

ON THE OPERATOR ACZÉL INEQUALITY AND ITS REVERSE

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Abstract. In this paper, we present some operator and eigenvalue inequalities involving operator monotone, doubly concave and doubly convex functions. These inequalities provide some variants of the operator Aczél inequality and its reverse via the generalized Kantorovich constant.

1. Introduction

Let $B(\mathcal{H})$ denote the C^* -algebra of all bounded linear operators on a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. An operator $A \in B(\mathcal{H})$ is called *positive* if $\langle Ax, x \rangle \geq 0$ for every $x \in \mathcal{H}$ and then we write $A \geq 0$. For self-adjoint operators $A, B \in B(\mathcal{H})$, we say $A \leq B$ if $B - A \geq 0$. Also, we say A is *strictly positive* and we write $A > 0$, if $\langle Ax, x \rangle > 0$ for every $x \in \mathcal{H}$. Let f be a continuous real function on $(0, \infty)$. Then f is said to be *operator monotone* (more precisely, operator monotone increasing) if $A \geq B$ implies $f(A) \geq f(B)$ for strictly positive operators A, B , and *operator monotone decreasing* if $-f$ is operator monotone or $A \geq B$ implies $f(A) \leq f(B)$. Also, f is said to be *operator convex* if $f((1 - \alpha)A + \alpha B) \leq (1 - \alpha)f(A) + \alpha f(B)$ for all strictly positive operators A, B and $\alpha \in [0, 1]$, and *operator concave* if $-f$ is operator convex.

In 1956, Aczél [1] proved that if $a_i, b_i (1 \leq i \leq n)$ are positive real numbers such that $a_1^2 - \sum_{i=2}^n a_i^2 > 0$ and $b_1^2 - \sum_{i=2}^n b_i^2 > 0$, then

$$\left(a_1 b_1 - \sum_{i=2}^n a_i b_i \right)^2 \geq \left(a_1^2 - \sum_{i=2}^n a_i^2 \right) \left(b_1^2 - \sum_{i=2}^n b_i^2 \right).$$

Aczél's inequality has important applications in the theory of functional equations in non-Euclidean geometry [1, 13] and considerable attention has been given to this inequality involving its generalizations, variations and applications. See [6, 12] and references therein. Popoviciu [12] first presented an exponential extension of Aczél's inequality as follows:

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THEOREM 1. ([12]) *Let $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} = 1, a_1^p - \sum_{i=2}^n a_i^p > 0$, and $b_1^q - \sum_{i=2}^n b_i^q > 0$. Then*

$$a_1 b_1 - \sum_{i=2}^n a_i b_i \geq \left(a_1^p - \sum_{i=2}^n a_i^p \right)^{\frac{1}{p}} \left(b_1^q - \sum_{i=2}^n b_i^q \right)^{\frac{1}{q}}.$$

Aczél’s and Popoviciu’s inequalities were sharpened and a variant of Aczél’s inequality in inner product spaces was given by Dragomir [6]. Recently, Moslehian in [11] proved an operator version of the classical Aczél inequality involving α -geometric mean

$$A \sharp_{\alpha} B = A^{1/2} (A^{-1/2} B A^{-1/2})^{\alpha} A^{1/2} \tag{1}$$

for $A > 0, B \geq 0$ and $\alpha \in [0, 1]$, in the following form:

THEOREM 2. ([11]) *Let J be an interval of $(0, \infty)$, let $f : J \rightarrow (0, \infty)$ be operator decreasing and operator concave on $J, \frac{1}{p} + \frac{1}{q} = 1, p, q > 1$ and let $A, B \in \mathbb{B}(\mathcal{H})$ be positive invertible operators with spectra contained in J . Then*

$$f(A \sharp_{\frac{1}{q}} B^q) \geq f(A^p) \sharp_{\frac{1}{q}} f(B^q), \tag{2}$$

$$\langle f(A \sharp_{\frac{1}{q}} B^q) x, x \rangle \geq \langle f(A^p) x, x \rangle^{\frac{1}{p}} \langle f(B^q) x, x \rangle^{\frac{1}{q}}, \tag{3}$$

for any vector $x \in \mathcal{H}$.

After that, Kaleibary and Furuichi in [10] provided a reverse of the operator Aczél inequality using the Kantorovich constant as follows.

THEOREM 3. ([10, Theorem 1]) *Let g be a non-negative operator monotone decreasing function on $(0, \infty), \frac{1}{p} + \frac{1}{q} = 1, p, q > 1$, and $0 < s A^p \leq B^q \leq t A^p$ for some scalars $0 < s \leq t$. Then, for all $x \in \mathcal{H}$*

$$g(A \sharp_{\frac{1}{q}} B^q) \leq \max\{K(s)^R, K(t)^R\} g(A^p) \sharp_{\frac{1}{q}} g(B^q), \tag{4}$$

$$\langle g(A \sharp_{\frac{1}{q}} B^q) x, x \rangle \leq \max\{K(s)^R, K(t)^R\} \langle g(A^p) x, x \rangle^{\frac{1}{p}} \langle g(B^q) x, x \rangle^{\frac{1}{q}}, \tag{5}$$

where $R = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$ and $K(h) = \frac{(h+1)^2}{4h}, h > 0$ is the Kantorovich constant.

In this paper, we first investigate some operator and eigenvalue inequalities involving operator monotone, doubly concave and doubly convex functions. Then we provide another type of operator Aczél inequalities along with their reverse using the obtained results. Since for a nonnegative continuous function f defined on $(0, \infty)$ the operator concavity is equivalent to the operator monotonicity, the assumptions on f in Theorem 2 seem to be slightly strong, in the special case of $J = (0, \infty)$. Hence, we aim to prove a variant of Theorem 2 with a reduced condition on f . As an application, we present a counterpart of the classical Aczél inequality stated in Theorem 1. These results are organized in Sections 2. Section 3 is devoted to study of Aczél type inequality involving doubly concave functions. In Sections 4, we show several eigenvalue inequalities

involving α -geometric mean and doubly convex functions. The obtained eigenvalue inequalities allow us to study the reverse of operator Aczél inequality via the generalized Kantorovich constant $K(w, \alpha)$. The assumptions of doubly convexity (concavity) will be discussed in more details in later sections.

2. A variant of the operator Aczél inequality

In this section, we present a variant of the operator Aczél inequality by using several reverse Young's inequalities. Let A and B be strictly positive operators. For each $\alpha \in [0, 1]$, the α -arithmetic mean is defined as $A \nabla_{\alpha} B := (1 - \alpha)A + \alpha B$ and the α -geometric mean is defined in (1). Clearly if $AB = BA$, then $A \sharp_{\alpha} B = A^{1-\alpha} B^{\alpha}$. Some basic properties of the arithmetic and geometric means can be found in [7]. It is well-known as the Young inequality [14]:

$$A \sharp_{\alpha} B \leq A \nabla_{\alpha} B.$$

The research on the Young inequality is interesting and there are several multiplicative and additive reverses of this inequality [2]. We give here some reverse inequalities for the operators with the sandwich condition $0 < sA \leq B \leq tA$.

LEMMA 1. ([10, Lemma 2]) *Let $0 < sA \leq B \leq tA$ for some scalars $0 < s \leq t$ and $\alpha \in [0, 1]$. Then*

$$A \nabla_{\alpha} B \leq \max\{K^R(s), K^R(t)\} (A \sharp_{\alpha} B), \quad (6)$$

where $K(\cdot)$ is the Kantorovich constant defined in Theorem 3 and $R = \max\{\alpha, 1 - \alpha\}$.

The function $K(\cdot)$ is decreasing on $(0, 1)$ and increasing on $[1, \infty)$, $K(t) = K(\frac{1}{t})$, and $K(t) \geq 1$ for every $t > 0$ [7].

LEMMA 2. ([10, Proposition 1]) *Let g be a non-negative operator monotone decreasing function on $(0, \infty)$ and $0 < sA \leq B \leq tA$ for some scalars $0 < s \leq t$. Then, for all $\alpha \in [0, 1]$*

$$\frac{1}{c} g(A \sharp_{\alpha} B) \leq g(c(A \sharp_{\alpha} B)) \leq g(A) \sharp_{\alpha} g(B), \quad (7)$$

where $c = \max\{K^R(s), K^R(t)\}$ with the Kantorovich constant $K(\cdot)$ and $R = \max\{\alpha, 1 - \alpha\}$.

LEMMA 3. *Let f be a non-negative operator monotone function on $(0, \infty)$ and $0 < sA \leq B \leq tA$ for some constants $0 < s \leq t$. Then, for all $\alpha \in [0, 1]$ we have*

$$cf(A \sharp_{\alpha} B) \geq f(c(A \sharp_{\alpha} B)) \geq f(A) \sharp_{\alpha} f(B),$$

where $c = \max\{K^R(s), K^R(t)\}$ with the Kantorovich constant $K(\cdot)$ and $R = \max\{\alpha, 1 - \alpha\}$.

Proof. First note that since f is analytic on $(0, \infty)$, we may assume that $f(x) > 0$ for all $x > 0$; otherwise f is identically zero. Also, since f is an operator monotone

function on $(0, \infty)$, so $\frac{1}{f}$ is a non-negative operator monotone decreasing function on $(0, \infty)$. By Applying Lemma 2 for $g = \frac{1}{f}$ we have

$$\frac{1}{c}f(A\sharp_{\alpha}B)^{-1} \leq f(c(A\sharp_{\alpha}B))^{-1} \leq f(A)^{-1}\sharp_{\alpha}f(B)^{-1} = (f(A)\sharp_{\alpha}f(B))^{-1}.$$

Reversing the all sides gives the desired inequality. \square

THEOREM 4. *Let f be a non-negative operator monotone function on $(0, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1, p, q > 1$, and $0 < sA^p \leq B^q \leq tA^p$ for some scalars $0 < s \leq t$. Then we have*

$$f(A^p\sharp_{1/q}B^q) \geq \frac{1}{c}f(A^p)\sharp_{1/q}f(B^q), \tag{8}$$

and

$$\langle f(A^p\sharp_{1/q}B^q)x, x \rangle \geq \frac{1}{c}\langle f(A^p)x, x \rangle^{1/p}\langle f(B^q)x, x \rangle^{1/q}, \tag{9}$$

for all $x \in \mathcal{H}$, in which $c = \max\{K^R(s), K^R(t)\}$ with the Kantorovich constant $K(\cdot)$ and $R = \max\{1/p, 1/q\}$.

Proof. Putting $A := A^p, B := B^q$ and $\alpha := 1/q$ in Lemma 3, we have (8). Also, using the first inequality of Lemma 3 with $A := A^p, B := B^q$ and $\alpha := 1/q$, we have

$$\begin{aligned} c\langle f(A^p\sharp_{1/q}B^q)x, x \rangle &\geq \langle f(c(A^p\sharp_{1/q}B^q))x, x \rangle \\ &\geq \langle f(A^p\nabla_{1/q}B^q)x, x \rangle \quad (\text{operator monotonicity of } f \text{ with (6)}) \\ &\geq \langle (\frac{1}{p}f(A^p) + \frac{1}{q}f(B^q))x, x \rangle \quad (\text{operator concavity of } f) \\ &= \frac{1}{p}\langle f(A^p)x, x \rangle + \frac{1}{q}\langle f(A^q)x, x \rangle \\ &\geq \langle f(A^p)x, x \rangle^{1/p}\langle f(B^q)x, x \rangle^{1/q} \quad (\text{AM-AG inequality}) \end{aligned}$$

which implies (9). \square

REMARK 1.

- (a) The constant c in Theorem 4 can be replaced by the constant $\acute{c} := \max\{S(s), S(t)\}$

where $S(x) := \frac{x^{\frac{1}{x-1}}}{e \log x^{\frac{1}{x-1}}}$ for $x > 0$ with $x \neq 1$ is the so-called Specht’s ratio. See

[8, Theorem 1]. In addition, for $\alpha \in (0, 1)$ we have no ordering between the estimates $K^R(h), R = \max\{\alpha, 1 - \alpha\}$ and $S(h)$ for $h > 0$ with $h \neq 1$ in general. Because we have numerical examples such that $K^{0.6}(0.01) - S(0.01) \simeq -1.30357$ and $K^{0.6}(5.0) - S(5.0) \simeq 0.0556589$. For $\alpha = 1$, although $K(h) \geq S(h)$ for $h > 0$ with $h \neq 1$, it cannot be satisfied in the condition $\alpha := 1/q$ with $1/q + 1/p = 1$ of Theorem 4.

- (b) Our results given in both (8) and (9) are weaker than ones in both (2) and (3), since $K(h) \geq 1$ (and also $S(h) \geq 1$) for $h > 0$ with $h \neq 1$. But our assumption for the function f in Theorem 4 is better than one for the function f in Theorem 2.

COROLLARY 1. Let $1/p + 1/q = 1$ with $p, q > 1$. For commuting positive invertible operators A and B with spectra contained in $(1, \infty)$ such that $sA^p \leq B^q \leq tA^p$ for some scalars $0 < s \leq t$, we have for any unit vector $x \in \mathcal{H}$,

$$\|(AB)^{1/2}x\|^2 - 1 \geq \frac{1}{c} \left(\|A^{p/2}x\|^2 - 1 \right)^{1/p} \left(\|B^{q/2}x\|^2 - 1 \right)^{1/q},$$

where the constant c is given in Theorem 4.

Proof. Taking $f(t) = t - 1$ on $(1, \infty)$ in Theorem 4, we get the desired result. \square

From (8), we also have the following corollaries.

COROLLARY 2. Let $1/p + 1/q = 1$ with $p, q > 1$ and f be an operator monotone function on $(0, \infty)$. For commuting positive invertible operators A and B such that $sA^p \leq B^q \leq tA^p$ for some scalars $0 < s \leq t$, we have

$$f(AB) \geq \frac{1}{c} f(A^p)^{1/p} f(B^q)^{1/q},$$

where the constant c is given in Theorem 4.

COROLLARY 3. Let $1/p + 1/q = 1$ with $p, q > 1$ and f be a non-negative increasing function on $(0, \infty)$ and a_i, b_i be positive numbers such that $0 < sa_i^p \leq b_i^q \leq ta_i^p$ for some scalars $0 < s \leq t$. Then we have

$$\sum_{i=1}^n f(a_i b_i) \geq \frac{1}{c} \left(\sum_{i=1}^n f(a_i^p) \right)^{1/p} \left(\sum_{i=1}^n f(b_i^q) \right)^{1/q}, \tag{10}$$

where the constant c is given in Theorem 4.

The following result provides a counterpart of Theorem 1.

COROLLARY 4. Let $1/p + 1/q = 1$ with $p, q > 1$. For positive numbers x_i and y_i such that $\sum_{i=2}^n x_i^p \geq x_1^p$, $\sum_{i=2}^n y_i^q \geq y_1^q$, $\sum_{i=2}^n x_i y_i \geq x_1 y_1$ and $0 < s \left(\frac{x_i}{x_1} \right)^p \leq \left(\frac{y_i}{y_1} \right)^q \leq t \left(\frac{x_i}{x_1} \right)^p$ for some scalars $0 < s \leq t$. Then we have

$$\sum_{i=2}^n x_i y_i - x_1 y_1 \geq \frac{1}{c} \left(\sum_{i=2}^n x_i^p - x_1^p \right)^{1/p} \left(\sum_{i=2}^n y_i^q - y_1^q \right)^{1/q},$$

where the constant c is given in Theorem 4.

Proof. Firstly we note that the inequality (10) is true for any $n \in \mathbb{N}$, as $i = 1, \dots, n - 1$ so that we may relabel as it is true for $i = 2, \dots, n$. Take a function $f(t) := t - \frac{1}{n-1}$, ($n \geq 2$) on $(\frac{1}{n-1}, \infty)$ in Corollary 3, then $f(t)$ is non-negative and monotone increasing on $(1, \infty)$. Then we obtain the inequality:

$$\sum_{i=2}^n a_i b_i - 1 \geq \frac{1}{c} \left(\sum_{i=2}^n a_i^p - 1 \right)^{1/p} \left(\sum_{i=2}^n b_i^q - 1 \right)^{1/q}.$$

Let $x_1, y_1 > 0$. Putting $a_i := \frac{x_i}{x_1}$ and $b_i := \frac{y_i}{y_1}$ for positive numbers x_i and y_i for $i = 2, \dots, n$ in the above, we obtain

$$\sum_{i=2}^n x_i y_i - x_1 y_1 \geq \frac{1}{c} \left(\sum_{i=2}^n x_i^p - x_1^p \right)^{1/p} \left(\sum_{i=2}^n y_i^q - y_1^q \right)^{1/q},$$

under the assumptions $\sum_{i=2}^n x_i^p \geq x_1^p$, $\sum_{i=2}^n y_i^q \geq y_1^q$ and $\sum_{i=2}^n x_i y_i \geq x_1 y_1$. \square

3. Aczél inequalities with the generalized Kantorovich constant for doubly concave functions

In the next we study an analogous of Theorem 4, with the generalized Kantorovich constant $K(w, \alpha)$. For this purpose, the assumption of doubly concavity of $f(t)$ is needed.

DEFINITION 1. A non-negative continuous function $f(t)$ defined on a positive interval $J \subset [0, \infty)$, is said to be doubly concave if:

1. $f(t)$ is concave in the usual sense;
2. $f(t)$ is geometrically concave, i.e., $g(x^\alpha y^{1-\alpha}) \geq g(x)^\alpha g(y)^{1-\alpha}$ for all $x, y \in I$, and $\alpha \in [0, 1]$.

If $f(t)$ and $g(t)$ are doubly concave on J , then so is their geometric mean $f(t)^\alpha g(t)^{1-\alpha}$ for $\alpha \in [0, 1]$ and their minimum $\min\{f(t), g(t)\}$. These properties say that there are a lot of doubly concave functions. The most important examples of doubly concave functions on $J = [0, \infty)$ are $t \mapsto t^p$ with exponent $p \in [0, 1]$. Other simple examples are $t \mapsto t/(t+1)$, $t \mapsto t/\sqrt{t+1}$ and $t \mapsto 1 - e^{-t}$. On $J = [1, \infty)$, the functions $\log t$ and $(t-1)^p, p \in [0, 1]$, and on $J = [0, 1]$, the function $-t \log t$ are also doubly concave. For more examples see [4].

Now, we are ready to give a result via the constant $K(w, a)$ occurring in the following lemma.

LEMMA 4. ([5, Lemma 8]) Let $A, B > 0$ with $0 < sA \leq B \leq tA$ for some scalars $0 < s \leq t$ with $w = t/s$. Then, for all vectors x and all $\alpha \in [0, 1]$

$$\langle A \sharp_\alpha B x, x \rangle \leq \langle Ax, x \rangle^{1-\alpha} \langle Bx, x \rangle^\alpha \leq K^{-1}(w, \alpha) \langle A \sharp_\alpha B x, x \rangle,$$

where $K(w, \alpha)$ is the generalized Kantorovich constant defined for $w > 0$ by:

$$K(w, \alpha) := \frac{(w^\alpha - w)}{(\alpha - 1)(w - 1)} \left(\frac{\alpha - 1}{\alpha} \frac{w^\alpha - 1}{w^\alpha - w} \right)^\alpha. \tag{11}$$

It is known that $K(w, \alpha) \in (0, 1]$ for $\alpha \in [0, 1]$. See [7] for some important properties of $K(w, \alpha)$.

LEMMA 5. ([9, Theorem 1]) *Let f be an increasing doubly concave function on $[0, \infty)$ and $0 < sA \leq B \leq tA$ for some scalars $0 < s \leq t$ with $w = t/s$. Then for all $\alpha \in [0, 1]$ and $k = 1, 2, \dots, n$,*

$$K^{-1}(w, \alpha) \lambda_k(f(A \#_\alpha B)) \geq \lambda_k(f(K^{-1}(w, \alpha)(A \#_\alpha B))) \geq \lambda_k(f(A) \#_\alpha f(B)),$$

where $K(w, \alpha)$ is the generalized Kantorovich constant defined as (11).

This statement is equivalent to the existence of a unitary operator U satisfying the following inequality:

$$f(A \#_\alpha B) \geq K(w, \alpha) U (f(A) \#_\alpha f(B)) U^*. \tag{12}$$

Also, from the proof of [9, Theorem 1] it is inferred that the right hand side inequality holds for an increasing geometrically concave function too.

THEOREM 5. *Let f be an increasing doubly concave function on $[0, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1, p, q > 1$, and $0 < sA^p \leq B^q \leq tA^p$ for some scalars $0 < s \leq t$ with $w = t/s$. Then, there is a unitary operator U such that*

$$f(A^p \#_{1/q} B^q) \geq K(w, 1/q) U (f(A^p) \#_{1/q} f(B^q)) U^*, \tag{13}$$

where $K(w, \alpha)$ is the generalized Kantorovich constant defined as (11). In addition, if f is an operator monotone function and $s \leq 1 \leq t$, then for all $x \in \mathcal{H}$

$$\langle f(A^p \#_{1/q} B^q) Ux, Ux \rangle \geq K^2(w, 1/q) \langle f(A^p)x, x \rangle^{1/p} \langle f(B^q)x, x \rangle^{1/q}. \tag{14}$$

Proof. Putting $A := A^p, B := B^q$ and $\alpha := 1/q$ in Lemma 5, we have (13). For the inequality (14), we first note that since f is an operator monotone function, $sA^p \leq B^q \leq tA^p$ implies $f(sA^p) \leq f(B^q) \leq f(tA^p)$. Since $s \leq 1 \leq t$, by the cocavity of f we have $sf(A^p) \leq f(B^q) \leq tf(A^p)$ and so the condition number of operators $f(A^p)$ and $f(B^q)$ is also w . Now we have

$$\begin{aligned} \langle U^* f(A^p \#_{1/q} B^q) Ux, x \rangle &\geq K(w, 1/q) \langle (f(A^p) \#_{1/q} f(B^q))x, x \rangle && \text{(by (13))} \\ &\geq K^2(w, 1/q) \langle f(A^p)x, x \rangle^{1/p} \langle f(B^q)x, x \rangle^{1/q} && \text{(Lemma 4).} \end{aligned}$$

□

COROLLARY 5. *Let $1/p + 1/q = 1$ with $p, q > 1$. For commuting positive invertible operators A and B with spectra contained in $(1, \infty)$ such that $sA^p \leq B^q \leq tA^p$ for $0 < s < 1 < t$, there is a unitary operator U that for any unit vector $x \in \mathcal{H}$,*

$$\|U^*(AB)^{1/2}Ux\|^2 - 1 \geq K^2(w, 1/q) \left(\|A^{p/2}x\|^2 - 1 \right)^{1/p} \left(\|B^{q/2}x\|^2 - 1 \right)^{1/q}.$$

Proof. Taking $f(t) = t - 1$ on $(1, \infty)$ in the inequality (14), we get the desired result. Note that this function is both operator monotone and doubly concave function on $(1, \infty)$. So, we have

$$\begin{aligned} \langle U^*(A \#_{1/q} B - 1)Ux, x \rangle &= \langle U^*(AB - 1)Ux, x \rangle = \langle U^*ABUx, x \rangle - \langle x, x \rangle \\ &= \| (U^*ABU)^{1/2}x \|^2 - 1 = \| U^*(AB)^{1/2}Ux \|^2 - 1. \end{aligned}$$

The right hand side of the inequality is obtained in a similar way. \square

4. Reverse inequalities with the generalized Kantorovich constant for doubly convex functions

Theorem 3 provided a reverse of an operator Aczél inequality with the Kantorovich constant $K(t)$. Also, it has been proved for a non-negative operator decreasing function g . In this section we are going to present some another reverse of an operator Aczél inequality via generalized Kantorovich constant $K(w, \alpha)$. For this aim we need doubly convex functions.

DEFINITION 2. A non-negative continuous function $g(t)$ defined on a positive interval $J \subset [0, \infty)$, is said doubly convex if:

1. $g(t)$ is convex in the usual sense;
2. $g(t)$ is geometrically convex, i.e., $g(x^\alpha y^{1-\alpha}) \leq g(x)^\alpha g(y)^{1-\alpha}$ for all $x, y \in J$, and $\alpha \in [0, 1]$.

Given real numbers $c_i \geq 0$ and $\alpha_i \in (-\infty, 0] \cup [1, \infty), i = 1, \dots, n$, the function $g(t) := \sum_{i=1}^n c_i t^{\alpha_i}$ is doubly convex on $(0, \infty)$. See [4].

LEMMA 6. ([3, p. 58] (The Minimax Principle) *Let A be a Hermitian operator on \mathcal{H} . Then*

$$\lambda_k(A) = \min_{\dim \mathcal{F} = n-k+1} \max \{ \langle Ah, h \rangle; h \in \mathcal{F}, \|h\| = 1 \},$$

where \mathcal{F} is a subspace of \mathcal{H} .

The following result gives an analogous of Lemma 2 with the constant $K(w, \alpha)$. Also, in what follows the capital letters A, B means $n \times n$ matrices or bounded linear operators on an n -dimensional complex Hilbert space \mathcal{H} .

PROPOSITION 1. *Let g be an increasing doubly convex function on $(0, \infty)$ and A, B be positive definite matrices such that $0 < sg(A) \leq g(B) \leq tg(A)$ for some scalars $0 < s \leq t$. Then, for all $\alpha \in [0, 1]$ and $k = 1, 2, \dots, n$*

$$\lambda_k(g(A \#_\alpha B)) \leq K^{-1}(w, \alpha) \lambda_k(g(A) \#_\alpha g(B)), \tag{15}$$

where $K(w, \alpha)$ is the generalized Kantorovich constant defined as (11).

Proof. We will use the following observation which follows from the standard Jensen’s inequality: for any vector x whose norm is less than or equal to one, since g is convex $\langle g(A)x, x \rangle \geq g(\langle Ax, x \rangle)$. For any integer k less than or equal to the dimension of the space, we have a subspace \mathcal{F} of dimension $n - k + 1$ such that

$$\begin{aligned} & \lambda_k(g(A)\sharp_\alpha g(B)) \\ &= \max_{x \in \mathcal{F}: \|x\|=1} \langle g(A)\sharp_\alpha g(B)x, x \rangle \quad (\text{minmax principle}) \\ &\geq \max_{x \in \mathcal{F}: \|x\|=1} K(w, \alpha) \langle g(A)x, x \rangle^{1-\alpha} \langle g(B)x, x \rangle^\alpha \quad (\text{Lemma 4}) \\ &= \max_{h \in \mathcal{F}: \|x\|=1} K(w, \alpha) (g\langle Ax, x \rangle)^{1-\alpha} (g\langle Bx, x \rangle)^\alpha \quad (\text{convexity of } g) \\ &\geq \max_{x \in \mathcal{F}: \|x\|=1} K(w, \alpha) g(\langle Ax, x \rangle^{1-\alpha} \langle Bx, x \rangle^\alpha) \quad (\text{geometrically convexity of } g) \\ &\geq \max_{x \in \mathcal{F}: \|x\|=1} K(w, \alpha) g(\langle A\sharp_\alpha Bx, x \rangle) \quad (\text{Lemma 4}) \\ &= K(w, \alpha) \max_{x \in \mathcal{F}: \|x\|=1} \langle g(A\sharp_\alpha B)x, x \rangle \quad (\text{monotonicity of } g) \\ &\geq K(w, \alpha) \lambda_k(g(A\sharp_\alpha B)), \quad (\text{minmax principle}). \quad \square \end{aligned}$$

REMARK 2. We know that the above statement is equivalent to the existence of a unitary operator U satisfying in the following inequality:

$$g(A\sharp_\alpha B) \leq K^{-1}(w, \alpha) U(g(A)\sharp_\alpha g(B))U^*. \tag{16}$$

This result provides a reverse of the inequality (12) for doubly convex functions.

Applying Proposition 1 we achieve the following reverse operator Aczél inequality.

THEOREM 6. *Let g be an increasing doubly convex function on $(0, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1, p, q > 1$ and $sg(A^p) \leq g(B^q) \leq tg(A^p)$ for some scalars $0 < s \leq t$. Then, there is a unitary operator U such that for all $x \in \mathcal{H}$*

$$g(A\sharp_{\frac{1}{q}} B^q) \leq K^{-1}(w, 1/q) U(g(A^p)\sharp_{\frac{1}{q}} g(B^q))U^*, \tag{17}$$

$$\langle g(A\sharp_{\frac{1}{q}} B^q)Ux, Ux \rangle \leq K^{-1}(w, 1/q) \langle g(A^p)x, x \rangle^{\frac{1}{p}} \langle g(B^q)x, x \rangle^{\frac{1}{q}}. \tag{18}$$

Proof. Letting $\alpha := \frac{1}{q}$ and replacing A^p and B^q with A and B in the inequality (16), we reach the first inequality. For the second, we have

$$\begin{aligned} \langle U^* g(A\sharp_{\frac{1}{q}} B^q)Ux, x \rangle &\leq K^{-1}(w, 1/q) \langle (g(A^p)\sharp_{\frac{1}{q}} g(B^q))x, x \rangle \quad (\text{by (17)}) \\ &\leq K^{-1}(w, 1/q) \langle g(A^p)x, x \rangle^{\frac{1}{p}} \langle g(B^q)x, x \rangle^{\frac{1}{q}}, \quad (\text{Lemma 4}). \quad \square \end{aligned}$$

REMARK 3. Theorem 6 is a conjugate of Theorem 5, which gives a reverse operator Aczél inequality relevant to the generalized Kantorovich constant.

In the following, we will present another reverse of Aczél inequality involving decreasing geometrically convex functions. Note that x^p for $p < 0$ on $(0, \infty)$ and $\csc(x)$ on $(0, \frac{\pi}{2})$ are examples of decreasing geometrically convex functions.

PROPOSITION 2. *Let g be a decreasing geometrically convex function on $(0, \infty)$ and $0 < sA \leq B \leq tA$ for some scalars $0 < s \leq t$ with $w = t/s$. Then for all $\alpha \in [0, 1]$ and $k = 1, 2, \dots, n$,*

$$\lambda_k(g(K^{-1}(w, \alpha)(A\#_{\alpha}B))) \leq \lambda_k(g(A)\#_{\alpha}g(B)).$$

Proof. Since g is an decreasing geometrically convex function $(0, \infty)$, so $f = 1/g$ is an increasing geometrically concave function on $(0, \infty)$ as follows:

$$f(x)^{\alpha} f(y)^{1-\alpha} = \frac{1}{g(x)^{\alpha} g(y)^{1-\alpha}} \leq \frac{1}{g(x^{\alpha} y^{1-\alpha})} = f(x^{\alpha} y^{1-\alpha}).$$

Furthermore, according to Lemma 5 for every increasing geometrically concave function f

$$\lambda_k(f(A)\#_{\alpha}f(B)) \leq \lambda_k(f(K^{-1}(w, \alpha)(A\#_{\alpha}B))).$$

Now, by applying this inequality for the function $f = 1/g$ we have

$$\lambda_k(g(A)^{-1}\#_{\alpha}g(B)^{-1}) \leq \lambda_k(g(K^{-1}(w, \alpha)(A\#_{\alpha}B))^{-1}),$$

where $k = 1, 2, \dots, n$. Thanks to the property $A^{-1}\#_{\alpha}B^{-1} = (A\#_{\alpha}B)^{-1}$ we can write

$$\lambda_k((g(A)\#_{\alpha}g(B))^{-1}) \leq \lambda_k(g(K^{-1}(w, \alpha)(A\#_{\alpha}B))^{-1}).$$

On the other hand, for every operator $A > 0$, $\lambda_k(A^{-1}) = \lambda_{n-k+1}^{-1}(A)$. Hence

$$\lambda_{n-k+1}^{-1}(g(A)\#_{\alpha}g(B)) \leq \lambda_{n-k+1}^{-1}(g(K^{-1}(w, \alpha)(A\#_{\alpha}B))).$$

This inequality is equevalent to the following one

$$\lambda_j(g(A)\#_{\alpha}g(B)) \geq \lambda_j(g(K^{-1}(w, \alpha)(A\#_{\alpha}B))),$$

for $j = 1, 2, \dots, n$ as desired. \square

THEOREM 7. *Let g be a decreasing doubly convex function on $(0, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1, p, q > 1$ and $0 < sA^p \leq B^q \leq tA^p$ for some constants $0 < s \leq t$. Then, there is a unitary operator U such that for all $x \in \mathcal{H}$*

$$\begin{aligned} g\left(K(w, \alpha)(A^p\#_{\frac{1}{q}}B^q)\right) &\leq U(g(A^p)\#_{\frac{1}{q}}g(B^q))U^*, \\ \left\langle g\left(K(w, \alpha)(A^p\#_{\frac{1}{q}}B^q)\right)Ux, Ux \right\rangle &\leq \langle g(A^p)x, x \rangle^{\frac{1}{p}} \langle g(B^q)x, x \rangle^{\frac{1}{q}}. \end{aligned}$$

Proof. The proof is similar to that of Theorem 6 by applying Proposition 2. \square

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