_i x)}{\psi'\left(\phi\left(1-F(\lambda_i x)\right)\right)}\\
&\quad-\frac{\psi'\left(\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)\right)}{\psi\left(\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)\right)}\times\sum_{i=1}^{n+1}\frac{-\lambda_i F'(\lambda_i x)}{\psi'\left(\phi\left(1-F(\lambda_i x)\right)\right)}\\
&=\frac{\psi'\left(\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)\right)}{\psi\left(\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)\right)}\times\frac{\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)}{\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)}\times\sum_{i=1}^{n}\frac{-\lambda_i F'(\lambda_i x)}{\psi'\left(\phi\left(1-F(\lambda_i x)\right)\right)}\\
&\quad-\frac{\psi'\left(\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)\right)}{\psi\left(\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)\right)}\times\frac{\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)}{\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)}\times\sum_{i=1}^{n+1}\frac{-\lambda_i F'(\lambda_i x)}{\psi'\left(\phi\left(1-F(\lambda_i x)\right)\right)}.
\end{aligned}$$

As the convexity of $-ln(\psi(e^x))$ implies that $\frac{t\psi'(t)}{\psi(t)}$ is decreasing in $t \geqslant 0$, we have, for all x,

$$\frac{\psi'\left(\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)\right)\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)}{\psi\left(\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)\right)}$$
$$\leqslant \frac{\psi'\left(\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)\right)\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)}{\psi\left(\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)\right)} \leqslant 0.$$

Because

$$\sum_{i=1}^{n}\frac{-\lambda_i F'(\lambda_i x)}{\psi'\left[\phi\left(1-F(\lambda_i x)\right)\right]} \geqslant 0,$$

to prove $\left(\frac{\bar{F}_{X_{1:n}}}{\bar{F}_{X_{1:n+1}}}\right)' \geqslant 0$ it suffices to verify, for all x,

$$\frac{\sum_{i=1}^{n}\frac{-\lambda_i F'(\lambda_i x)}{\psi'(\phi(1-F(\lambda_i x)))}}{\sum_{i=1}^{n}\phi\left(1-F(\lambda_i x)\right)} \leqslant \frac{\sum_{i=1}^{n+1}\frac{-\lambda_i F'(\lambda_i x)}{\psi'(\phi(1-F(\lambda_i x)))}}{\sum_{i=1}^{n+1}\phi\left(1-F(\lambda_i x)\right)},$$

which is equivalent to showing that, for all x,

$$\sum_{i=1}^{n}\frac{-\lambda_{n+1}F'(\lambda_{n+1}x)\phi\left(1-F(\lambda_i x)\right)}{\psi'\left(\phi\left(1-F(\lambda_{n+1}x)\right)\right)} \geqslant \sum_{i=1}^{n}\frac{-\lambda_i F'(\lambda_i x)\phi\left(1-F(\lambda_{n+1}x)\right)}{\psi'\left(\phi\left(1-F(\lambda_i x)\right)\right)}$$
$$\iff 0 \geqslant \sum_{i=1}^{n}\phi\left(1-F(\lambda_{n+1}x)\right)\phi\left(1-F(\lambda_i x)\right)$$
$$\times\left[\frac{\lambda_{n+1}F'(\lambda_{n+1}x)}{\psi'\left(\phi\left(1-F(\lambda_{n+1}x)\right)\right)\phi\left(1-F(\lambda_{n+1}x)\right)} - \frac{\lambda_i F'(\lambda_i x)}{\psi'\left(\phi\left(1-F(\lambda_i x)\right)\right)\phi\left(1-F(\lambda_i x)\right)}\right]$$
$$\iff 0 \geqslant \sum_{i=1}^{n}\phi\left(1-F(\lambda_{n+1}x)\right)\phi\left(1-F(\lambda_i x)\right)$$
$$\times\left[-\frac{d\left(\ln(\phi(1-F(\lambda_{n+1}x)))\right)}{dx} + \frac{d\left(\ln(\phi(1-F(\lambda_i x)))\right)}{dx}\right]. \tag{6}$$

Due to the condition $\lambda_i \leqslant \lambda_{n+1}$, for all i, (6) holds if

$$\frac{d^2\ln(\phi(1-F(x)))}{dx^2}$$

is positive. Thus, it suffices that $\phi(1-F(x))$ is log-convex.

(ii) The proof is similar to that of Part (i) and is therefore omitted for the sake of brevity. □

EXAMPLE 4. Here again, it should be mentioned that the condition "$-\ln\psi(e^x)$ is convex and $\phi(1-F(x))$ is log-concave" in Part (i) of Theorem 4 are satisfied by some survival Archimedean copulas and some marginals. For example, consider $\psi(x) = (x+1)^{-1}$. Thus, $-\ln(\psi(e^x))$ is convex. On the other hand, let us set $F(x) = 1 - e^{-\lambda x}, (\lambda > 0)$, then $\phi(1-F(x)) = \frac{1}{1-F(x)} - 1 = e^{\lambda x} - 1$. Now, it is easy to verify that $\phi(1-F(x))$ is log-concave.

5. Concluding remarks

In this paper, we have established the hazard rate order and reversed hazard rate order of series and parallel systems when the components follow general scale model under Archimedean copula. Several examples are also presented for illustrations.

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