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MAPPING PROPERTIES OF MULTILINEAR FRACTIONAL MAXIMAL OPERATORS IN METRIC MEASURE SPACES

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Abstract. In this paper, we introduce two kinds of multilinear fractional maximal operators in metric measure spaces. We prove that these operators map product Morrey spaces to Morrey spaces, and map product Lebesgue spaces to the fractional Hajłasz spaces under certain restrictions on the underlying metric measure space. We also introduce a kind of discrete multilinear fractional maximal operator, which is constructed in terms of coverings and partitions of unities and has better regularity. With the aid of Poincaré inequality, we establish the Sobolev bounds for the above operators.

1. Introduction

Multilinear maximal operator and its fractional version are crucial tools in the multilinear Calderón-Zygmund and Potential theory (see [2, 16]). The primary purpose of this paper is to investigate the smoothing and mapping properties for the multilinear fractional maximal operators on Morrey spaces, fractional Hajłasz spaces and Sobolev spaces (called the Newtonian spaces) in metric measure spaces. It should be pointed out that various Sobolev type spaces on metric measure spaces play a central role in analysis on metric spaces (see [7, 8, 9, 14, 18, 22, 24, 26, 27] for example).

In what follows, we assume that $X=(X,d,\mu)$ is a metric measure space equipped with a metric d and a Borel regular outer measure μ, which satisfies $0<\mu(E)<\infty$ whenever U is nonempty, open and bounded. The measure μ is doubling, if there exists a fixed constant $c_1>0$, called the doubling constant, such that

$$\mu(B(x,2r))\leqslant c_1\mu(B(x,r)) \tag{1.1}$$

for every ball $B(x,r)=\{y\in X: d(y,x)<r\}$. Note that the doubling condition implies that

$$\frac{\mu(B(y,r))}{\mu(B(x,R))}\geqslant c_2\Big(\frac{r}{R}\Big)^Q$$

for every $r\in(0,R]$ and $y\in B(x,R)$. Here the constants c_2 and $Q>1$ depend only on c_1. Actually, we may take $Q=\log_2 c_1$.

We now introduce the multilinear fractional maximal operators.

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DEFINITION 1.1. (Multilinear fractional maximal operators). For $\alpha \geqslant 0$, m, $\kappa \geqslant 1$ and $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in L^1_{\mathrm{loc}}(X)$, the multilinear fractional maximal operators are defined by

$$\mathfrak{M}^{\kappa}_{\alpha}(\vec{f})(x) = \sup_{r>0} r^{\alpha} \prod_{j=1}^{m} \frac{1}{\mu(B(x,\kappa r))} \int_{B(x,r)} |f_j(y)| d\mu(y)$$

and

$$\widetilde{\mathfrak{M}}^{\kappa}_{\alpha}(\vec{f})(x) = \sup_{r>0} \mu(B(x,\kappa r))^{\alpha} \prod_{j=1}^{m} \frac{1}{\mu(B(x,\kappa r))} \int_{B(x,r)} |f_j(y)| d\mu(y).$$

When $m = 1$, we denote $\mathfrak{M}^{\kappa}_{\alpha} = \mathscr{M}^{\kappa}_{\alpha}$ and $\widetilde{\mathfrak{M}}^{\kappa}_{\alpha} = \widetilde{\mathscr{M}}^{\kappa}_{\alpha}$. When $\alpha = 0$ and $\kappa = 1$, the operator $\mathscr{M}^{\kappa}_{\alpha}$ or $\widetilde{\mathscr{M}}^{\kappa}_{\alpha}$ reduces to the usual Hardy-Littlewood maximal operator $\mathscr{M}$. By the Hardy-Littlewood maximal function theorem for the doubling measures (see [6]), we see that $\mathscr{M}$ is bounded on $L^p(X)$ for $1 < p \leqslant \infty$ and maps $L^1(X)$ to $L^{1,\infty}(X)$. These facts together with the similar arguments as in the proof of [10, Theorem 3.2] yield the following result.

THEOREM 1.1. *Let* $\kappa \geqslant 2$, $p > 1$, $0 \leqslant \alpha \leqslant 1/p$ *and* $q = p/(1-\alpha p)$. *Then*

$$\|\widetilde{\mathscr{M}}^{\kappa}_{\alpha} f\|_{L^q(X)} \lesssim_{\alpha,p} \|f\|_{L^p(X)}$$

for every $f \in L^p(X)$. *For* $\kappa = 1$, *the same result holds, under the additional restriction that* μ *is doubling.*

It was pointed out in [21] that Theorem 1.1 is note true, in general, if $1 \leqslant \kappa < 2$ and $\alpha = 0$. If we assume the measure μ satisfies a lower bound condition, that is, there exists a constant $c_3 > 0$ such that

$$\mu(B(x,r)) \geqslant c_3 r^Q \tag{1.2}$$

for all $x \in X$ and $r > 0$. Then it holds that

$$\mathfrak{M}^{\kappa}_{\alpha}(\vec{f})(x) \leqslant c_3^{-1} \widetilde{\mathfrak{M}}^{\kappa}_{\alpha/Q}(\vec{f})(x) \tag{1.3}$$

for all $x \in X$ and $\kappa \geqslant 1$. If we assume the measure μ satisfies a upper bound condition, that is, there exists a constant $c_4 > 0$ such that

$$\mu(B(x,r)) \leqslant c_4 r^Q \tag{1.4}$$

for all $x \in X$ and $r > 0$. Then we have

$$\widetilde{\mathfrak{M}}^{\kappa}_{\alpha}(\vec{f})(x) \leqslant c_4^{\alpha} \kappa^{Q} \mathfrak{M}^{\kappa}_{\alpha Q}(\vec{f})(x)$$

for all $x \in X$ and $\kappa \geqslant 1$.

Applying Theorem 1.1 and (1.3), we can get the following result immediately.

THEOREM 1.2. *Let* $\kappa \geqslant 2$ *and* $\vec{f} = (f_1,\ldots,f_m)$ *with each* $f_j \in L^{p_j}(X)$ *for* $p_j > 1$.

(i) *If* $0 \leqslant \alpha \leqslant \sum_{i=1}^m 1/p_i$ *and* $1/q = \sum_{i=1}^m 1/p_i - \alpha \leqslant 1$. *Then*

$$\|\widetilde{\mathfrak{M}}_\alpha^\kappa(\vec{f})\|_{L^q(X)} \lesssim_{\alpha,p_1,\ldots,p_m} \prod_{j=1}^m \|f_j\|_{L^{p_j}(X)}.$$

(ii) *Assume that the measure* μ *satisfies the lower bound condition* (1.2)*. If* $0 \leqslant \alpha \leqslant \sum_{i=1}^m Q/p_i$ *and* $1/q = \sum_{i=1}^m 1/p_i - \alpha/Q \leqslant 1$. *Then*

$$\|\mathfrak{M}_\alpha^\kappa(\vec{f})\|_{L^q(X)} \lesssim_{\alpha,c_3,Q,p_1,\ldots,p_m} \prod_{j=1}^m \|f_j\|_{L^{p_j}(X)}.$$

For $1 \leqslant \kappa < 2$, *the same result holds, under the additional restriction that* μ *is doubling.*

Observe that part (i) of Theorem 1.2 follows easily from Theorem 1.1 and the following inequality

$$\widetilde{\mathfrak{M}}_\alpha^\kappa(\vec{f})(x) \leqslant \prod_{j=1}^m \widetilde{\mathscr{M}}_{\alpha_j}^\kappa f_j(x)$$

for all $x \in X$ and $\kappa \geqslant 1$, where $\alpha = \sum_{j=1}^m \alpha_j$ with each $\alpha_j \geqslant 0$. Particularly, when μ is doubling, the case $1 \leqslant \kappa < 2$ follows from the results for the case $\kappa = 2$ and the following inequality

$$\widetilde{\mathfrak{M}}_\alpha^\kappa(\vec{f})(x) \leqslant \widetilde{\mathfrak{M}}_\alpha^1(\vec{f})(x) \leqslant c_1^m \widetilde{\mathfrak{M}}_\alpha^2(\vec{f})(x)$$

for all $x \in X$ and $1 \leqslant \kappa < 2$. Part (ii) of Theorem 1.2 follows from part (i) of Theorem 1.2 and (1.3).

The rest of this paper is organized as follows. Section 2 is devoted to proving the boundedness of the multilinear fractional maximal operators on Morrey spaces. In Section 3, we study the regularity properties of the multilinear fractional maximal operators. We shall prove that the multilinear fractional maximal function of a vector-valued function $\vec{f} = (f_1,\ldots,f_m) \in L^{p_1}(X) \times \cdots \times L^{p_m}(X)$ has a generalized gradient under an annular decay property. This result can be viewed a generalization of the main result of [17] to the metric setting and a multilinear case of [12, Theorem 4.2]. In Section 4 we introduce the discrete multilinear fractional maximal operator, which has better regularity. In Section 5, we show that the discrete multilinear fractional maximal operator maps the product Sobolev spaces to Sobolev spaces, which is a generalization of the main result of [17] to the metric setting and a multilinear case of [10, Theorems 6.1 and 6.3].

Throughout this paper, if there exists a constant $c > 0$ depending only on ϑ such that $A \leqslant cB$, we then write $A \lesssim_\vartheta B$ or $B \gtrsim_\vartheta A$; and if $A \lesssim_\vartheta B \lesssim_\vartheta A$, we then write $A \sim_\vartheta B$.

2. Boundedness on Morrey spaces

In this section we study the action of the multilinear fractional maximal operators on product Morrey spaces. Let us recall the definition of Morrey spaces.

DEFINITION 2.1. (Morrey spaces). Let $1 \leqslant p < \infty$, $\beta \in \mathbb{R}$ and $\kappa \geqslant 1$. A locally integrable function f belongs to the Morrey space $L^{p,\beta,\kappa}(X)$, if

$$\|f\|_{L^{p,\beta,\kappa}(X)} = \sup_{x\in X, r>0} r^{-\beta}\Big(\frac{1}{\mu(B(x,\kappa r))}\int_{B(x,r)} |f(y)|^p d\mu(y)\Big)^{1/p} < \infty.$$

See [19] for the properties. Another way to define the Morrey spaces is the following

$$\widetilde{L}^{p,\beta,\kappa}(X) := \{f \in L^1_{\text{loc}}(X) : \|f\|_{\widetilde{L}^{p,\beta,\kappa}(X)} < \infty\},$$

where

$$\|f\|_{\widetilde{L}^{p,\beta,\kappa}(X)} = \sup_{x\in X, r>0} \mu(B(x,\kappa r))^{-\beta}\Big(\frac{1}{\mu(B(x,\kappa r))}\int_{B(x,r)} |f(y)|^p d\mu(y)\Big)^{1/p}.$$

Chiarenza and Frasca [5] firstly established the boundedness of the usual Hardy-Littlewood maximal operator on Morrey spaces in the Euclidean setting. Later on, the above result was extended to nondoubling metric spaces setting in [19]. Recently, Heikkinen et al. [10] extended the result of [19] to the fractional case.

THEOREM 2.1. ([10], [19]). *Let* $0 \leqslant \alpha < -\beta$, $1 < p < \infty$ *and* $q = p\beta/(\alpha+\beta)$. *Then*

$$\|\mathscr{M}^2_\alpha f\|_{L^{q,\alpha+\beta,4}(X)} \lesssim_{\alpha,\beta,p} \|f\|_{L^{p,\beta,2}(X)},$$

$$\|\widetilde{\mathscr{M}}^2_\alpha f\|_{\widetilde{L}^{q,\alpha+\beta,4}(X)} \lesssim_{\alpha,\beta,p} \|f\|_{\widetilde{L}^{p,\beta,2}(X)}.$$

In this section we shall extend Theorem 2.1 to the multilinear case.

THEOREM 2.2. *Let* $\kappa \geqslant 2$, $\alpha = \sum_{i=1}^m \alpha_i$ *and* $\beta = \sum_{i=1}^m \beta_i$ *with each* $0 \leqslant \alpha_i < -\beta_i$ *and* $1/p = \sum_{i=1}^m (\beta_i+\alpha_i)/(p_i\beta_i)$ *and* $p_i > 1$. *Then*

$$\|\mathfrak{M}^\kappa_\alpha(\vec{f})\|_{L^{p,\alpha+\beta,4}(X)} \lesssim_{\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_m,p_1,\dots,p_m} \prod_{j=1}^m \|f_j\|_{L^{p_j,\beta_j,2}(X)}, \tag{2.1}$$

$$\|\widetilde{\mathfrak{M}}^\kappa_\alpha(\vec{f})\|_{\widetilde{L}^{p,\alpha+\beta,4}(X)} \lesssim_{\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_m,p_1,\dots,p_m} \prod_{j=1}^m \|f_j\|_{\widetilde{L}^{p_j,\beta_j,2}(X)}. \tag{2.2}$$

For $1 \leqslant \kappa < 2$, *the same results hold, under the additional restriction that* μ *is doubling.*

Proof. We only prove (2.1) and (2.2) is analogous. One can easily check that

$$\mathfrak{M}^2_\alpha(\vec{f})(x) \leqslant \prod_{j=1}^m \mathscr{M}^2_{\alpha_j} f_j(x) \tag{2.3}$$

for all $x \in X$, where $\alpha = \sum_{j=1}^m \alpha_j$ with each $\alpha_j \geqslant 0$. Fix $x \in X$ and $r > 0$. Inequality (2.3) together with Hölder's inequality yields that

$$\int_{B(x,r)} |\mathfrak{M}^2_\alpha(\vec{f})(y)|^p d\mu(y) \leqslant \prod_{j=1}^m \Big(\int_{B(x,r)} |\mathscr{M}^2_{\alpha_j} f_j(y)|^{pq_j}\Big)^{1/q_j}, \tag{2.4}$$

where $\sum_{j=1}^m 1/q_j = 1$ and $q_j > 1$. For $\gamma < 0$, we write $\gamma = \sum_{j=1}^m \gamma_j$ with $\gamma_j = \alpha_j + \beta_j$. (2.4) together with Theorem 2.1 implies that

$$\begin{aligned}
& r^{-\gamma}\Big(\frac{1}{\mu(B(x,4r))}\int_{B(x,r)} |\mathfrak{M}^2_\alpha(\vec{f})(y)|^p d\mu(y)\Big)^{1/p} \\
&\leqslant \prod_{j=1}^m r^{-\gamma_j}\Big(\frac{1}{\mu(B(x,4r))}\int_{B(x,r)} |\mathscr{M}^2_{\alpha_j} f_j(y)|^{pq_j} d\mu(y)\Big)^{1/(pq_j)} \\
&\leqslant \prod_{j=1}^m \|\mathscr{M}^2_{\alpha_j} f_j\|_{L^{pq_j,\gamma_j,4}(X)} \\
&\lesssim_{\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_m,p_1,\dots,p_j} \prod_{j=1}^m \|f_j\|_{L^{p_j,\beta_j,2}(X)},
\end{aligned} \tag{2.5}$$

where we take $pq_j = p_j\beta_j/(\alpha_j+\beta_j)$. Then (2.1) with $\kappa = 2$ follows from (2.5). Obviously, $\mathfrak{M}^\kappa_\alpha(\vec{f})(x) \leqslant \mathfrak{M}^2_\alpha(\vec{f})(x)$ for all $\kappa \geqslant 2$ and $x \in X$. Thus (2.1) holds for $\kappa > 2$. When $1 \leqslant \kappa < 2$ and μ satisfies the doubling condition (1.1), we have

$$\mathfrak{M}^\kappa_\alpha(\vec{f})(x) \leqslant \mathfrak{M}^1_\alpha(\vec{f})(x) \leqslant c_1^m \mathfrak{M}^2_\alpha(\vec{f})(x)$$

for all $x \in X$. This yields (2.1) for $1 \leqslant \kappa < 2$. $\square$

REMARK 2.2. Theorem 2.2 extends Theorem 2.1, which corresponds to the case $m = 1$ and $\kappa = 2$.

3. Boundedness on fractional Hajłasz spaces

In this section we shall prove that the multilinear fractional maximal operators have certain smoothing properties. More precisely, we shall prove these operators are bounded from product Lebesgue spaces to certain fractional Hajłasz spaces.

DEFINITION 3.1. (Fractional Hajłasz spaces). Let $s \geqslant 0$. We say that a measurable function $g \geqslant 0$ is a s-Hajłasz gradient of a measurable function u, if there exists $E \subset X$ with $\mu(E) = 0$ such that

$$|u(x) - u(y)| \leqslant d(x,y)^s (g(x) + g(y)) \tag{3.1}$$

for all $x,y\in X\backslash E$. The collection of all s-Hajłasz gradients of u is denoted by $\mathscr{D}^s(u)$. Let $1\leqslant p<\infty$. A homogeneous Hajłasz space $\dot{M}^{s,p}(X)$ consists of measurable function u such that

$$\|u\|_{\dot{M}^{s,p}(X)}=\inf_{g\in\mathscr{D}^s(u)}\|g\|_{L^p(X)}<\infty.$$

The Sobolev space $M^{s,p}(X)$ is $\dot{M}^{s,p}(X)\cup L^p(X)$ equipped with the norm

$$\|u\|_{M^{s,p}(X)}=\Big(\|u\|^p_{L^p(X)}+\|u\|^p_{\dot{M}^{s,p}(X)}\Big)^{1/p}. \tag{3.2}$$

Note that $M^{s,p}(X)$ is a Banach space (see [8, Theorem 8.3]). These spaces were introduced by Hajłasz in [7] for $s=1$ and Yang [25] for $s>0$.

DEFINITION 3.2. (Annular decay properties). Let $0<\delta\leqslant 1$. We say that the metric measure space X satisfies the δ-annular decay property, if there exists a constant $c_5>0$ such that for all $x\in X$, $R>0$ and $0<h<R$, it holds that

$$\mu(B(x,R)\setminus B(x,R-h))\leqslant c_5\Big(\frac{h}{R}\Big)^{\delta}\mu(B(x,R)). \tag{3.3}$$

See for instance [4, 20] and [9, Chapter 9] for examples and for more information on these and related conditions.

We shall establish the following result.

THEOREM 3.1. *Let $\kappa\geqslant 1$, X satisfy the δ-annular decay property (3.3) and μ satisfy the doubling condition (1.1). Let $\vec{f}=(f_1,\dots,f_m)$ with each $f\in L^{p_j}(X)$ for $p_j>1$. Then*

(i) *Let $\delta\leqslant\alpha<mQ$ and $0<1/q_1=\sum_{i=1}^m 1/p_i-(\alpha-\delta)/Q\leqslant 1$. Then $2^{\delta}mc_1^{2m}(1+c_5)\mathfrak{M}^{\kappa}_{\alpha-\delta}(\vec{f})$ is a generalized δ-gradient of $\mathfrak{M}^{\kappa}_{\alpha}(\vec{f})$. Moreover, if μ satisfies the lower bound condition (1.2), then*

$$\|\mathfrak{M}^{\kappa}_{\alpha}(\vec{f})\|_{\dot{M}^{\delta,q_1}(X)}\lesssim_{\alpha,m,Q,c_1,c_3,c_5,p_1,\dots,p_m}\prod_{j=1}^m\|f_j\|_{L^{p_j}(X)}. \tag{3.4}$$

(ii) *Let $\beta\in(0,\delta/Q]$, $0\leqslant\alpha-\beta\leqslant\sum_{i=1}^m 1/p_i$ and $1/q_2=\sum_{i=1}^m 1/p_i-\alpha+\beta\leqslant 1$. Assume that μ satisfies the upper bound condition (1.4). Then $2^{\delta}\kappa^{Q\beta}mc_1^{2m}c_4^{\beta}(c_5+1)\widetilde{\mathfrak{M}}^{\kappa}_{\alpha-\beta}(\vec{f})$ is a generalized $Q\beta$-gradient of $\widetilde{\mathfrak{M}}^{\kappa}_{\alpha}(\vec{f})$. Moreover,*

$$\|\widetilde{\mathfrak{M}}^{\kappa}_{\alpha}(\vec{f})\|_{\dot{M}^{Q\beta,q_2}(X)}\lesssim_{\alpha,\beta,\kappa,m,Q,c_1,c_4,c_5,p_1,\dots,p_m}\prod_{j=1}^m\|f_j\|_{L^{p_j}(X)}. \tag{3.5}$$

Proof. We first prove (i). It suffices to show that

$$|\mathfrak{M}^{\kappa}_{\alpha}(\vec{f})(x)-\mathfrak{M}^{\kappa}_{\alpha}(\vec{f})(y)|\leqslant 2^{\delta}mc_1^{2m}(1+c_5)d(x,y)^{\delta}(\mathfrak{M}^{\kappa}_{\alpha-\delta}(\vec{f})(x)+\mathfrak{M}^{\kappa}_{\alpha-\delta}(\vec{f})(y)) \tag{3.6}$$

for all $x,y\in X$.

Fix $x,y\in X$, we first assume that $\mathfrak{M}_{\alpha}^{\kappa}(\vec{f})(x)\geqslant \mathfrak{M}_{\alpha}^{\kappa}(\vec{f})(y)$. Given $\varepsilon>0$, there exists $r=r(\varepsilon,x)>0$ such that

$$\mathfrak{M}_{\alpha}^{\kappa}(\vec{f})(x)\leqslant r^{\alpha}\prod_{l=1}^{m}\frac{1}{\mu(B(x,\kappa r))}\int_{B(x,r)}|f_l(z)|d\mu(z)+\varepsilon. \tag{3.7}$$

Fix $l\in\{1,2,\dots,m\}$. By (1.1) and the fact that $B(x,r)\subset B(y,r+d(x,y))$, it holds that

$$\begin{aligned}
&\frac{1}{\mu(B(x,\kappa r))}\int_{B(x,r)}|f_l(z)|d\mu(z)\\
&\leqslant \frac{\mu(B(y,\kappa(r+d(x,y))))}{\mu(B(x,\kappa r))}\frac{1}{\mu(B(y,\kappa(r+d(x,y))))}\int_{B(y,r+d(x,y))}|f_l(z)|d\mu(z)\\
&\leqslant \frac{\mu(B(x,(2\kappa+1)r))}{\mu(B(x,\kappa r))}\frac{1}{\mu(B(y,\kappa(r+d(x,y))))}\int_{B(y,r+d(x,y))}|f_l(z)|d\mu(z)\\
&\leqslant c_1^2\frac{1}{\mu(B(y,\kappa(r+d(x,y))))}\int_{B(y,r+d(x,y))}|f_l(z)|d\mu(z).
\end{aligned} \tag{3.8}$$

When $r\leqslant d(x,y)$, (3.7) together with (3.8) implies that

$$\begin{aligned}
&\mathfrak{M}_{\alpha}^{\kappa}(\vec{f})(x)-\mathfrak{M}_{\alpha}^{\kappa}(\vec{f})(y)\\
&\leqslant r^{\alpha}\prod_{l=1}^{m}\frac{1}{\mu(B(x,\kappa r))}\int_{B(x,r)}|f_l(z)|d\mu(z)+\varepsilon\\
&\leqslant c_1^{2m}(r+d(x,y))^{\alpha}\prod_{l=1}^{m}\frac{1}{\mu(B(y,\kappa(r+d(x,y))))}\int_{B(y,r+d(x,y))}|f_l(z)|d\mu(z)+\varepsilon\\
&\leqslant 2^{\delta}c_1^{2m}d(x,y)^{\delta}\mathfrak{M}_{\alpha-\delta}^{\kappa}(\vec{f})(y)+\varepsilon.
\end{aligned} \tag{3.9}$$

When $r>d(x,y)$. (3.7) gives that

$$\begin{aligned}
&\mathfrak{M}_{\alpha}^{\kappa}(\vec{f})(x)-\mathfrak{M}_{\alpha}^{\kappa}(\vec{f})(y)\\
&\leqslant r^{\alpha}\prod_{l=1}^{m}\frac{1}{\mu(B(x,\kappa r))}\int_{B(x,r)}|f_l(z)|d\mu(z)\\
&\quad-(r+d(x,y))^{\alpha}\prod_{l=1}^{m}\frac{1}{\mu(B(y,\kappa(r+d(x,y))))}\int_{B(y,r+d(x,y))}|f_l(z)|d\mu(z)+\varepsilon\\
&\leqslant \sum_{l=1}^{m}\prod_{\mu=1}^{l-1}\Big(\frac{1}{\mu(B(x,\kappa r))}\int_{B(x,r)}|f_{\mu}(z)|d\mu(z)\Big)\\
&\quad\times\prod_{\nu=l+1}^{m}\Big(\frac{1}{\mu(B(x,\kappa(r+d(x,y))))}\int_{B(y,r+d(x,y))}|f_{\nu}(z)|d\mu(z)\Big)\\
&\quad\times\Big(\frac{r^{\alpha}}{\mu(B(x,\kappa r))}\int_{B(x,r)}|f_l(z)|d\mu(z)\\
&\quad-\tfrac{(r+d(x,y))^{\alpha}}{\mu(B(y,\kappa(r+d(x,y))))}\textstyle\int_{B(y,r+d(x,y))}|f_l(z)|d\mu(z)\Big)+\varepsilon.
\end{aligned} \tag{3.10}$$

Fix $1 \leqslant l \leqslant m$. By the doubling property and the δ-annular decay property, one has

$$\begin{aligned}
&\frac{1}{\mu(B(x,\kappa r))}-\frac{1}{\mu(B(y,\kappa(r+d(x,y))))}\\
&\leqslant \Big(\frac{\mu(B(y,\kappa(r+d(x,y)))\backslash B(x,\kappa r))}{\mu(B(x,\kappa r))}\Big)\frac{1}{\mu(B(y,\kappa(r+d(x,y))))}\\
&\leqslant \Big(\frac{\mu(B(x,\kappa r+(\kappa+1)d(x,y))\backslash B(x,\kappa r))}{\mu(B(x,\kappa r+(\kappa+1)d(x,y)))}\Big)\frac{\mu(B(x,\kappa r+(\kappa+1)d(x,y)))}{\mu(B(x,\kappa r))}\\
&\quad\times\frac{1}{\mu(B(y,\kappa(r+d(x,y))))}\\
&\leqslant c_5\Big(\frac{(\kappa+1)d(x,y)}{\kappa r+(\kappa+1)d(x,y)}\Big)^{\delta}\frac{\mu(B(x,4\kappa r))}{\mu(B(x,\kappa r))}\frac{1}{\mu(B(y,\kappa(r+d(x,y))))}\\
&\leqslant 2^{\delta}c_1^2c_5\Big(\frac{d(x,y)}{r+d(x,y)}\Big)^{\delta}\frac{1}{\mu(B(y,\kappa(r+d(x,y))))}.
\end{aligned}\tag{3.11}$$

It follows that

$$\begin{aligned}
&\frac{r^{\alpha}}{\mu(B(x,\kappa r))}\int_{B(x,r)}|f_l(z)|d\mu(z)-\frac{(r+d(x,y))^{\alpha}}{\mu(B(y,\kappa(r+d(x,y))))}\int_{B(y,r+d(x,y))}|f_l(z)|d\mu(z)\\
&\leqslant (r+d(x,y))^{\alpha}\Big(\frac{1}{\mu(B(x,\kappa r))}-\frac{1}{\mu(B(y,\kappa(r+d(x,y))))}\Big)\int_{B(y,r+d(x,y))}|f_l(z)|d\mu(z)\\
&\leqslant 2^{\delta}c_1^2c_5d(x,y)^{\delta}(r+d(x,y))^{\alpha-\delta}\frac{1}{\mu(B(y,\kappa(r+d(x,y))))}\int_{B(y,r+d(x,y))}|f_l(z)|d\mu(z).
\end{aligned}$$

This together with (3.8) and (3.10) implies that

$$\begin{aligned}
&\mathfrak{M}_{\alpha}^{\kappa}(\vec{f})(x)-\mathfrak{M}_{\alpha}^{\kappa}(\vec{f})(y)\\
&\leqslant 2^{\delta}c_1^2c_5d(x,y)^{\delta}(r+d(x,y))^{\alpha-\delta}\\
&\quad\times\sum_{l=1}^{m}\prod_{\mu=1}^{l-1}\Big(\frac{c_1^2}{\mu(B(y,\kappa(r+d(x,y))))}\int_{B(y,r+d(x,y))}|f_{\mu}(z)|d\mu(z)\Big)\\
&\quad\times\prod_{\nu=l+1}^{m}\Big(\frac{1}{\mu(B(x,\kappa(r+d(x,y))))}\int_{B(y,r+d(x,y))}|f_{\nu}(z)|d\mu(z)\Big)\\
&\quad\times\frac{1}{\mu(B(y,\kappa(r+d(x,y))))}\int_{B(y,r+d(x,y))}|f_l(z)|d\mu(z)+\varepsilon\\
&\leqslant 2^{\delta}mc_1^{2m}c_5d(x,y)^{\delta}\mathfrak{M}_{\alpha-\delta}^{\kappa}(\vec{f})(y)+\varepsilon,
\end{aligned}\tag{3.12}$$

Then (3.12) and (3.9) yield that

$$\mathfrak{M}_{\alpha}^{\kappa}(\vec{f})(x)-\mathfrak{M}_{\alpha}^{\kappa}(\vec{f})(y)\leqslant 2^{\delta}mc_1^{2m}(1+c_5)d(x,y)^{\delta}\mathfrak{M}_{\alpha-\delta}^{\kappa}(\vec{f})(y)+\varepsilon\tag{3.13}$$

if $\mathfrak{M}_{\alpha}^{\kappa}(\vec{f})(x)\geqslant\mathfrak{M}_{\alpha}^{\kappa}(\vec{f})(y)$. Similarly, we can get

$$\mathfrak{M}_{\alpha}^{\kappa}(\vec{f})(y)-\mathfrak{M}_{\alpha}^{\kappa}(\vec{f})(x)\leqslant 2^{\delta}mc_1^{2m}(1+c_5)d(x,y)^{\delta}\mathfrak{M}_{\alpha-\delta}^{\kappa}(\vec{f})(x)+\varepsilon\tag{3.14}$$

if $\mathfrak{M}_{\alpha}^{\kappa}(\vec{f})(x)<\mathfrak{M}_{\alpha}^{\kappa}(\vec{f})(y)$. If follows from (3.13) and (3.14) that

$$|\mathfrak{M}_{\alpha}^{\kappa}(\vec{f})(x)-\mathfrak{M}_{\alpha}^{\kappa}(\vec{f})(y)|\leqslant 2^{\delta}mc_1^{2m}(1+c_5)d(x,y)^{\delta}(\mathfrak{M}_{\alpha-\delta}^{\kappa}(\vec{f})(x)+\mathfrak{M}_{\alpha-\delta}^{\kappa}(\vec{f})(y))+\varepsilon.\tag{3.15}$$

Letting $\varepsilon \to 0$, inequality (3.15) leads to (3.6). From (3.6) we see that $2^{\delta} m c_1^{2m}(1+c_5)\mathfrak{M}^{\kappa}_{\alpha-\delta}(\vec{f})$ is a generalized δ-gradient of $\mathfrak{M}^{\kappa}_{\alpha}(\vec{f})$. This together with (ii) of Theorem 1.2 yields (3.4).

We now prove (ii). The proof is analogous to (i). Let $x, y \in X$ and $0 < \beta \leqslant \delta/Q$. We want to show that

$$\begin{aligned}
&|\widetilde{\mathfrak{M}}^{\kappa}_{\alpha}(\vec{f})(x) - \widetilde{\mathfrak{M}}^{\kappa}_{\alpha}(\vec{f})(y)| \\
&\leqslant 2^{\delta} \kappa^{Q\beta} m c_1^{2m} c_4^{\beta}(c_5+1) d(x,y)^{Q\beta} (\widetilde{\mathfrak{M}}^{\kappa}_{\alpha-\beta}(\vec{f})(x) + \widetilde{\mathfrak{M}}^{\kappa}_{\alpha-\beta}(\vec{f})(y)).
\end{aligned} \tag{3.16}$$

Assume that $\widetilde{\mathfrak{M}}^{\kappa}_{\alpha}(\vec{f})(x) \geqslant \widetilde{\mathfrak{M}}^{\kappa}_{\alpha}(\vec{f})(y)$. Given $\varepsilon > 0$, there exists $r = r(\varepsilon, x) > 0$ such that

$$\widetilde{\mathfrak{M}}^{\kappa}_{\alpha}(\vec{f})(x) \leqslant (\mu(B(x,\kappa r)))^{\alpha-m} \prod_{l=1}^{m} \int_{B(x,r)} |f_l(z)| d\mu(z) + \varepsilon. \tag{3.17}$$

When $r > d(x,y)$. We get easily from (3.17) that

$$\begin{aligned}
&\widetilde{\mathfrak{M}}^{\kappa}_{\alpha}(\vec{f})(x) - \widetilde{\mathfrak{M}}^{\kappa}_{\alpha}(\vec{f})(y) \\
&\leqslant (\mu(B(x,\kappa r)))^{\alpha-m} \prod_{l=1}^{m} \int_{B(x,r)} |f_l(z)| d\mu(z) \\
&\quad -(\mu(B(y,\kappa(r+d(x,y)))))^{\alpha-m} \prod_{l=1}^{m} \int_{B(y,r+d(x,y))} |f_l(z)| d\mu(z) + \varepsilon \\
&\leqslant \sum_{l=1}^{m} \prod_{\mu=1}^{l-1} \Big(\frac{1}{\mu(B(x,\kappa r))} \int_{B(x,r)} |f_\mu(z)| d\mu(z) \Big) \\
&\quad \times \prod_{\nu=l+1}^{m} \Big(\frac{1}{\mu(B(x,\kappa(r+d(x,y))))} \int_{B(y,r+d(x,y))} |f_\nu(z)| d\mu(z) \Big) \\
&\quad \times \Big((\mu(B(x,\kappa r)))^{\alpha-1} \int_{B(x,r)} |f_l(z)| d\mu(z) \\
&\quad -(\mu(B(y,\kappa(r+d(x,y)))))^{\alpha-1} \int_{B(y,r+d(x,y))} |f_l(z)| d\mu(z) \Big) + \varepsilon.
\end{aligned} \tag{3.18}$$

Fix $l \in \{1, 2, \ldots, m\}$. (3.11) leads to

$$\begin{aligned}
&(\mu(B(x,\kappa r)))^{\alpha-1} \int_{B(x,r)} |f_l(z)| d\mu(z) - (\mu(B(y,\kappa(r+d(x,y)))))^{\alpha-1} \\
&\quad \times \int_{B(y,r+d(x,y))} |f_l(z)| d\mu(z) \\
&\leqslant (\mu(B(y,\kappa(r+d(x,y)))))^{\alpha} \Big(\frac{1}{\mu(B(x,\kappa r))} - \frac{1}{\mu(B(y,\kappa(r+d(x,y))))} \Big) \\
&\quad \times \int_{B(y,r+d(x,y))} f_l(z) d\mu(z) \\
&\leqslant 2^{\delta} c_1^2 c_5 \Big(\frac{d(x,y)}{(r+d(x,y))} \Big)^{\delta} (\mu(B(y,\kappa(r+d(x,y)))))^{\alpha-1} \int_{B(y,r+d(x,y))} |f_l(z)| d\mu(z).
\end{aligned} \tag{3.19}$$

It follows form (3.8), (3.18) and (3.19) that

$$\widetilde{\mathfrak{M}}_{\alpha}^{\kappa}(\vec{f})(x)-\widetilde{\mathfrak{M}}_{\alpha}^{\kappa}(\vec{f})(y)\leqslant 2^{\delta}mc_1^{2m}c_5\Big(\frac{d(x,y)}{(r+d(x,y))}\Big)^{\delta}(\mu(B(y,\kappa(r+d(x,y)))))^{\alpha-m}\times\prod_{l=1}^{m}\int_{B(y,r+d(x,y))}|f_l(z)|d\mu(z)+\varepsilon. \tag{3.20}$$

Note that $\delta-Q\beta\geqslant 0$. Then by (1.4) we get

$$\begin{aligned}&\Big(\frac{d(x,y)}{(r+d(x,y))}\Big)^{\delta}(\mu(B(y,\kappa(r+d(x,y)))))^{\beta}\\&\leqslant d(x,y)^{Q\beta}d(x,y)^{\delta-Q\beta}\frac{(\mu(B(y,\kappa(r+d(x,y)))))^{\beta}}{(r+d(x,y))^{\delta}}\\&\leqslant c_4^{\beta}\kappa^{Q\beta}d(x,y)^{Q\beta}\Big(\frac{d(x,y)}{r+d(x,y)}\Big)^{\delta-Q\beta}\leqslant c_4^{\beta}\kappa^{Q\beta}d(x,y)^{Q\beta}.\end{aligned} \tag{3.21}$$

Then (3.20) and (3.21) may yield that

$$\widetilde{\mathfrak{M}}_{\alpha}^{\kappa}(\vec{f})(x)-\widetilde{\mathfrak{M}}_{\alpha}^{\kappa}(\vec{f})(y)\leqslant 2^{\delta}\kappa^{Q\beta}mc_1^{2m}c_4^{\beta}c_5d(x,y)^{Q\beta}\widetilde{\mathfrak{M}}_{\alpha-\beta}^{\kappa}(\vec{f})(y)+\varepsilon \tag{3.22}$$

when $r>d(x,y)$. If $r\leqslant d(x,y)$, if follows from (3.8), (1.4) and (3.17) that

$$\begin{aligned}&\widetilde{\mathfrak{M}}_{\alpha}^{\kappa}(\vec{f})(x)-\widetilde{\mathfrak{M}}_{\alpha}^{\kappa}(\vec{f})(y)\\&\leqslant(\mu(B(x,\kappa r)))^{\alpha}\prod_{l=1}^{m}\frac{1}{\mu(B(x,\kappa r))}\int_{B(x,r)}|f_l(z)|d\mu(z)+\varepsilon\\&\leqslant c_1^{2m}(\mu(B(x,\kappa(r+d(x,y)))))^{\alpha}\prod_{l=1}^{m}\frac{1}{\mu(B(y,\kappa(r+d(x,y))))}\int_{B(y,r+d(x,y))}|f_l(z)|d\mu(z)+\varepsilon\\&\leqslant c_1^{2m}c_4^{\beta}(\kappa(r+d(x,y)))^{Q\beta}(\mu(B(x,\kappa(r+d(x,y)))))^{\alpha-\beta}\\&\quad\times\prod_{l=1}^{m}\frac{1}{\mu(B(y,\kappa(r+d(x,y))))}\int_{B(y,r+d(x,y))}|f_l(z)|d\mu(z)+\varepsilon\\&\leqslant 2^{Q\beta}\kappa^{Q\beta}c_1^{2m}c_4^{\beta}d(x,y)^{Q\beta}\widetilde{\mathfrak{M}}_{\alpha-\beta}^{\kappa}(\vec{f})(y)+\varepsilon.\end{aligned}$$

This combines with (3.21) implies that

$$\widetilde{\mathfrak{M}}_{\alpha}^{\kappa}(\vec{f})(x)-\widetilde{\mathfrak{M}}_{\alpha}^{\kappa}(\vec{f})(y)\leqslant 2^{\delta}\kappa^{Q\beta}mc_1^{2m}c_4^{\beta}c_5d(x,y)^{Q\beta}\widetilde{\mathfrak{M}}_{\alpha-\beta}^{\kappa}(\vec{f})(y)+\varepsilon. \tag{3.23}$$

Similarly, when $\widetilde{\mathfrak{M}}_{\alpha}^{\kappa}(\vec{f})(x)<\widetilde{\mathfrak{M}}_{\alpha}^{\kappa}(\vec{f})(y)$, we can conclude that

$$\widetilde{\mathfrak{M}}_{\alpha}^{\kappa}(\vec{f})(y)-\widetilde{\mathfrak{M}}_{\alpha}^{\kappa}(\vec{f})(x)\leqslant 2^{\delta}\kappa^{Q\beta}mc_1^{2m}c_4^{\beta}(c_5+1)d(x,y)^{Q\beta}\widetilde{\mathfrak{M}}_{\alpha-\beta}^{\kappa}(\vec{f})(x)+\varepsilon. \tag{3.24}$$

We get from (3.23) and (3.24) that

$$\begin{aligned}&|\widetilde{\mathfrak{M}}_{\alpha}^{\kappa}(\vec{f})(x)-\widetilde{\mathfrak{M}}_{\alpha}^{\kappa}(\vec{f})(y)|\\&\leqslant 2^{\delta}\kappa^{Q\beta}mc_1^{2m}c_4^{\beta}(c_5+1)d(x,y)^{Q\beta}(\widetilde{\mathfrak{M}}_{\alpha-\beta}^{\kappa}(\vec{f})(x)+\widetilde{\mathfrak{M}}_{\alpha-\beta}^{\kappa}(\vec{f})(y))+\varepsilon.\end{aligned} \tag{3.25}$$

Letting $\varepsilon\to 0$, (3.25) leads to (3.16). Hence, the function $2^{\delta}\kappa^{Q\beta}mc_1^{2m}c_4^{\beta}(c_5+1)\widetilde{\mathfrak{M}}_{\alpha-\beta}^{\kappa}(\vec{f})$ is a generalized $Q\beta$-gradient of $\widetilde{\mathfrak{M}}_{\alpha}^{\kappa}(\vec{f})$. This together with part (i) of Theorem 1.2 yields (3.5). □

REMARK 3.3. Theorem 3.1 can be seen as a generalization of the main result of [17] to the metric setting. Part (i) of Theorem 3.1 extends [12, Theorem 4.2], which corresponds to the case $m = \kappa = 1$. It should be pointed out that part (ii) of Theorem 3.1 is new even in the special case $m = 1$, $\kappa = 1$ and $\alpha = 0$.

REMARK 3.4. Under the assumptions of Theorem 3.1, we see that $\mathfrak{M}_\alpha^\kappa(\vec{f}) \in M_{\mathrm{loc}}^{\delta,q_1}(X)$ and $\widetilde{\mathfrak{M}}_\alpha^\kappa(\vec{f}) \in M_{\mathrm{loc}}^{\delta,q_2}(X)$, where $1/q_1^* = \sum_{i=1}^m 1/p_i - \alpha/Q$ and $1/q_2^* = \sum_{i=1}^m 1/p_i - \alpha$. Moreover, for all open sets $A \subset X$ with $\mu(A) < \infty$, the following inequalities are valid.

$$\|\mathfrak{M}_\alpha^\kappa(\vec{f})\|_{M^{\delta,q}(A)} \lesssim_{\alpha,Q,c_3,p_1,\dots,p_m} \left(\mu(A)^{1/q_1 - 1/q_1^*} + 1\right) \prod_{l=1}^m \|f_l\|_{L^{p_l}(A)},$$

$$\|\widetilde{\mathfrak{M}}_\alpha^\kappa(\vec{f})\|_{M^{Q\beta,q}(A)} \lesssim_{\alpha,p_1,\dots,p_m} \left(\mu(A)^{1/q_2 - 1/q_2^*} + 1\right) \prod_{l=1}^m \|f_l\|_{L^{p_l}(A)}.$$

4. The discrete multilinear fractional maximal operator

Let us begin with the construction of the discrete multilinear fractional maximal operator. Let $r > 0$. Since the measure μ is doubling, there are balls $B(x_i, 6r)$, $i = 1, 2, \dots,$ such that

$$X = \bigcup_{i=1}^\infty B(x_i, r) \quad \text{and} \quad \sum_{i=1}^\infty \chi_{B(x_i,6r)} \leqslant N < \infty,$$

where $\chi_{B(x_i,6r)}$ is the characteristic function of the ball $B(x_i, 6r)$ and the constant N depends only on the doubling constant. This means that the dilated balls $B(x_i, 6r)$, $i = 1, 2, \dots,$ are of bounded overlap. We construct a partition of unity subordinate to the covering $B(x_i, 6r)$, $i = 1, 2, \dots,$ of X. Indeed, there is a family of functions φ_i, $i = 1, 2, \dots,$ such that $0 \leqslant \varphi_i \leqslant 1$, $\varphi_i = 0$ in $X \setminus B(x_i, 6r)$, $\varphi \geqslant \nu > 0$ in $B(x_i, 3r)$, φ_i is Lipschitz with constant L/r with ν and L depending only on the doubling constant, and

$$\sum_{i=1}^\infty \varphi_i(x) = 1$$

for every $x \in X$.

In what follows, we denote by $A_B(f) = \frac{1}{\mu(B)} \int_B f(x)\, d\mu(x)$ the integrable average of f over the set B. We now introduce the discrete multilinear fractional maximal operator.

DEFINITION 4.1. (Discrete multilinear fractional maximal operator). The discrete convolution of a locally integrable function f at the scale $3r$ is

$$D_r(f)(x) = \sum_{i=1}^\infty \varphi_i(x) A_{B(x_i,3r)}(|f|)$$

for every $x \in X$. Let r_j, $j = 1,2,\dots$ be an enumeration of the positive rationals and let balls $B(x_i, r_j)$, $i = 1,2,\dots$ be a covering of X as above. The discrete multilinear fractional maximal operator of $\vec{f} = (f_1,\dots,f_m)$ in X is

$$\mathfrak{M}^*_\alpha(\vec{f})(x) = \sup_{j\geqslant 1} r_j^\alpha \prod_{l=1}^m D_{r_j}(f_l)(x)$$

for every $x \in X$.

When $m = 1$, the operator $\mathfrak{M}^*_\alpha$ reduces to the discrete fractional maximal function $\mathscr{M}^*_\alpha$ studied in [11]. Specially, the Hardy-Littlewood type discrete maximal function corresponds to the special case $\alpha = 0$ of $\mathscr{M}^*_\alpha$, which was investigated by many authors (see [1, 14, 15] for example).

The following result shows that the discrete multilinear fractional maximal function is equivalent with two-sided estimates to the standard multisublinear fractional maximal function.

THEOREM 4.1. *Let $\vec{f} = (f_1,\dots,f_m)$ with each $f_j \in L^1_{\mathrm{loc}}(X)$. Assume that the measure μ satisfies the doubling condition (1.1). Then*

$$\mathfrak{M}^*_\alpha(\vec{f})(x) \sim_{c_1} \mathfrak{M}^1_\alpha(\vec{f})(x)$$

for every $x \in X$.

Proof. Observing that for each $x \in X$, there exists $i = i(x)$ such that $x \in B(x_i, r_j)$. It follows that $B(x, r_j) \subset B(x_i, 2r_j)$. This together with the fact that $\varphi_i \geqslant \nu$ on $B(x_i, r_j)$ yields that

$$r_j^\alpha \prod_{l=1}^m A_{B(x,r_j)}(|f_l|) \lesssim_{c_1} r_j^\alpha \prod_{l=1}^m A_{B(x_i,3r_j)}(|f_l|) \lesssim_{c_1} r_j^\alpha \prod_{l=1}^m \varphi_i(x) A_{B(x_i,3r_j)}(|f_l|) \lesssim_{c_1} \mathfrak{M}^*_\alpha(\vec{f})(x) \tag{4.1}$$

for all $x \in X$. Taking the supremum on the left side of (4.1), one has

$$\mathfrak{M}^1_\alpha(\vec{f})(x) \lesssim_{c_1} \mathfrak{M}^*_\alpha(\vec{f})(x) \tag{4.2}$$

for all $x \in X$. On the other hand, fix $x \in X$ and a positive rational number r_j. Since $\varphi_i = 0$ on $X \setminus B(x_i, 6r_j)$ and $B(x_i, 3r_j) \subset B(x, 9r_j)$ for every $x \in B(x_i, 6r_j)$, by the doubling condition (1.1), it holds that

$$\begin{aligned}
r_j^\alpha \prod_{l=1}^m D_{r_j}(f_l)(x) &= r_j^\alpha \prod_{l=1}^m \Big(\sum_{i=1}^\infty \varphi_i(x) A_{B(x_i,3r_j)}(|f_l|) \Big) \\
&\leqslant r_j^\alpha \prod_{l=1}^m \Big(\sum_{i=1}^\infty \varphi_i(x) \frac{\mu(B(x,9r_j))}{\mu(B(x_i,3r_j))} A_{B(x,9r_j)}(|f_l|) \Big) \\
&\lesssim_{c_1} r_j^\alpha \prod_{l=1}^m A_{B(x,9r_j)}(|f_l|) \\
&\lesssim_{c_1} \mathfrak{M}^1_\alpha(\vec{f})(x).
\end{aligned}$$

It follows that

$$\mathfrak{M}^*_\alpha(\vec{f})(x) \lesssim_{c_1} \mathfrak{M}^1_\alpha(\vec{f})(x)$$

for every $x \in X$. This together with (4.2) yields our claim. □

Applying Theorems 4.1 and 1.2, we obtain

THEOREM 4.2. *Assume that the measure μ satisfies the doubling condition (1.1) and the lower bound condition (1.2). Let $\vec{f} = (f_1,\dots,f_m)$ with each $f_j \in L^{p_j}(X)$ for $p_j > 1$, $0 \leqslant \alpha \leqslant \sum_{i=1}^m Q/p_i$ and $\frac{1}{q} = \sum_{i=1}^m 1/p_i - \alpha/Q \leqslant 1$. Then*

$$\|\mathfrak{M}^*_\alpha(\vec{f})\|_{L^q(X)} \lesssim_{\alpha,Q,c_1,c_3,p_1,\dots,p_m} \prod_{j=1}^m \|f_j\|_{L^{p_j}(X)}.$$

5. Boundedness on Sobolev spaces

A nonnegative Borel function g on X is an upper gradient of a function u on an open set $\Omega \subset X$, if for all rectifiable paths $\gamma\colon [0,1] \to X$, we have

$$|u(\gamma(0)) - u(\gamma(1))| \leqslant \int_\gamma g dx, \tag{5.1}$$

whenever both $u(\gamma(0))$ and $u(\gamma(1))$ are finite, and $\int_\gamma g ds = \infty$ otherwise. The assumption that g is a Borel function is needed in the definition of the path integral. If g is merely a μ-measurable function and inequality (5.1) holds for p-almost every path, then g is said to be a p-weak upper gradient of u. By saying that (5.1) holds for p-almost every path we mean that it fails only for a path family with zero p-modulus. A family Γ of curves is of zero p-modulus if there is a non-negative Borel measurable function $\rho \in L^p(X)$ such that for all curves $\gamma \in \Gamma$, the path integral $\int_\gamma \rho dx = \infty$. If we redefine a p-weak upper gradient on a set of measure zero we obtain an upper gradient of the same function. If g is a p-weak upper gradient of u, then there is a sequence g_i, $i = 1,\dots,$ of upper gradients of u such that $\int_X |g_i - g|^p d\mu \to 0$ as $i \to \infty$. Hence every p-weak upper gradient can be approximated by upper gradients in the $L^p(X)$-norm. If u has an upper gradient that belongs to $L^p(X)$ with $p \geqslant 1$, then it has a minimal p-weak upper gradient g_u in the sense that for every p-weak upper gradient g of u, $g_u \leqslant g$ almost everywhere.

We now recall the Sobolev space in metric space, which is called Newtonian space usually.

DEFINITION 5.1. (Sobolev spaces in metric space). We define the first order Sobolev spaces on the metric space X using the p-weak upper gradients. These spaces are called Newtonian spaces. For $1 \leqslant p < \infty$ and $u \in L^p(X)$, let

$$\|u\|_{N^{1,p}(X)} = \Big(\int_X |u|^p d\mu + \inf_g \int_X g^p d\mu\Big)^{1/p},$$

where the infimum is taken over all p-weak upper gradients of u. Then Newtonian space on X is the quotient space

$$N^{1,p}(X)=\{u:\|u\|_{N^{1,p}(X)}<\infty\}/\sim,$$

where $u\sim v$ if and only if $\|u-v\|_{N^{1,p}(X)}=0$. The same definition applies to subsets of X as well.

The notion of a p-weak upper gradient is used to prove that $N^{1,p}(X)$ is a Banach space. We refer to [3, 22, 23] for the properties of Newtonian spaces

Before showing our main results of this section, we shall present the following useful proposition, which followed from [3].

PROPOSITION 5.1. ([3]). *Let $u=\sup_i u_i$ and $g=\sup_i g_i$, where g_i is a p-weak upper gradients of u_i. If u is finite almost everywhere, then g is a p-weak upper gradient of u.*

We shall establish the following result.

THEOREM 5.2. *Assume that the measure μ satisfies the lower bound condition (1.4) and that μ is doubling. Let $\vec{f}=(f_1,\dots,f_m)$ with each $f_j\in L^{p_j}(X)$ for $p_j>1$, $1\leqslant\alpha\leqslant\sum_{i=1}^m Q/p_i$ and $1/q=\sum_{i=1}^m 1/p_i-(\alpha-1)/Q\leqslant 1$. Then, there exists a constant $C=C(L,m,c_1)>0$ such that $C\mathfrak{M}^*_{\alpha-1}(\vec{f})$ is a weak upper gradient of $\mathfrak{M}^*_{\alpha}(\vec{f})$.*

Proof. Fix $r>0$, we first consider the weak upper gradient of $r^\alpha\prod_{l=1}^m D_r(f_l)$. One can easily check that

$$\begin{aligned}&\Big|r^\alpha\prod_{l=1}^m D_r(f_l)(x)-r^\alpha\prod_{l=1}^m D_r(f_l)(y)\Big|\\&\leqslant r^\alpha\sum_{l=1}^m\prod_{\mu=1}^{l-1}D_r(f_\mu)(y)\prod_{\nu=l+1}^m D_r(f_l)(x)|D_r(f_l)(x)-D_r(f_l)(y)|.\end{aligned}\tag{5.2}$$

Since each φ_i is $\frac{L}{r}$-Lipschitz continuous and has a supported in $B(x_i,6r)$, then

$$\begin{aligned}|D_r(f_l)(x)-D_r(f_l)(y)|&=\Big|\sum_{i=1}^\infty\varphi_i(x)A_{B(x_i,3r)}(|f_l|)-\sum_{i=1}^\infty\varphi_i(y)A_{B(x_i,3r)}(|f_l|)\Big|\\&\leqslant\sum_{i=1}^\infty|\varphi_i(x)-\varphi_i(y)|A_{B(x_i,3r)}(|f_l|)\\&\leqslant Lr^{-1}d(x,y)\sum_{i=1}^\infty A_{B(x_i,3r)}(|f_l|).\end{aligned}\tag{5.3}$$

If $x\in B(x_i,r)$, then $B(x_i,3r)\subset B(x,9r)\subset B(x_i,15r)$ and

$$A_{B(x_i,3r)}(|f_l|)\leqslant c_1^3A_{B(x,9r)}(|f_l|).\tag{5.4}$$

By the bounded overlap property of the balls $B(x_i,6r)$, $i=1,\dots,$ and (5.4), one finds that

$$D_r(f_l)(x)=\sum_{i=1}^{\infty}\varphi_i(x)A_{B(x_i,3r)}(|f_l|)\chi_{B(x_i,6r)}(x)\leqslant Nc_1^3A_{B(x,9r)}(|f_l|). \tag{5.5}$$

Similarly, it holds that

$$D_r(f_l)(y)\leqslant Nc_1^3A_{B(x,9r)}(|f_l|). \tag{5.6}$$

Then by (5.3), (5.4) and the bounded overlap property of the balls $B(x_i,6r)$, $i=1,\dots,$ we have

$$\begin{aligned}
&|D_r(f_l)(x)-D_r(f_l)(y)|\\
&\leqslant Lr^{-1}d(x,y)\sum_{i=1}^{\infty}A_{B(x_i,3r)}(|f_l|)\chi_{B(x_i,r)}(x)\\
&\leqslant Lr^{-1}d(x,y)\sum_{i=1}^{\infty}A_{B(x_i,3r)}(|f_l|)\chi_{B(x_i,6r)}(x)\leqslant LNc_1^3r^{-1}d(x,y)A_{B(x,9r)}(|f_l|).
\end{aligned} \tag{5.7}$$

It follows from (5.2) and (5.5)-(5.7) that

$$\begin{aligned}
&\left|r^{\alpha}\prod_{l=1}^{m}D_r(f_l)(x)-r^{\alpha}\prod_{l=1}^{m}D_r(f_l)(y)\right|\\
&\leqslant d(x,y)Lm(Nc_1^3)^m r^{\alpha-1}\prod_{l=1}^{m}A_{B(x,9r)}(|f_l|)\\
&\leqslant d(x,y)9^{1-\alpha}Lm(Nc_1^3)^m\mathfrak{M}^1_{\alpha-1}(\vec{f})(x).
\end{aligned}$$

Therefore, the function $9^{1-\alpha}Lm(Nc_1^3)^m\mathfrak{M}^1_{\alpha-1}(\vec{f})$ is a weak upper gradient of $r^{\alpha}\prod_{l=1}^{m}D_r(f_l)$. This together with Theorem 4.1 implies that there exists a constant $C=C(L,m,c_1)>0$ such that $C\mathfrak{M}^*_{\alpha-1}(\vec{f})$ is a weak upper gradient of $r^{\alpha}\prod_{l=1}^{m}D_r(f_l)$. By Theorem 4.2, we see that $\mathfrak{M}^*_{\alpha-1}(\vec{f})\in L^q(X)$. Hence, the function $\mathfrak{M}^*_{\alpha-1}(\vec{f})$ is finite almost everywhere. Applying Proposition 5.1, we can get our desired conclusion. □

REMARK 5.2. The following Sobolev type theorem is a generalization of the main result of [17, Theorem 2.3] to the metric setting. Theorem 5.2 implies [10, Theorem 6.1] when $m=1$. With the assumptions of Theorem 5.2, we see that $\mathfrak{M}^*_{\alpha}(\vec{f})\in N^{1,q^*}_{\mathrm{loc}}(X)$ and

$$\|\mathfrak{M}^*_{\alpha}(\vec{f})\|_{N^{1,q^*}(A)}\lesssim_{\alpha,Q,L,m,c_1,c_3,p_1,\dots,p_m}\left(\mu(A)^{1/q^*-1/q}+1\right)\prod_{j=1}^{m}\|f_j\|_{L^{p_j}(A)}$$

for all open sets $A\subset X$ with $\mu(A)<\infty$. Here $1/q^*=\sum_{i=1}^{m}1/p_i-\alpha/Q\leqslant 1$.

In order to establish next result, we need the following definition.

DEFINITION 5.3. ($(1,p)$-Poincaré inequality). We say that X supports a (weak) $(1,p)$-Poincaré inequality if there exist constant $c_6>0$ and $\lambda\geqslant 1$ such that for all balls $B(x,r)\subset X$, for all locally integrable functions u on X and for all p-weak upper gradients g of u,

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)}|u-A_{B(x,r)}(u)|d\mu\leqslant c_6 r\Big(\frac{1}{\mu(B(x,\lambda r))}\int_{B(x,\lambda r)}g^p d\mu\Big)^{1/p}.$$

Note that since p-weak upper gradients can be approximated by upper gradients in the $L^p(X)$-norm, it would be enough to require the Poincaré inequality for upper gradients only. By Hölder's inequality we see that if X supports a $(1,p)$-Poincaré inequality, then it supports a $(1,q)$-Poincaré inequality for every $q>p$. It was shown in [13] that if X is complete and μ is doubling, then a $(1,p)$-Poincaré inequality implies a $(1,r)$-Poincaré inequality for some $r<p$. Hence the $(1,p)$-Poincaré inequality is a self improving condition.

We now establish the boundedness of the discrete multisublinear fractional maximal function in Sobolev spaces.

THEOREM 5.3. *Assume that the measure μ satisfies the doubling condition* (1.1) *and the lower bound condition* (1.2)*. Let $\vec{f}=(f_1,\dots,f_m)$ with each $f_j\in N^{1,p_j}(X)$ for $1<p_j<\infty$, $1\leqslant\alpha\leqslant\sum_{i=1}^m Q/p_i$ and $1/q=\sum_{i=1}^m 1/p_i-\alpha/Q\leqslant 1$. Assume that X is a complete metric space which supports a $(1,p)$-Poincaré inequality with $p=\min_{1\leqslant i\leqslant m}p_i$. Then $\mathfrak{M}_\alpha^*(\vec{f})\in N^{1,q}(X)$. Moreover,*

$$\|\mathfrak{M}_\alpha^*(\vec{f})\|_{N^{1,q}(X)}\lesssim_{\alpha,Q,c_1,c_3,c_6,p_1,\dots,p_m}\prod_{l=1}^m\|f_l\|_{N^{1,p_l}(X)}.$$

Proof. Let $f_l\in N^{1,p_l}(X)$ and $g_l\in L^{p_l}(X)$ be a weak upper gradient of f_l. Fix $j\geqslant 1$ and $r>0$, let $x,y\in B(x_j,r)$ and $I_j=\{i:B(x_i,6r)\cap B(x_j,r)\neq\emptyset\}$. On the basis of the bounded overlap of the balls $B(x_i,6r)$, the set I_j is finite and the cardinality does not depend on j. One can easily check that

$$\begin{aligned}
&\Big|r^\alpha\prod_{l=1}^m D_r(f_l)(x)-r^\alpha\prod_{l=1}^m D_r(f_l)(y)\Big|\\
&=r^\alpha\Big|\prod_{l=1}^m\sum_{i=1}^\infty\varphi_i(x)A_{B(x_i,3r)}(|f_l|)-\prod_{l=1}^m\sum_{i=1}^\infty\varphi_i(y)A_{B(x_i,3r)}(|f_l|)\Big|\\
&\leqslant r^\alpha\sum_{l=1}^m\prod_{\mu=1}^{l-1}\sum_{i=1}^\infty\varphi_i(y)A_{B(x_i,3r)}(|f_\mu|)\prod_{\nu=l+1}^m\sum_{i=1}^\infty\varphi_i(x)A_{B(x_i,3r)}(|f_\nu|)\\
&\quad\times\Big|\sum_{i=1}^\infty(\varphi_i(x)-\varphi_i(y))(A_{B(x_i,3r)}(|f_l|)-A_{B(x_j,3r)}(|f_l|))\Big|.
\end{aligned}\tag{5.8}$$

It was shown in [10] that

$$\begin{aligned}
&\Big|\sum_{i=1}^\infty(\varphi_i(x)-\varphi_i(y))(A_{B(x_i,3r)}(|f_l|)-A_{B(x_j,3r)}(|f_l|))\Big|\\
&\lesssim_{c_6}d(x,y)\Big(\tfrac{1}{\mu(B(x_j,10\lambda r))}\textstyle\int_{B(x_j,10\lambda r)}g_l^{r_l}d\mu\Big)^{1/r_l}
\end{aligned}\tag{5.9}$$

for some $r_l \in (1, p_l)$ and $\lambda \geqslant 1$. Let $\alpha = \sum_{i=1}^m \alpha_i$ with $\alpha_i \geqslant 0$. Then (5.4) leads to

$$r^{\alpha_l} \sum_{i=1}^{\infty} \varphi_i(x) A_{B(x_i,3r)}(|f_l|) \leqslant c_1^3 r^{\alpha_l} A_{B(x,9r)}(|f_l|) \sum_{i=1}^{\infty} \varphi_i(x) \leqslant c_1^3 \mathscr{M}^1_{\alpha_\mu}(f_l)(x). \tag{5.10}$$

Note that $B(x, 9r) \subset B(y, 11r) \subset B(x, 13r)$. This together with (5.4) and (1.1) implies that

$$\begin{aligned} & r^{\alpha_l} \sum_{i=1}^{\infty} \varphi_i(y) A_{B(x_i,3r)}(|f_l|) \\ \leqslant\; & c_1^3 r^{\alpha_l} A_{B(x,9r)}(|f_l|) \sum_{i=1}^{\infty} \varphi_i(x) \\ \leqslant\; & c_1^3 r^{\alpha_l} A_{B(x,9r)}(|f_l|) = c_1^3 r^{\alpha_l} \frac{\mu(B(y,11r))}{\mu(B(x,9r))} A_{B(y,11r)}(|f_l|) \leqslant c_1^4 \mathscr{M}^1_{\alpha_l}(f_l)(x). \end{aligned} \tag{5.11}$$

It follows from (5.8)-(5.11) that

$$\begin{aligned} & \left| r^{\alpha} \prod_{l=1}^m D_r(f_l)(x) - r^{\alpha} \prod_{l=1}^m D_r(f_l)(y) \right| \\ \lesssim_{c_1,c_6}\; & d(x,y) \sum_{l=1}^m \prod_{\substack{1 \leqslant \mu \leqslant m \\ \mu \neq l}} \mathscr{M}^1_{\alpha_\mu}(f_\mu)(x) r^{\alpha_l} \left(\frac{1}{\mu(B(x_j, 10\lambda r))} \int_{B(x_j,10\lambda r)} g_l^{r_l} d\mu \right)^{1/r_l}. \end{aligned}$$

Since the pointwise Lipschitz constant of a function is a weak upper gradient, we set that

$$g_r(x) = C \sum_{l=1}^m \prod_{\substack{1 \leqslant \mu \leqslant m \\ \mu \neq l}} \mathscr{M}^1_{\alpha_\mu}(f_\mu)(x) r^{\alpha_l} \sum_{j=1}^m \left(\frac{1}{\mu(B(x_j, 10\lambda r))} \int_{B(x_j,10\lambda r)} g_l^{r_l} d\mu \right)^{1/r_l} \chi_{B(x_j,6r)}(x)$$

is a weak upper gradient of $r^{\alpha} \prod_{l=1}^m D_r(f_l)$. By the bounded overlap of the balls, there exists a constant $C = C(c_1, c_6) > 0$ such that

$$g_r(x) \leqslant C(c_1,c_6) \sum_{l=1}^m (\mathscr{M}^*_{\alpha_l r_l}(g_l^{r_l})(x))^{1/r_l} \prod_{\substack{1 \leqslant \mu \leqslant m \\ \mu \neq l}} \mathscr{M}^1_{\alpha_\mu}(f_\mu)(x) =: C(c_1,c_6) G(\vec{f})(x).$$

By Theorems 1.2 and 4.1 with $m = 1$, the Hölder's inequality and the Minkowski's inequality, it holds that

$$\begin{aligned} \|G(\vec{f})\|_{L^q(X)} &\leqslant \prod_{l=1}^m \|(\mathscr{M}^1_{\alpha_l r_l}(g_l^{r_l}))^{1/r_l}\|_{L^{q_l}(X)} \prod_{\substack{1 \leqslant \mu \leqslant m \\ \mu \neq l}} \|\mathscr{M}^1_{\alpha_\mu}(f_\mu)\|_{L^{q_\mu}(X)} \\ &\lesssim_{\alpha,Q,c_1,c_3,p_1,\dots,p_m} \prod_{l=1}^m \|g_l\|_{L^{p_l}(X)} \prod_{\substack{1 \leqslant \mu \leqslant m \\ \mu \neq l}} \|f_\mu\|_{L^{p_\mu}(X)}. \end{aligned} \tag{5.12}$$

Here $1/q = \sum_{i=1}^m 1/q_i$ and $q_i = Qp_i/(Q - \alpha_i p_i)$. Then $G(\vec{f})$ is finite almost everywhere. Applying Proposition 5.1, we obtain that $C(c_1,c_6)G(\vec{f})$ is a q-weak upper gradient of $\mathfrak{M}^*_\alpha(\vec{f})$. Note that $g_l^{r_l} \in L^{p_l/r_l}(X)$ and $p_l/r_l > 1$. By Theorem 4.2 we can get

$$\|\mathfrak{M}^*_\alpha(\vec{f})\|_{L^q(X)} \lesssim_{\alpha,Q,c_1,c_3,p_1,\dots,p_m} \prod_{j=1}^m \|f_j\|_{L^{p_j}(X)}.$$

This together with (5.12) yields that

$$\|\mathfrak{M}^*_\alpha(\vec{f})\|_{N^{1,q}(X)} \lesssim_{\alpha,Q,c_1,c_3,c_6,p_1,\dots,p_m} \prod_{j=1}^m \|f_j\|_{N^{1,p_j}(X)}.$$

Then Theorem 5.3 is proved. □

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