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SHARP INEQUALITIES FOR HERMITIAN TOEPLITZ DETERMINANTS FOR STRONGLY STARLIKE AND STRONGLY CONVEX FUNCTIONS

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Abstract. Sharp upper and lower bounds are found of the second and third order Hermitian Toeplitz determinants for the classes of strongly starlike and strongly convex functions of order α ($\alpha \in [0,1)$).

1. Introduction

Let $\mathscr{H}$ be the class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathscr{A}$ be its subclass of functions f of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 = 1,\ z \in \mathbb{D}. \tag{1}$$

Let $\mathscr{S}$ be the subclass of $\mathscr{A}$ of all univalent functions. Given $\alpha \in (0,1]$, let $\mathscr{S}^*_\alpha$ and $\mathscr{S}^c_\alpha$ denote the subclasses of $\mathscr{A}$ of all functions f satisfying

$$\left|\operatorname{Arg}\frac{zf'(z)}{f(z)}\right| \leqslant \alpha\frac{\pi}{2}, \quad z \in \mathbb{D}, \tag{2}$$

and

$$\left|\operatorname{Arg}\left(1+\frac{zf''(z)}{f'(z)}\right)\right| \leqslant \alpha\frac{\pi}{2}, \quad z \in \mathbb{D}, \tag{3}$$

respectively, and the so-called strongly starlike and strongly convex of order α. If $\alpha = 1$, then (2) defines the class of starlike functions denoted by $\mathscr{S}^*$ ([1],[17]), and (3) specifies the class of convex functions denoted by $\mathscr{S}^c$ ([19]).

The class of strongly starlike functions was introduced by Stankiewicz [20] and [21], and independently by Brannan and Kirwan [4] (see also [9, Vol. I, pp. 137-142]). Stankiewicz [21] presented an external geometrical characterization of strongly starlike functions. Brannan and Kirwan found a geometrical condition called δ-visibility which is sufficient for functions to be strongly starlike. In turn, Ma and Minda [15] gave the internal characterization of functions in $\mathscr{S}^*_\alpha$ basing on the concept of k-starlike

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domains. Further results regarding the geometry of strongly starlike functions were presented in [13, Chapter IV], [14] and [22].

Given $q,n \in \mathbb{N}$, the Hermitian Toeplitz matrix $T_{q,n}(f)$ of a function $f \in \mathscr{A}$ of the form (1) is defined by

$$T_{q,n}(f) := \begin{bmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ \overline{a}_{n+1} & a_n & \dots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{a}_{n+q-1} & \overline{a}_{n+q-2} & \dots & a_n \end{bmatrix},$$

where $\overline{a}_k := \overline{a_k}$. Let $\det T_{q,n}(f)$ denote the determinant of $T_{q,n}(f)$. In particular,

$$\det T_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ \overline{a}_2 & 1 & a_2 \\ \overline{a}_3 & \overline{a}_2 & 1 \end{vmatrix} = 1 + 2\operatorname{Re}\left(a_2^2\overline{a}_3\right) - 2|a_2|^2 - |a_3|^2. \tag{4}$$

In recent years many papers have been devoted to the estimation of determinants whose entries are coefficients of functions in the class $\mathscr{A}$ or its subclasses. Hankel matrices i.e., square matrices which have constant entries along the reverse diagonal (see e.g., [6], [7] and [11], with further references), and the symmetric Toeplitz determinant (see [2]) are of particular interest.

In [8] and [10], research was instigated into the study of Hermitian Toeplitz determinants which elements are the coefficients of univalent functions, noting that Hermitian Toeplitz matrices play an important role in functional analysis, applied mathematics as well as in physics and technical sciences.

In this paper we continue this research by finding the sharp upper and lower bounds of the second and third Hermitian Toeplitz determinants for the classes of strongly starlike and strongly convex functions of order α.

We note that the problem of finding sharp bounds for the modulus of the coefficients a_n of strongly starlike functions is far from easy, with sharp bounds known only when $n = 2$ and 3 [4], for $n = 4$ [16], and a partial solution in the case $n = 5$ [3]. On the other hand, a complete solution in the case $n = 5$ has recently been obtained in [12], when the coefficients a_n are real.

Let $\mathscr{P}$ be the class of all $p \in \mathscr{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{5}$$

which have positive real part.

In the proof of the main result we will use the following lemma which contains the Carathéodory result for c_1 [5], and the well-known formula for c_2 (e.g., [18, p. 166]).

LEMMA 1. *If $p \in \mathscr{P}$ is of the form (5), then*

$$c_1 = 2\zeta_1, \tag{6}$$

and

$$c_2 = 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2 \tag{7}$$

for some $\zeta_1, \zeta_2 \in \overline{\mathbb{D}}$.

We first recall the following observation ([10]). Given a compact subclass $\mathscr{F}$ of $\mathscr{A}$, let $A_2(\mathscr{F}) := \max\{|a_2| : f \in \mathscr{F}\}$. If $f \in \mathscr{A}$, then

$$\det T_{2,1}(f) = 1 - |a_2|^2,$$

and the result below is clear. Equality for the lower bound is attained by a function in $\mathscr{F}$ which is extremal for $A_2(\mathscr{F})$ and for the upper bound when f is the identity function.

THEOREM 1. *Let $\mathscr{F}$ be a compact subclass of $\mathscr{A}$. If the identity is an element of $\mathscr{F}$, then*

$$1 - A_2^2(\mathscr{F}) \leqslant \det T_{2,1}(f) \leqslant 1.$$

Both inequalities are sharp.

2. Strongly starlike functions of order α

We compute the sharp upper and lower bounds of $\det T_{2,1}(f)$ and $\det T_{3,1}(f)$ for functions in the class $\mathscr{S}_\alpha^*$.

Let $\alpha \in (0,1]$. Since $A_2(\mathscr{S}_\alpha^*) = 2\alpha$ ([4]) with the extremal function f satisfying

$$\frac{zf'(z)}{f(z)} = \left(\frac{1+z}{1-z}\right)^\alpha, \quad z \in \mathbb{D}, \tag{8}$$

and since the identity function belongs to the class $\mathscr{S}_\alpha^*$, by Theorem 1 we deduce

THEOREM 2. *Let $\alpha \in (0,1]$. If $f \in \mathscr{S}_\alpha^*$, then*

$$1 - 4\alpha^2 \leqslant \det T_{2,1}(f) \leqslant 1.$$

Both inequalities are sharp.

In particular, for $\alpha = 1$ i.e., for starlike functions we have the following [8]

COROLLARY 1. *If $f \in \mathscr{S}^*$, then*

$$-3 \leqslant \det T_{2,1}(f) \leqslant 1.$$

Both inequalities are sharp.

We next compute the upper and lower bounds of $\det T_{3,1}(f)$.

THEOREM 3. *Let $\alpha \in (0,1]$. If $f \in \mathscr{S}_\alpha^*$, then*

$$\det T_{3,1}(f) \leqslant \begin{cases} 1, & 0 < \alpha < \sqrt{8/15}, \\ (5\alpha^2 - 1)(3\alpha^2 - 1), & \sqrt{8/15} \leqslant \alpha \leqslant 1, \end{cases} \tag{9}$$

and

$$\det T_{3,1}(f) \geqslant \begin{cases} (5\alpha^2 - 1)(3\alpha^2 - 1), & 0 < \alpha < (\sqrt{241} - 1)/30, \\ -\dfrac{(4\alpha^2 + \alpha - 1)^2}{(5\alpha - 1)(3\alpha + 1)}, & (\sqrt{241} - 1)/30 \leqslant \alpha \leqslant 1. \end{cases} \tag{10}$$

All inequalities are sharp.

Proof. Fix $\alpha \in (0,1]$ and let $f \in \mathscr{S}^*_\alpha$ be of the form (1). Then by (2),

$$\frac{zf'(z)}{f(z)} = (p(z))^\alpha, \quad z \in \mathbb{D}, \tag{11}$$

for a certain $p \in \mathscr{P}$ of the form (5). Substituting the series (1) and (5) into (11), by equating coefficients we obtain

$$a_2 = \alpha c_1, \quad a_3 = \frac{\alpha}{2}\left(c_2 + \frac{3\alpha - 1}{2}c_1^2\right). \tag{12}$$

Since the class $\mathscr{S}^*_\alpha$ and $\det T_{3,1}(f)$ are rotationally invariant, by (6) we may assume that $c_1 \in [0,2]$, i.e., that $\zeta_1 \in [0,1]$. Furthermore, (4) with (12), (6) and (7) gives

$$\begin{aligned}\det T_{3,1}(f) =& 1 - 2\alpha^2 c_1^2 + \frac{1}{16}\alpha^2(3\alpha-1)(5\alpha+1)c_1^4 \\ &- \frac{1}{4}\alpha^2|c_2|^2 + \frac{1}{4}\alpha^2(1+\alpha)c_1^2\operatorname{Re}(c_2) \\ =& 15\alpha^4\zeta_1^4 - 8\alpha^2\zeta_1^2 + 1 - \alpha^2(1-\zeta_1^2)^2|\zeta_2|^2 + 2\alpha^3(1-\zeta_1^2)\zeta_1^2\operatorname{Re}(\zeta_2)\end{aligned} \tag{13}$$

for some $\zeta_1, \zeta_2 \in \overline{\mathbb{D}}$.

We now define

$$F(x,y,t) := 15\alpha^4x^2 - 8\alpha^2x + 1 + 2\alpha^3(1-x)xy\cos t - \alpha^2(1-x)^2y^2$$

for $x, y \in [0,1]$ and $t \in [0,2\pi]$.

If $\zeta_2 \neq 0$, then $\zeta_2 = |\zeta_2|e^{i\theta}$ for a unique $\theta \in [0,2\pi)$. Thus by (13),

$$\det T_{3,1}(f) = F(\zeta_1^2, |\zeta_2|, \theta). \tag{14}$$

If $\zeta_2 = 0$, then by (13),

$$\det T_{3,1}(f) = F(\zeta_1^2, 0, \theta) = F(\zeta_1^2, 0, 0). \tag{15}$$

We therefore find the maximum and minimum values of F.

A. Clearly

$$\begin{aligned}F(x,y,t) &\leqslant F(x,y,0) \\ &= 15\alpha^4x^2 - 8\alpha^2x + 1 + 2\alpha^3(1-x)xy - \alpha^2(1-x)^2y^2 \\ &=: G(x,y),\end{aligned} \tag{16}$$

for $x, y \in [0,1]$ and $t \in [0,2\pi]$.

(a) For $x = 1$,

$$G(1,y) = 15\alpha^4 - 8\alpha^2 + 1 = (5\alpha^2-1)(3\alpha^2-1), \quad y \in [0,1].$$

(b) Let $x \in [0,1)$. Set

$$y_w := \frac{\alpha x}{1-x}.$$

(c) Next note that $y_w > 1$ is valid if, and only if, $1/(\alpha+1) < x < 1$. Then

$$\begin{aligned}G(x,y) &\leqslant G(x,1) \\ &= \alpha^2(5\alpha+1)(3\alpha-1)x^2 + 2\alpha(\alpha-3)x + 1 - \alpha^2, \quad y \in [0,1].\end{aligned}$$

(c1) If $\alpha = 1/3$, then

$$G(x,1) = -\frac{16}{27}x + \frac{8}{9} \leqslant \frac{4}{9}, \quad x \in [3/4,1).$$

(c2) For $\alpha \in (0,1] \setminus \{1/3\}$, let

$$x_w := \frac{3-\alpha}{(5\alpha+1)(3\alpha-1)}.$$

(c3) Next suppose that $\alpha \in (0,1/3)$. Then $x_w < 0$ and therefore

$$G(x,1) \leqslant G\left(\frac{1}{\alpha+1},1\right) = \frac{(4\alpha^2-\alpha-1)^2}{(\alpha+1)^2}, \quad x \in [0,1). \tag{17}$$

(c4) Finally suppose that $\alpha \in (1/3,1]$. Then $x_w > 0$, and $x_w > 1$ if, and only if, $\alpha \in (0,\alpha_1)$, where $\alpha_1 := (1+\sqrt{241})/30 \approx 0.551$. Consequently, for $\alpha \in (1/3,\alpha_1)$, (17) holds.

Next we check when the condition $1/(\alpha+1) \leqslant x_w \leqslant 1$ holds. The inequality $x_w \leqslant 1$ is true if, and only if, $\alpha \in [\alpha_1,1]$. In turn, the inequality

$$\frac{3-\alpha}{(5\alpha+1)(3\alpha-1)} \geqslant \frac{1}{\alpha+1}$$

which is equivalent to $4\alpha^2-\alpha-1 \leqslant 0$ holds if and only if $\alpha \in (1/3,\alpha_2]$, where $\alpha_2 := (1+\sqrt{17})/8 \approx 0.6404$. Therefore $1/(\alpha+1) \leqslant x_w \leqslant 1$ is true only when $\alpha \in [\alpha_1,\alpha_2]$. Thus for $x \in [0,1)$,

$$\begin{aligned} G(x,1) &\leqslant \max\left\{G\left(\frac{1}{\alpha+1},1\right), G(1,1)\right\} \\ &= \max\left\{\frac{(4\alpha^2-\alpha-1)^2}{(\alpha+1)^2}, (5\alpha^2-1)(3\alpha^2-1)\right\}. \end{aligned}$$

Since the inequality

$$\frac{(4\alpha^2-\alpha-1)^2}{(\alpha+1)^2} \geqslant (5\alpha^2-1)(3\alpha^2-1)$$

is equivalent to

$$15\alpha^3 + 30\alpha^2 - 9\alpha - 8 \leqslant 0,$$

which holds if, and only if, $\alpha \in (1/3,\alpha_3]$, where $\alpha_3 \approx 0.585$, we obtain for $x \in [0,1)$,

$$G(x,1) \leqslant \begin{cases} \dfrac{(4\alpha^2-\alpha-1)^2}{(\alpha+1)^2}, & \alpha \in [\alpha_1,\alpha_3], \\ (5\alpha^2-1)(3\alpha^2-1), & \alpha \in (\alpha_3,\alpha_2]. \end{cases}$$

For $\alpha \in (\alpha_2,1]$ we have $x_w < 1/(\alpha+1)$, and it follows that

$$G(x,1) \leqslant G(1,1) = (5\alpha^2-1)(3\alpha^2-1), \quad x \in [0,1).$$

(d) It remains to consider the case $y_w \leqslant 1$ which holds only when $0 \leqslant x \leqslant 1/(\alpha+1)$. Thus

$$G(x,y) \leqslant G(x,y_w) = (4\alpha^2 x - 1)^2 \leqslant 1, \quad y \in [0,1],$$

since, as easy to check the last inequality, i.e., the inequality $|4\alpha^2 x-1| \leqslant 1$ is true for $0 \leqslant x \leqslant 1/(\alpha+1)$.

B. Clearly

$$\begin{aligned} F(x,y,t) &\geqslant F(x,y,\pi) \\ &= 15\alpha^4x^2 - 8\alpha^2x + 1 - 2\alpha^3(1-x)xy - \alpha^2(1-x)^2y^2 \\ &=: H(x,y), \end{aligned} \tag{18}$$

for $x,y \in [0,1]$ and $t \in [0,2\pi]$.

(a) For $x=1$,

$$H(1,y) = 15\alpha^4 - 8\alpha^2 + 1 = (5\alpha^2-1)(3\alpha^2-1), \quad y \in [0,1].$$

(b) Let $x \in [0,1)$ and set

$$y'_w := -\frac{\alpha x}{1-x}.$$

(c) Since $y'_w < 0$,

$$\begin{aligned} H(x,y) &\geqslant H(x,1) \\ &= \alpha(5\alpha-1)(3\alpha+1)x^2 - 2\alpha^2(\alpha+3)x + 1 - \alpha^2, \quad x \in [0,1). \end{aligned}$$

(c1) When $\alpha = 1/5$,

$$H(x,1) = -\frac{32}{125}x + \frac{4}{25} \geqslant \frac{88}{125}, \quad x \in [0,1).$$

(c2) For $\alpha \in (0,1] \setminus \{1/5\}$, let

$$x'_w := \frac{\alpha+3}{(5\alpha-1)(3\alpha+1)}.$$

(c3) Suppose that $\alpha \in (0,1/5)$. Then $x'_w < 0$ and therefore

$$H(x,1) \geqslant H(1,1) = (5\alpha^2-1)(3\alpha^2-1), \quad x \in [0,1). \tag{19}$$

(c4) Suppose that $\alpha \in (1/5,1]$. Then $x'_w > 0$, and $x'_w > 1$ if and only if $15\alpha^2 + \alpha - 4 < 0$, i.e., for $\alpha \in (1/5,\alpha_4)$, where $\alpha_4 := (\sqrt{241}-1)/30 \approx 0.484$. Consequently, for $\alpha \in (1/5,\alpha_4)$, (19) holds.

For $\alpha \in [\alpha_4,1]$, we have $0 < x'_w \leqslant 1$, and therefore

$$H(x,1) \geqslant H(x'_w,1) = -\frac{(4\alpha^2+\alpha-1)^2}{(5\alpha-1)(3\alpha+1)}.$$

C. Note that for $\zeta_2 = 0$, by (15),

$$\det T_{3,1}(f) = F(\zeta_1^2,0,0) = G(\zeta_1^2,0) = H(\zeta_1^2,0). \tag{20}$$

Thus summarizing we obtain (9) from (14), (20) and (16), and from (14), (20) and (18) we obtain inequality (10), which establish Theorem 3.

D. It therefore remains to show that the inequalities are sharp. Clearly, the identity function is extremal for the first inequality in (9). The function f given by (8) for which $a_2 = 2\alpha$ and $a_3 = 3\alpha^2$ is extremal for the second inequality in (9) and for the first one in (10). Let now $(\sqrt{241}-1)/30 \leqslant \alpha \leqslant 1$. Set

$$\tau := \sqrt{\frac{\alpha+3}{(5\alpha-1)(3\alpha+1)}}.$$

Since $\tau \leqslant 1$, the function

$$\tilde{p}(z) := \frac{1-z^2}{1-2\tau z+z^2} = 1+2\tau z+(4\tau^2-2)z^2+\cdots, \quad z \in \mathbb{D},$$

belongs to $\mathscr{P}$. Thus the function f given by (11), where p is replaced by $\tilde{p}$, being of the form (1) with

$$a_2 = 2\alpha\tau, \quad a_3 = \alpha\left((3\alpha+1)\tau^2-1\right),$$

belongs to $\mathscr{S}^*_\alpha$ and is extremal for the second inequality in (10). Thus the proof of Theorem 3 is complete. □

In particular, for $\alpha = 1$ i.e., for starlike functions we deduce the following [8]

COROLLARY 2. *If $f \in \mathscr{S}^*$, then*

$$-1 \leqslant \det T_{3,1}(f) \leqslant 8.$$

Both inequalities are sharp.

3. Convex functions of order α

We compute the upper and lower sharp bounds for $\det T_{2,1}(f)$ and $\det T_{3,1}(f)$ in the class of strongly convex functions of order α.

Let $\alpha \in (0,1]$. Since by (24) below, $A_2(\mathscr{S}^c_\alpha) = \alpha$ with extreme function f satisfying

$$1+\frac{zf''(z)}{f'(z)} = \left(\frac{1+z}{1-z}\right)^\alpha, \quad z \in \mathbb{D}, \tag{21}$$

and since the identity function belongs to the class $\mathscr{S}^c_\alpha$, Theorem 1 gives

THEOREM 4. *Let $\alpha \in (0,1]$. If $f \in \mathscr{S}^c_\alpha$, then*

$$1-\alpha^2 \leqslant \det T_{2,1}(f) \leqslant 1.$$

Both inequalities are sharp.

In particular, for $\alpha = 0$ i.e., for convex functions we deduce [8]

COROLLARY 3. *If $f \in \mathscr{S}^c$, then*

$$0 \leqslant \det T_{2,1}(f) \leqslant 1.$$

Both inequalities are sharp.

We next consider $\det T_{3,1}(f)$ and prove the following

THEOREM 5. *Let* $\alpha \in (0,1]$. *If* $f \in \mathscr{S}^c_\alpha$, *then*

$$(1-\alpha^2)^2 \leqslant \det T_{3,1}(f) \leqslant 1. \tag{22}$$

Both inequalities are sharp.

Proof. Fix $\alpha \in (0,1]$ and let $f \in \mathscr{S}^c(\alpha)$ be of the form (1). Then by (3),

$$(1-\alpha)f'(z) + zf''(z) = (1-\alpha)p(z)f'(z), \quad z \in \mathbb{D}, \tag{23}$$

for a certain $p \in \mathscr{P}$ of the form (5). Substituting (1) and (5) into (23), and equating coefficients we obtain

$$a_2 = \frac{1}{2}\alpha c_1, \quad a_3 = \frac{1}{6}\alpha\left(c_2 + \frac{3\alpha-1}{2}c_1^2\right). \tag{24}$$

Since the class $\mathscr{S}^c_\alpha$ and $\det T_{3,1}(f)$ are rotationally invariant, we may assume that $c := c_1 \in [0,2]$, i.e., that $\zeta_1 \in [0,1]$. Furthermore, (4) with (24), (6) and (7) gives

$$\begin{aligned}
\det T_{3,1}(f) =& \frac{1}{144}\left(144 - 72\alpha^2 c^2 + \alpha^2(9\alpha^2-1)c^4 - 4\alpha^2|c_2|^2 + 4\alpha^2 c^2 \operatorname{Re}(c_2)\right)\\
=& 1 - 2\alpha^2\zeta_1^2 + \alpha^4\zeta_1^4 - \frac{\alpha^2}{9}(1-\zeta_1^2)^2|\zeta_2|^2\\
=& (1-\alpha^2\zeta_1^2)^2 - \frac{\alpha^2}{9}(1-\zeta_1^2)^2|\zeta_2|^2\\
\leqslant& (1-\alpha^2\zeta_1^2)^2 \leqslant 1,
\end{aligned}$$

which gives the upper bound in (22).

To prove the lower bound in (22) we observe that

$$\begin{aligned}
\det T_{3,1}(f) &= (1-\alpha^2\zeta_1^2)^2 - \frac{\alpha^2}{9}(1-\zeta_1^2)^2|\zeta_2|^2\\
&\geqslant (1-\alpha^2\zeta_1^2)^2 - \frac{\alpha^2}{9}(1-\zeta_1^2)^2 \geqslant (1-\alpha^2)^2.
\end{aligned} \tag{25}$$

Now note that the last inequality, written as

$$(1-\alpha^2\zeta_1^2)^2 - (1-\alpha^2)^2 \geqslant \frac{\alpha^2}{9}(1-\zeta_1^2)^2,$$

is equivalent to

$$\alpha^2(1-\zeta_1^2)\left(17 - 9\alpha^2 - (9\alpha^2-1)\zeta_1^2\right) \geqslant 0,$$

which is true since

$$17 - 9\alpha^2 \geqslant (9\alpha^2-1)\zeta_1^2.$$

The above inequality is clearly true for $\alpha \in (0,1/3]$, and it is also true for $\alpha \in (1/3,1]$ since

$$17 - 9\alpha^2 \geqslant 9\alpha^2 - 1 \geqslant (9\alpha^2-1)\zeta_1^2.$$

Thus (25), i.e., the left inequality in (22) is established.

Equality for the upper bound in (22) holds for the identity, and for the lower bound for the function f given by (21) for which $a_2 = \alpha$ and $a_3 = \alpha^2$. $\square$

In paricular, for the class of convex functions we deduce [8]

COROLLARY 4. *If $f \in \mathscr{S}^c$, then*

$$0 \leqslant \det T_{3,1}(f) \leqslant 1.$$

Both inequalities are sharp.

We end by noting that Theorems 4 and 5 suggest the following conjecture.

CONJECTURE 1. *Let $\alpha \in (0,1]$ and $q \in \mathbb{N} \setminus \{1\}$. If $f \in \mathscr{S}^c_\alpha$, then*

$$(1-\alpha^2)^{q-1} \leqslant \det T_{q,1}(f) \leqslant 1.$$

Both inequalities are sharp.

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