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COMPLETE MONOTONICITY OF SOME FUNCTIONS INVOLVING k–DIGAMMA FUNCTION WITH APPLICATION

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Abstract. We present several complete monotonicity properties involving k-digamma function with single parameter. These established results provide a k-generalization for the known results obtained by Burić and Elezović in [5]. Finally, we give an application to the generalized Nielsen's β-function and pose two open problems.

1. Introduction

The Euler gamma function is defined for all positive real numbers x by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt.$$

It is common knowledge that the logarithmic derivative of $\Gamma(x)$ is called the psi or digamma function, and $\psi^{(m)}(x)$ for $m\in\mathbb{N}$ are known as the polygamma functions. The gamma, digamma and polygamma functions play an important role in the theory of special functions, and have applications in many other branches, such as statistics, fractional differential equations, mathematical physics and theory of infinite series. The reader may see references [6, 7, 8]. some of the work about the complete monotonicity, convexity and concavity, and inequalities of these special functions may refer to [1, 2, 3, 4, 9, 10, 11, 12, 19, 20, 21, 22, 25].

In 2007, Diaz and Pariguan [7] defined the $k-$analogue of the gamma function for $k>0$ and $x>0$ as

$$\Gamma_k(x)=\int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt=\lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{x}{k}-1}}{x(x+k)\cdots(x+(n-1)k)},$$

where $\lim_{k\to1}\Gamma_k(x)=\Gamma(x)$. Similarly, we may define the $k-$analogue of the digamma and polygamma functions as

$$\psi_k(x)=\frac{d}{dx}\ln\Gamma_k(x)\quad\text{and}\quad\psi_k^{(m)}(x)=\frac{d^m}{dx^m}\psi_k(x).$$

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It is well known that the $k-$analogues of the digamma and polygamma functions satisfy the following recursive formula and series identities (see [7])

$$\Gamma_k(x+k) = x\Gamma_k(x), \quad x > 0, \tag{1.1}$$

$$\psi_k(x) = \frac{\ln k - \gamma}{k} - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{nk(nk+x)} \tag{1.2}$$

and

$$\psi_k^{(m)}(x) = (-1)^{m+1} m! \sum_{n=0}^{\infty} \frac{1}{(nk+x)^{m+1}} \tag{1.3}$$

$$= (-1)^{m+1} \int_0^{\infty} \frac{1}{1-e^{-kt}} t^m e^{-xt}\, dt. \tag{1.4}$$

For more properties of these functions, the reader may see the references [14, 15, 16, 26].

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and satisfies $(-1)^n f^{(n)}(x) \geqslant 0$ for $x \in I$ and $n \geqslant 0$. For the background and application, the reader may see [23]. A characterization of completely monotonic functions is given by the Bernstein-Widder theorem which reads that a function $f(x)$ on $x \in [0,\infty)$ is completely monotonic if and only if there exists a bounded and non-decreasing function $g(t)$ such that the integral

$$f(x) = \int_0^{\infty} e^{-xt}\, dg(t)$$

converges for $x \in [0,\infty)$. That is, a function $f(x)$ is completely monotonic on $x \in [0,\infty)$ if and only if it is a Laplace transform of a bounded and non-decreasing measure $g(t)$. From above theorem it follows that completely monotonic functions on $[0,\infty)$ are always strictly completely monotonic unless they are constant (see [18]).

Recently, Burić and Elezović [5] studied complete monotonicity properties of some functions involving the psi function and they proved necessary and sufficient conditions for these functions to be complete monotonic. It is a natural question if these results can be generalized to k-digamma functions. The main aim of this paper is to generalize theorems proved by Burić and Elezović and establish monotonicity properties of some functions involving k-digamma function.

2. Main results

LEMMA 2.1. ([5]) *Let φ be bounded and continuous at 0. Suppose that for all positive x we have*

$$\int_0^{\infty} e^{-xt} \varphi(t)\, dt \geqslant 0. \tag{2.1}$$

Then, it holds $\varphi(0) \geqslant 0$.

THEOREM 2.1. *Let k, a, b, c, d be some given positive numbers and $a \leqslant c$. Then the function*

$$f_{k,1}(x) = \psi_k(ax+b) - \psi_k(cx+d) + \log\left(\frac{c}{a}\right). \tag{2.2}$$

is completely monotonic on $(0,\infty)$ *if and only if*

$$\lambda \leqslant \frac{k(c-a)}{2}. \tag{2.3}$$

where $\lambda = k(c-a)+ad-bc$.

Proof. Direct calculation yields

$$\begin{aligned}(-1)^n f_{k,1}^{(n)}(x) &= (-1)^n\left[a^n\psi_k^{(n)}(ax+b)-c^n\psi_k^{(n)}(cx+d)\right]\\ &= \int_0^\infty \frac{c^nt^ne^{-(cx+d)t}}{1-e^{-kt}}dt-\int_0^\infty \frac{a^nt^ne^{-(ax+b)t}}{1-e^{-kt}}dt.\end{aligned} \tag{2.4}$$

The substitutions $t=au$ and $t=cu$ turn (2.4) into

$$\begin{aligned}(-1)^n f_{k,1}^{(n)}(x) &= \int_0^\infty \frac{c^na^{n+1}u^ne^{-(cx+d)au}}{1-e^{-kau}}du-\int_0^\infty \frac{a^nc^{n+1}u^ne^{-(ax+b)cu}}{1-e^{-kcu}}du\\ &= \int_0^\infty (act)^n\left[\frac{ae^{-adt}}{1-e^{-kat}}-\frac{ce^{-bct}}{1-e^{-kct}}\right]e^{-acxt}dt\end{aligned}.$$

Denote

$$g_{k,1}(t) = \frac{ae^{-adt}}{1-e^{-kat}}-\frac{ce^{-bct}}{1-e^{-kct}}. \tag{2.5}$$

It can be written as follows

$$g_{k,1}(t) = e^{(ka-ad)t}\left(\frac{a}{e^{kat}-1}-\frac{ce^{\lambda t}}{e^{kct}-1}\right) \tag{2.6}$$

where λ is defined by

$$\lambda = k(c-a)+ad-bc. \tag{2.7}$$

In order to prove Theorem 1.1, we shall show $g_{k,1}(t)\geqslant 0$. This is equivalent to

$$h_{k,1}(t) = a\left(e^{kct}-1\right)-ce^{\lambda t}\left(e^{kat}-1\right)\geqslant 0. \tag{2.8}$$

For $\lambda\leqslant 0$ and $a\leqslant c$, a simple computation gives

$$h_{k,1}(t) = \sum_{n=1}^{\infty}\frac{k^nc^{n+1}a}{n!}\left(c^{n-1}-a^{n-1}e^{\lambda t}\right)\geqslant 0.$$

If $\lambda>0$, we get

$$\begin{aligned}h_{k,1}(t) &= a\sum_{n=1}^{\infty}\frac{k^nc^nt^n}{n!}-c\sum_{i=0}^{\infty}\frac{\lambda^it^i}{i!}\sum_{j=1}^{\infty}\frac{k^ja^jt^j}{j!}\\ &= \sum_{n=1}^{\infty}\left(\frac{ak^nc^nt^n}{n!}-c\sum_{j=1}^{n-1}\frac{\lambda^{n-j}}{(n-j)!}\frac{k^ja^j}{j!}\right)t^n\\ &= \sum_{n=1}^{\infty}\frac{c\left[ak^nc^{n-1}-(ka+\lambda)^n+\lambda^n\right]}{n!}t^n.\end{aligned}$$

So, we only show that the following inequality holds true

$$ak^nc^{n-1}-(ka+\lambda)^n+\lambda^n\geqslant 0. \tag{2.9}$$

For $n=1$, the inequality(2.4) is trivial. For $n=2$, it is equivalent to (2.3). Now, we prove (2.9) for $n>2$ by mathematical induction.

In order to prove

$$ak^{n+1}c^n - (ka+\lambda)^{n+1} + \lambda^{n+1} \geqslant 0. \tag{2.10}$$

we only need to prove

$$\begin{aligned}(ka+\lambda)^{n+1} &= (ka+\lambda)^n (ka+\lambda)\\ &\leqslant \left(ak^n c^{n-1} + \lambda^n\right)(ka+\lambda)\\ &\leqslant \lambda^{n+1} + ak^{n+1}c^n\end{aligned}$$

by assumption of induction. This is sufficient to prove

$$k^{n+1}ac^{n-1} + k^{n-1}c^{n-1}\lambda + \lambda^n \leqslant k^{n+1}c^n. \tag{2.11}$$

Note that $ka \leqslant kc - 2\lambda$ and $\lambda \leqslant kc$, that we have

$$\begin{aligned}&k^{n+1}ac^{n-1} + k^{n-1}c^{n-1}\lambda + \lambda^n\\ &\leqslant k^n c^{n-1}(kc-2\lambda) + k^{n-1}c^{n-1}\lambda + \lambda^n\\ &\leqslant k^{n+1}c^n + \lambda\left[\lambda^{n-1} - (kc)^{n-1}\right]\\ &\leqslant k^{n+1}c^n\end{aligned}$$

This implies that the inequality (2.11) holds true.

On the other hand, if $f_{k,1}$ is completely monotonic on $(0,\infty)$, we can get

$$\int_0^\infty e^{-acxt} g_{k,1}(t)\,dt > 0$$

by taking $n=0$. Applying Lemma 2.1, we get $g_{k,1}(0) \geqslant 0$. A direct computation results in

$$\begin{aligned}g_{k,1}(0) &= \lim_{t\to 0}\left(\frac{a}{e^{kat}-1} - \frac{ce^{\lambda t}}{e^{kct}-1}\right)\\ &= \lim_{t\to 0}\frac{a\left(e^{kct}-1\right) - ce^{\lambda t}\left(e^{kat}-1\right)}{\left(e^{kat}-1\right)\left(e^{kct}-1\right)}\\ &= \frac{k(c-a)-2\lambda}{2k}\\ &\geqslant 0.\end{aligned}$$

That is $\lambda \leqslant \frac{k(c-a)}{2}$. The proof is complete. □

REMARK 2.1. If $k=1$, the Theorem 2.1 reduces to Theorem 1 in [5].

By Theorem 2.1, the following corollaries can be easily obtained.

COROLLARY 2.1. *Let k,a,b,c,d be given positive numbers. Then the function*

$$\psi_k(ax+b) - \psi_k(ax+d) \tag{2.12}$$

is completely monotonic on $(0,\infty)$ if and only if $d \leqslant b$.

COROLLARY 2.2. *Let k,a,c be given positive numbers. Then the function*

$$\psi_k(ax+1) - \psi_k(cx+1) + \log\left(\frac{c}{a}\right) \tag{2.13}$$

is completely monotonic if and only if $a \leqslant c$ and $k \leqslant 2$.

LEMMA 2.2. *For $k>0$ and $x>0$, the following duplication formula holds true*

$$\psi_k(2kx)=\frac{1}{2}\psi_k(kx)+\frac{1}{2}\psi_k\left(kx+\frac{k}{2}\right)+\frac{1}{k}\ln 2. \tag{2.14}$$

Proof. Using identity [7]

$$\Gamma_k(x)=k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right) \tag{2.15}$$

and Legendre relation

$$2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)=\pi^{\frac{1}{2}}\Gamma(2x), \tag{2.16}$$

we easily obtain

$$\Gamma_k(kx)=k^{x-1}\Gamma(x), \tag{2.17}$$

$$\Gamma_k\left(kx+\frac{k}{2}\right)=k^{x-\frac{1}{2}}\Gamma\left(x+\frac{1}{2}\right), \tag{2.18}$$

and

$$\Gamma_k(2kx)=\frac{(2k)^{2x-1}}{\pi^{\frac{1}{2}}}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right). \tag{2.19}$$

Combining (2.17),(2.18) with (2.19), we have

$$\left(\frac{\pi}{k}\right)^{\frac{1}{2}}\Gamma_k(2kx)=2^{2x-1}\Gamma_k(kx)\Gamma_k\left(kx+\frac{k}{2}\right). \tag{2.20}$$

Taking logarithm and differentiating on both sides of (2.20), we get (2.14). This completes the proof. □

COROLLARY 2.3. *Let $k>0$. Then the function*

$$\psi_k(x+\xi)-\frac{1}{2}\psi_k(kx)-\frac{1}{2}\psi_k\left(kx+\frac{k}{2}\right)+\frac{k-1}{k}\ln 2 \tag{2.21}$$

is completely monotonic on $(0,\infty)$ if and only if

$$\xi\geqslant\frac{2k-1}{4}.$$

Proof. Applying Lemma 2.2, we easily complete the proof. □

THEOREM 2.2. *If k,a,b,c,d are positive numbers, then the function*

$$f_{k,2}(x)=\psi_k(ax+b)-k\log(cx+d) \tag{2.22}$$

is completely monotonic on $(0,\infty)$ if and only if it holds

$$\mu\leqslant\frac{kc}{2} \tag{2.23}$$

where $\mu=kc+ad-bc$.

Proof. Using integral representation of k-psi function and identity

$$\log x = \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} dt. \tag{2.24}$$

we have

$$\begin{aligned}(-1)^n f_{k,2}^{(n)}(x) &= (-1)^n \left[a^n \psi_k^{(n)}(ax+b) - kc^n \log^{(n)}(cx+d)\right] dt \\ &= \int_0^\infty (act)^n e^{-acxt} g_{k,2}(t)\end{aligned}$$

where

$$g_{k,2}(t) = \frac{e^{-adt}}{t} - \frac{kce^{-bct}}{1-e^{-kct}}. \tag{2.25}$$

Direct calculation yields

$$g_{k,2}(t) = \frac{e^{-adt}}{t(e^{kct}-1)} h_{k,2}(t)$$

where

$$h_{k,2}(t) = e^{kct} - 1 - kcte^{\mu t}. \tag{2.26}$$

For the positivity of the function $g_{k,2}(t)$, it suffices to prove $h_{k,2}(t) \geqslant 0$ for $t \in (0,\infty)$. Now, let us prove $h_{k,2}(t) \geqslant 0$ in two cases.

Case 1. Suppose $\mu \leqslant 0$. For $k > 0$ and $t > 0$, we have

$$e^{kct} \geqslant 1 + kct. \tag{2.27}$$

This implies that $h_{k,2}(t) \geqslant 0$ holds true for $\mu \leqslant 0$.

Case 2. For $\mu > 0$, we have

$$\begin{aligned}h_{k,2}(t) &= \sum_{n=1}^\infty \frac{(kct)^n}{n!} - kct \sum_{n=0}^\infty \frac{(\mu t)^n}{n!} \\ &= kc \sum_{n=1}^\infty \frac{t^n}{n!}\left(k^{n-1}c^{n-1} - n\mu^{n-1}\right).\end{aligned}$$

So, we need to show

$$k^{n-1}c^{n-1} - n\mu^{n-1} \geqslant 0. \tag{2.28}$$

For $n = 1$, it is trivial. For $n = 2$, it is equivalent to (2.23). That is

$$2ad < (2b-k)c. \tag{2.29}$$

Let us assume that it is valid for n, and we shall show that it is also valid for $n+1$. Considering the condition(2.23), we have

$$\begin{aligned}(n+1)\mu^n &= n\mu^{n-1}\mu + \mu^n \\ &\leqslant k^{n-1}c^{n-1}\mu + \mu^n \\ &\leqslant k^{n-1}c^{n-1}\frac{kc}{2} + \frac{(kc)^n}{2} \\ &= k^n c^n.\end{aligned}$$

So, we prove (2.28) by induction.

Next, we prove necessity. Suppose f is completely monotonic. Then the following integral

$$\int_0^{\infty} e^{-acxt} g_{k,2}(t)dt \geqslant 0.$$

From Lemma 2.1, we have $g_{k,2}(0) \geqslant 0$. Using Taylor formula, we get

$$\begin{aligned}\lim_{t\to 0} g_{k,2}(t) &= \lim_{t\to 0} \frac{e^{-adt}\left(1-e^{-kct}\right)-kcte^{-bct}}{t\left(1-e^{-kct}\right)}\\ &= \lim_{t\to 0} \frac{(1-adt)\left(kct-\frac{k^2c^2t^2}{2}\right)-kct(1-bct)+o\left(t^2\right)}{t(kct)+o\left(t^2\right)}\\ &= bc-ad-\frac{kc}{2}.\end{aligned}$$

This is equivalent to (2.23). This completes the proof. □

REMARK 2.2. Taking $k=1$ in theorem 2.2, we get Theorem 2 in [5].

3. An application

In this section, we shall give an application to the generalized Nielsen's Beta function by using Theorem 2.1. The classical Nielsen's β-function can be defined as ([17])

$$\begin{aligned}\beta(x) &= \int_0^1 \frac{t^{x-1}}{1+t}dt = \int_0^{\infty} \frac{e^{-xt}}{1+e^{-t}}dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+x},\\ &= \frac{1}{2}\left\{\psi\left(\frac{x+1}{2}\right)-\psi\left(\frac{x}{2}\right)\right\},\end{aligned}$$

where $x \in (0,\infty)$. This function is closely related to other special functions such as hypergeometric function $F(a,b;c;x)$, beta function $B(x,y)$ and so on. Here we have two interesting equations:

$$\beta(x)+\beta(1-x) = B(x,1-x) \tag{3.1}$$

and

$$\beta(x) = \frac{1}{x^2}F(1,x;x+1;-1). \tag{3.2}$$

Very naturally, we may define the k-generalization of the Nielsen's β-function as

$$\begin{aligned}\beta_k(x) &= \int_0^1 \frac{t^{x-1}}{1+t^k}dt\\ &= \int_0^{\infty} \frac{e^{-xt}}{1+e^{-kt}}dt\\ &= \sum_{n=0}^{\infty}\left(\frac{1}{2nk+x}-\frac{1}{2nk+k+x}\right)\\ &= \frac{1}{2}\left\{\psi_k\left(\frac{x+k}{2}\right)-\psi_k\left(\frac{x}{2}\right)\right\}.\end{aligned}$$

By using Theorem 2.1, we easily obtain complete monotonicity of generalized Nielsen's β-function.

THEOREM 3.1. *For $k>0$ and $0<\alpha\leqslant 1$, the function $x^{\alpha-1}\beta_k(x)$ is complete monotonic on $(0,\infty)$.*

Proof. In Theorem 2.1, by taking $a=c=\frac{1}{2}, b=\frac{k}{2}, d=0$, we easily obtain the function $2\beta_k(x)$ is complete monotonic on $(0,\infty)$. Since the product of any two completely monotonic function is also completely monotonic on their domain, and the function $x^{\alpha-1}$ for $0<\alpha\leqslant 1$ is clearly completely monotonic on $(0,\infty)$, so we obtain that the function $x^{\alpha-1}\beta_k(x)$ is complete monotonic on $(0,\infty)$. This completes the proof. □

4. Open problems

Very recently, K. Nantomah, E. Prempeh and S. B. Twum[16] introduced a new definition of gamma function with two parameters as follows:

$$\Gamma_{p,k}(x)=\frac{(p+1)!k^{p+1}(pk)^{\frac{x}{k}-1}}{(x)_{p,k}}, x>0 \tag{4.1}$$

where $(x)_{p,k}=x(x+k)(x+2k)\dots(x+pk)$ and $\lim_{p\to\infty}\Gamma_{p,k}(x)=\Gamma_k(x)$. Furthermore, we naturally define the (p,k)- analogue of the digamma and polygamma functions as follows: $\psi_{p,k}(x)=\frac{\Gamma'_{p,k}(x)}{\Gamma_{p,k}(x)}$ and $\psi^{(m)}_{p,k}(x)=\frac{d^m}{dx^m}\psi_{p,k}(x)$. In [24], Yin established the following theorem:

THEOREM 4.1. (Theorem 4.1, [24]) *For $p\in\mathbf{N}, k>0$ and $\alpha\leqslant 1$, the function*

$$\delta_{p,k,\alpha}(x)=x^{\alpha}\left[\frac{1}{k}\ln\frac{pkx}{x+k(p+1)}-\psi_{p,k}(x)\right]$$

is complete monotonic on $(0,\infty)$.

Setting $p\mapsto\infty$, we get

THEOREM 4.2. *For $k>0$ and $\alpha\leqslant 1$, the function*

$$\delta_{k,\alpha}(x)=x^{\alpha}\left[\frac{1}{k}\ln x-\psi_k(x)\right]$$

is complete monotonic on $(0,\infty)$.

By proved results of Matejiv̌cka in [13], we may obtained complete monotonic degree of the function $\frac{1}{k}\ln x-\psi_k(x)$ as follows:

$$\deg^x_{cm}\left[\frac{1}{k}\ln x-\psi_k(x)\right]=1. \tag{4.2}$$

Very natural, we pose the following open problems:

OPEN PROBLEM 4.1. *If k,a,b,c,d are positive numbers, then determine α,k,a,b,c,d for which the function*

$$x^{\alpha}[\psi_k(ax+b)-k\log(cx+d)] \tag{4.3}$$

is completely monotonic on $(0,\infty)$.

OPEN PROBLEM 4.2. *Discuss complete monotonic degree of the function $\psi_k(ax+b)-k\log(cx+d)$ under the condition that the above open problem 4.1 is valid.*

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