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A FAMILY OF WILKER'S INEQUALITIES IN TWO PARAMETERS AND ITS APPLICATIONS

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Abstract. Based on the reparameterization technique, this paper presents a family of Wilker's Inequalities with an extra parameter α. By using Pade approximation, the optimal value of α within a local interval is given. And simple proofs are provided based on the bounds of the functions $\sin(\alpha x)$ and $\cos(\alpha x)$. With the constraint $\alpha = 1$, the new method can recover several previous results. In principle, the idea can be extended to more other forms of bounding functions, or even more other inequality types. The comparison results show that the results in this paper are much better than those of prevailing methods.

1. Introduction

The Wilker inequality, which involves the trigonometric function

$$f(x) = \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x}, \tag{1}$$

has caused wide interests of many researchers, see also [4, 5, 2, 19, 3, 24, 26, 12, 29, 28, 11, 25, 13, 27, 9, 18, 14, 22, 1, 15, 16, 30, 20, 10, 17, 32, 34, 35, 36, 33, 31, 23, 8] and the references therein. Let

$$\begin{cases} b(x) = \frac{8}{945}x^2 - \frac{16}{14175}x^4, c(x) = (\frac{160}{\pi^5} - \frac{16}{\pi^3})(\frac{\pi}{2} - x), \\ a(x) = \frac{8}{945}x^2, d(x) = (\frac{160}{\pi^5} - \frac{16}{\pi^3})(\frac{\pi}{2} - x) + (\frac{960}{\pi^6} - \frac{96}{\pi^4})(\frac{\pi}{2} - x)^2, \\ b_1(x) = \frac{8}{945}x^2 - \frac{\alpha}{14175}x^4 \text{ where } \alpha = \frac{480\pi^6 - 40320\pi^4 + 3628800}{\pi^8} \approx 17.15041, \\ L_1(x) = 2 + \frac{16}{\pi^4}x^3\tan x, R_1(x) = 2 + \frac{8}{45}x^3\tan x, \\ L_2(x) = 2 + (\frac{8}{45} - a(x))x^3\tan x, R_2(x) = 2 + (\frac{8}{45} - b_1(x))x^3\tan x, \\ L_3(x) = 2 + (\frac{16}{\pi^4} + c(x))x^3\tan x, R_3(x) = 2 + (\frac{16}{\pi^4} + d(x))x^3\tan x. \end{cases} \tag{2}$$

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Summer and Mortici [19, 25] had provided the following two-sided bounds such as

$$L_1(x) < f(x) < R_1(x), \ \ 0 < x < \pi/2, \tag{3}$$

$$2+\Big(\frac{8}{45}-a(x)\Big)x^3\tan x < f(x) < 2+\Big(\frac{8}{45}-b(x)\Big)x^3\tan x, \ 0<x<1. \tag{4}$$

Later, Nenezić, Malešević and Mortici [24] had presented the following improved two-sided bounds such as

$$L_2(x) < f(x) < R_2(x), \quad 0 < x < \pi/2. \tag{5}$$

$$L_3(x) < f(x) < R_3(x), \quad 0 < x < \pi/2. \tag{6}$$

Let

$$\begin{aligned} D_1(x) &= \tfrac{f(x)-L_1(x)}{x^4\cdot(\frac{\pi}{2}-x)}\cdot\cos(x), \quad & D_2(x) &= \tfrac{f(x)-R_1(x)}{x^6}\cdot\cos(x),\\ D_3(x) &= \tfrac{f(x)-L_2(x)}{x^8}\cdot\cos(x), \quad & D_4(x) &= \tfrac{f(x)-R_2(x)}{x^8\cdot(\frac{\pi}{2}-x)}\cdot\cos(x),\\ D_5(x) &= \tfrac{f(x)-L_3(x)}{x^4\cdot(\frac{\pi}{2}-x)^2}\cdot\cos(x), \quad & D_6(x) &= \tfrac{f(x)-R_3(x)}{x^4\cdot(\frac{\pi}{2}-x)^3}\cdot\cos(x). \end{aligned} \tag{7}$$

As shown in Fig. 1, for all $0<x<\pi/2$, one obtains that

$$\begin{cases} 0.00434 \approx \frac{320-32\pi^2}{\pi^6} < D_1(x) < 0.010303,\\ -8.465e-3 \approx -\frac{8}{945} < D_2(x) < \frac{-64\pi^4+5760}{45\pi^7} \approx -3.488e-3,\\ 7.7e-4 \approx \frac{64\pi^6-5376\pi^4+483840}{945\pi^9} < D_3(x) < \frac{16}{14175} \approx 1.1287e-3,\\ -1.06e-4 \approx \frac{256\pi^6-43008\pi^4-483840\pi^2+8709120}{945\pi^{10}} < D_4(x)\\ < \frac{32\pi^8-960\pi^6+80640\pi^4-7257600}{14175\pi^9} \approx -5.16e-5,\\ 1.14e-3 \approx \frac{32\pi^4+1440\pi^2-17280}{45\pi^6} < D_5(x) < \frac{-192\pi^2+1920}{\pi^7} \approx 8.28e-3,\\ -8.7165e-3 < D_6(x) < \frac{64\pi^4+11520\pi^2-120960}{45\pi^7} \approx -7.56e-3, \end{cases} \tag{8}$$

Chen, Ma, Jin and Wang [6] have provided an improved bounds for all $0<x<\pi/2$, by using the bounding functions in form of

$$L_4(x) = q_4(x)+q_5(x)\tan(x) < f(x) < p_4(x)+p_5(x)\tan(x) = R_4(x),$$

which obtains much tighter bounds, where $q_i(x)$ and $p_j(x)$ are polynomials of degree i and j, respectively.

This paper provides a family of bounding functions in form of

$$\begin{aligned} G_1(x,\alpha) &= 2\cos(\alpha x)+((-1+\alpha^2)+(-\tfrac{\alpha^4}{12}+\tfrac{\alpha^2}{6}+\tfrac{17}{180})x^2\\ &\quad +(\tfrac{\alpha^6}{360}-\tfrac{\alpha^4}{72}+\tfrac{7\alpha^2}{360}-\tfrac{127}{7560})x^4)x\sin(x), \end{aligned} \tag{9}$$

$$G_2(x,\alpha) = G_1(x,\alpha)+(-\tfrac{\alpha^8}{20160}+\tfrac{\alpha^6}{2160}-\tfrac{7\alpha^4}{4320}+\tfrac{31\alpha^2}{15120}+\tfrac{37}{129600})x^7\sin(x),$$

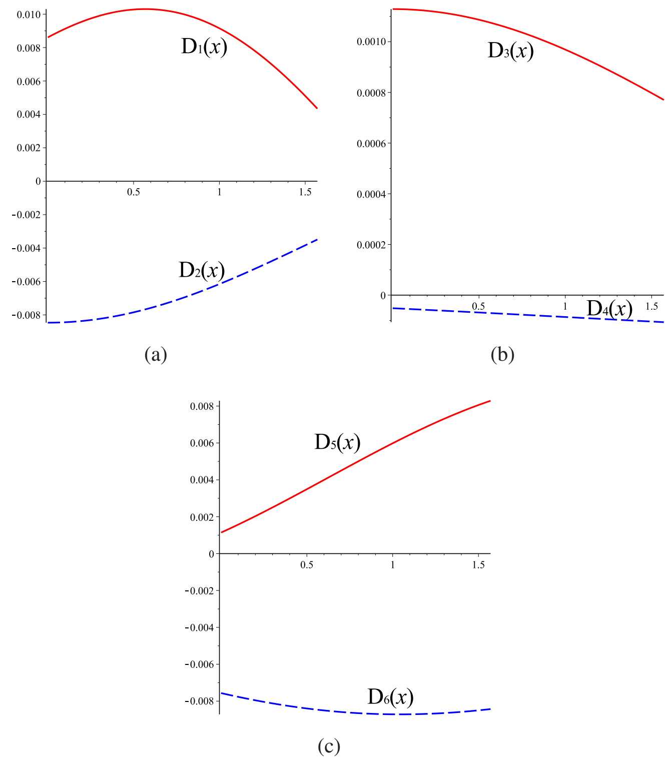


Figure 1: *Error plots of* $D_i(t)$ *in Eq. (7),* $i = 1, 2, \cdots, 6$.

for bounding

$$F(x) = f(x) \cdot \cos(x) \tag{10}$$

with a much better approximation effect. It can be verified that

$$E_i(x,\alpha) = F(x) - G_i(x,\alpha), \quad i = 1, 2,$$

$$E_1(0,\alpha) = E_1'(0,\alpha) = \cdots = E_1^{(7)}(0,\alpha), \quad E_2(0,\alpha) = E_1'(0,\alpha) = \cdots = E_2^{(9)}(0,\alpha),$$

$$E_1^{(8)}(0,\alpha) = -2\alpha^8 + \tfrac{56\alpha^6}{3} - \tfrac{196\alpha^4}{3} + \tfrac{248\alpha^2}{3} + \tfrac{518}{45},$$

$$E_2^{(10)}(0,\alpha) = 2\alpha^{10} - 30\alpha^8 + 196\alpha^6 - 620\alpha^4 + 762\alpha^2 - \tfrac{8182}{33},$$

where $E_i^{(j)}(x,\alpha)$ denotes the derivatives of $E_i(x,\alpha)$ in x.

The main results are as follows.

THEOREM 1. *For all* $0 < x < \pi/2$, *we have that*

$$\begin{aligned} L_5(x)\cdot\cos(x) &\triangleq G_1(x,\alpha_1) < F(x) < G_1(x,\alpha_2) \triangleq R_5(x)\cdot\cos(x), \\ L_6(x)\cdot\cos(x) &\triangleq G_2(x,\alpha_3) < F(x) < G_2(x,\alpha_4) \triangleq R_6(x)\cdot\cos(x), \end{aligned} \tag{11}$$

where $\alpha_1 \approx 1.81168$, $\alpha_2 \approx 1.87682$, $\alpha_3 \approx 1.29481$ *and* $\alpha_4 \approx 1.36071$ *are the roots of the following equations such that*

$$E_1(\tfrac{\pi}{2},\alpha_1)=0,\ \ E_1^{(8)}(0,\alpha_2)=0,\ \ E_2^{(10)}(0,\alpha_3)=0,\ \ E_2(\tfrac{\pi}{2},\alpha_4)=0. \tag{12}$$

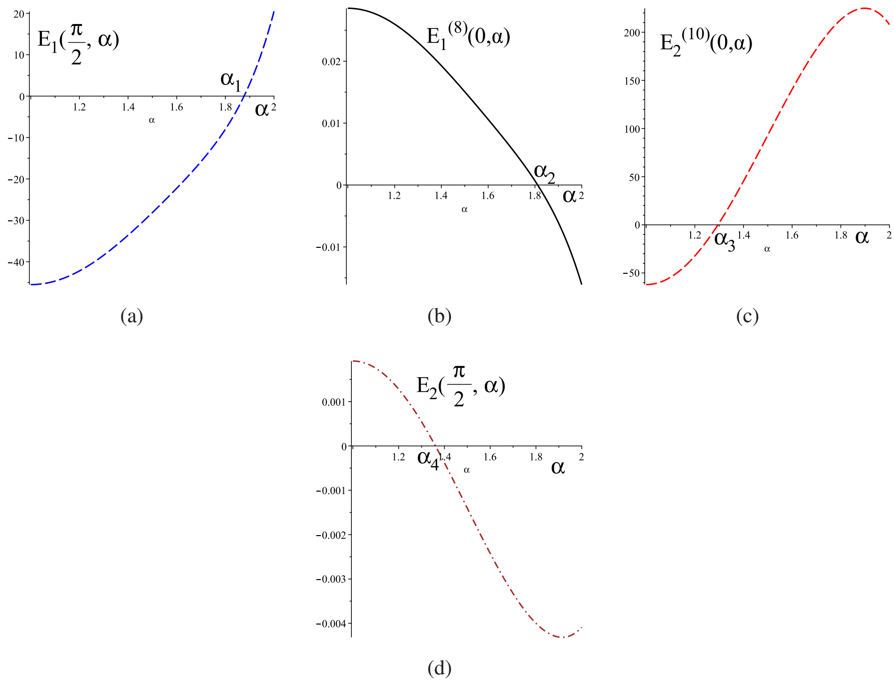


Figure 2: *Error plots of (a)* $E_1(\frac{\pi}{2},\alpha)$*; (b)* $E_1^{(8)}(0,\alpha)$*; (c)* $E_2^{(10)}(0,\alpha)$ *and (d)* $E_2(\frac{\pi}{2},\alpha)$ *in Eq. (12).*

As shown in Fig. 2, combining Theorem 1 with Eq. (12), for $\forall x \in (0,\pi/2)$ and $\alpha \in [1,2]$, α_1 is the minimum constant such that $E_1(\frac{\pi}{2},\alpha) \geqslant 0$ and $G_1(x,\alpha) \leqslant F(x)$, α_2 is the minimum constant such that $E_1^{(8)}(0,\alpha) \leqslant 0$ and $G_1(x,\alpha) \geqslant F(x)$, α_3 is the minimum constant such that $E_2^{(10)}(0,\alpha) \geqslant 0$ and $G_2(x,\alpha) \leqslant F(x)$, and α_4 is the minimum constant such that $E_2(\frac{\pi}{2},\alpha) \leqslant 0$ and $G_2(x,\alpha) \geqslant F(x)$.

Let

$$\begin{aligned} D_7(x) &= \tfrac{f(x)-G_1(x,\alpha)/\cos(x)}{x^8\cdot(\frac{\pi}{2}-x)}\cdot\cos(x), \quad D_8(x) = \tfrac{f(x)-G_1(x,\alpha_2)/\cos(x)}{x^{10}}\cdot\cos(x),\\ D_9(x) &= \tfrac{f(x)-G_2(x,\alpha_3)/\cos(x)}{x^{12}}\cdot\cos(x), \quad D_{10}(x) = \tfrac{f(x)-G_2(x,\alpha_4)/\cos(x)}{x^{11}\cdot(\frac{\pi}{2}-x)}\cdot\cos(x), \end{aligned} \tag{13}$$

Comparing the results of Eq. (7) as shown in Fig. 1, the results of Eq. (13) achieve much better approximation effect, as shown in Fig. 3.

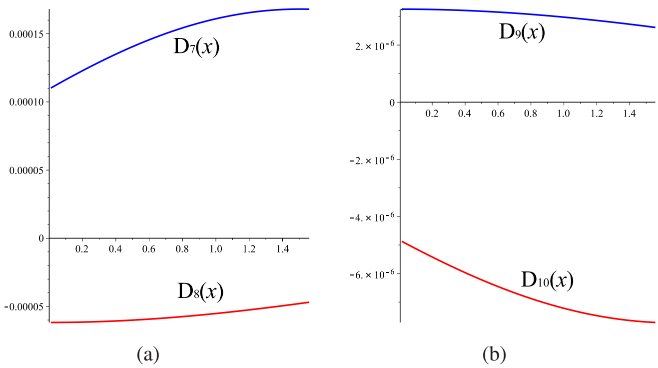


Figure 3: *Error plots of $D_i(t)$ in Eq. (13), $i = 7,8,9,10$.*

2. Proof of Theorem 1

In principle, the idea of the method in [21] may be applied for the proof of Theorem 1. In this section, we provide the following method for proving Theorem 1. For the sake of convenience, we introduce Theorem 3.5.1 in Page 67, Chapter 3.5 of [7] as follows.

THEOREM 2. *Let w_0, w_1, $\cdots$, w_r be $r+1$ distinct points in $[a,b]$, and n_0, $\cdots$, n_r be $r+1$ integers $\geqslant 0$. Let $N = n_0 + \cdots + n_r + r$. Suppose that $g(t)$ is a polynomial of degree N such that $g^{(i)}(w_j) = f^{(i)}(w_j)$, $i = 0,\cdots,n_j$, $j = 0,\cdots,r$. Then there exists $\xi_1(t) \in [a,b]$ such that $f(t) - g(t) = \dfrac{f^{(N+1)}(\xi_1(t))}{(N+1)!}\prod_{i=0}^{r}(t-w_i)^{n_i+1}$.*

We have the following lemmas.

LEMMA 3. *For $\alpha \in (0,2)$, $x \in (0,\pi/2)$, $n \geqslant 1$, we have that*

$$\begin{aligned} &\underline{C}_n(x,\alpha) < \cos(\alpha x) < \bar{C}_n(x,\alpha), \\ &\underline{S}_n(x,\alpha) < \sin(\alpha x) < \bar{S}_n(x,\alpha), \end{aligned} \tag{14}$$

where

$$\begin{aligned} &C_n(x,\alpha) = \sum_{i=0}^{2n-1} \frac{(-1)^i \cdot (\alpha x)^{2i}}{(2i)!}, \quad S_n(x,\alpha) = \sum_{i=0}^{2n} \frac{(-1)^i \cdot (\alpha x)^{2i+1}}{(2i+1)!}, \\ &\underline{C}_n(x,\alpha) = C_n(x,\alpha) + \frac{\cos(\frac{\alpha\pi}{2}) - C_n(\frac{\pi}{2},\alpha)}{(\frac{\pi}{2})^{4n}} x^{4n}, \\ &\bar{C}_n(x,\alpha) = C_n(x,\alpha) + \frac{\cos(\frac{\alpha\pi}{2}) - C_n(\frac{\pi}{2},\alpha)}{(\frac{\pi}{2})^{4n-1}} x^{4n-1}, \\ &\underline{S}_n(x,\alpha) = S_n(x,\alpha) + \frac{\sin(\frac{\alpha\pi}{2}) - S_n(\frac{\pi}{2},\alpha)}{(\frac{\pi}{2})^{4n+2}} x^{4n+2}, \\ &\bar{S}_n(x,\alpha) = S_n(x,\alpha) + \frac{\sin(\frac{\alpha\pi}{2}) - S_n(\frac{\pi}{2},\alpha)}{(\frac{\pi}{2})^{4n+3}} x^{4n+3}. \end{aligned} \tag{15}$$

Proof. Let $H_1(x,\alpha)=\cos(\alpha x)-\underset{\sim}{C}_n(x,\alpha)$, $H_2(x,\alpha)=\cos(\alpha x)-\bar{C}_n(x,\alpha)$, $H_3(x,\alpha)$ $=\sin(\alpha x)-\underset{\sim}{S}_n(x,\alpha)$ and $H_4(x,\alpha)=\sin(\alpha x)-\overline{S}_n(x,\alpha)$. It can be verified that

$$\begin{aligned}
H_1(0)=H_1^{'}(0)=\cdots=H_1^{(4n-1)}=H_1(\tfrac{\pi}{2})=0,\\
H_2(0)=H_2^{'}(0)=\cdots=H_2^{(4n-2)}=H_2(\tfrac{\pi}{2})=0,\\
H_3(0)=H_3^{'}(0)=\cdots=H_3^{(4n+1)}=H_3(\tfrac{\pi}{2})=0,\\
H_4(0)=H_4^{'}(0)=\cdots=H_4^{(4n+2)}=H_4(\tfrac{\pi}{2})=0.
\end{aligned} \tag{16}$$

Combining Eq. (16) with Theorem 2, for $\alpha\in(0,2)$, $x\in(0,\pi/2)$, $n\geqslant 1$, there exists $\xi_i(x)\in(0,\frac{\pi}{2})$ such that

$$\begin{aligned}
H_1(x)&=\frac{H_1^{(4n+1)}(\xi_1(x))}{(4n+1)!}x^{4n}(x-\tfrac{\pi}{2})=\frac{-\alpha^{4n+1}\cdot\sin(\alpha\xi_1(x))}{(4n+1)!}x^{4n}(x-\tfrac{\pi}{2})>0,\\
H_2(x)&=\frac{H_2^{(4n)}(\xi_2(x))}{(4n)!}x^{4n-1}(x-\tfrac{\pi}{2})=\frac{\alpha^{4n}\cdot\cos(\alpha\xi_1(x))}{(4n)!}x^{4n-1}(x-\tfrac{\pi}{2})<0,\\
H_3(x)&=\frac{H_3^{(4n+3)}(\xi_3(x))}{(4n+3)!}x^{4n+2}(x-\tfrac{\pi}{2})=\frac{-\alpha^{4n+3}\cdot\cos(\alpha\xi_3(x))}{(4n+3)!}x^{4n+2}(x-\tfrac{\pi}{2})>0,\\
H_4(x)&=\frac{H_4^{(4n+4)}(\xi_4(x))}{(4n+4)!}x^{4n+3}(x-\tfrac{\pi}{2})=\frac{\alpha^{4n+4}\cdot\sin(\alpha\xi_4(x))}{(4n+4)!}x^{4n+3}(x-\tfrac{\pi}{2})<0,
\end{aligned} \tag{17}$$

which leads to Eq. (14), and the proof of the lemma is completed. □

Let

$$\begin{aligned}
\lambda_{1,0} &= \frac{-45\alpha_1^8+420\alpha_1^6-1470\alpha_1^4+1860\alpha_1^2+259}{453600\pi}\approx 1.0948\times 10^{-4},\\
\lambda_{1,1} &= \frac{-45\alpha_1^8+420\alpha_1^6-1470\alpha_1^4+1860\alpha_1^2+259}{226800\pi^2}\approx 6.9697\times 10^{-5},\\
\lambda_{1,2} &= \frac{33\alpha_1^{10}\pi^2-6940\pi^2-11880\alpha_1^8+(-1386\pi^2+110880)\alpha_1^6+(5940\pi^2-388080)\alpha_1^4}{29937600\pi^3}\approx -1.1694\times 10^{-5},\\
&\quad+\frac{(-7887\pi^2+491040)\alpha_1^2+68376}{29937600\pi^3}\approx -1.1694\times 10^{-5},\\
\lambda_{1,3} &= \frac{33\alpha_1^{10}\pi^2-6940\pi^2-11880\alpha_1^8+(-1386\pi^2+110880)\alpha_1^6+(5940\pi^2-388080)\alpha_1^4+(-7887\pi^2+491040)\alpha_1^2+68376}{14968800\pi^4}\\
&\quad+\frac{(-7887\pi^2+491040)\alpha_1^2+68376}{14968800\pi^4}\approx -7.445\times 10^{-6},\\
\lambda_{1,4} &= \frac{-273\alpha_1^{12}\pi^4+660230\pi^4+144144\alpha_1^{10}\pi^2-30313920\pi^2-51891840\alpha_1^8+(36036\pi^4-6054048\pi^2+484323840)\alpha_1^6}{32691859200\pi^5}\\
&\quad+\frac{(-165165\pi^4+25945920\pi^2-1695133440)\alpha_1^4+(223860\pi^4-34450416\pi^2+2144862720)\alpha_1^2+298666368}{32691859200\pi^5}\\
&\approx 6.015\times 10^{-7},\\
\lambda_{1,5} &= \frac{(-156\alpha_1^{12}+20592\alpha_1^6-94380\alpha_1^4+127920\alpha_1^2+373757)\pi^{14}+(82368\alpha_1^{10}-3459456\alpha_1^6+14826240\alpha_1^4)\pi^{12}}{9340531200\pi^{16}}\\
&\quad+\frac{(-19685952\alpha_1^2-16679208)\pi^{12}+(-29652480\alpha_1^8+276756480\alpha_1^6-968647680\alpha_1^4+1225635840\alpha_1^2+84509568)\pi^{10}}{9340531200\pi^{16}}\\
&\quad+\frac{8124779520\pi^8-518088130560\pi^6+20723525222400\pi^4-382588157952000\pi^2+306070526361600\pi}{9340531200\pi^{16}}\\
&\quad+\frac{1224282105446400}{9340531200\pi^{16}}\approx 3.814\times 10^{-7},\\
\lambda_{1,6} &= \frac{(90\alpha_1^{14}-30030\alpha_1^6+141960\alpha_1^4-194460\alpha_1^2-1820149)\pi^{16}+(-65520\alpha_1^{12}+8648640\alpha_1^6-39639600\alpha_1^4)\pi^{14}}{1961511552000\pi^{17}}\\
&\quad+\frac{(53726400\alpha_1^2+156978120)\pi^{14}+(34594560\alpha_1^{10}-1452971520\alpha_1^6+6227020800\alpha_1^4-8268099840\alpha_1^2-7005398400)\pi^{12}}{1961511552000\pi^{17}}
\end{aligned}$$

$$+\frac{(-12454041600\alpha_1^8+116237721600\alpha_1^6-406832025600\alpha_1^4+514767052800\alpha_1^2+35563207680)\pi^{10}}{1961511552000\pi^{17}}$$
$$+\frac{3387499315200\pi^8-212017604198400\pi^6+8034351316992000\pi^4-128549621071872000\pi^2}{1961511552000\pi^{17}}$$
$$+\frac{128549621071872000\pi+257099242143744000}{1961511552000\pi^{17}}\approx -2.033\times 10^{-8},$$

$$\lambda_{1,7}=\frac{(-15015\alpha_1^6+70980\alpha_1^4-97230\alpha_1^2-818207)\pi^{16}+(4324320\alpha_1^6-19819800\alpha_1^4+26863200\alpha_1^2+61896120)\pi^{14}}{490377888000\pi^{18}}$$
$$+\frac{(-726485760\alpha_1^6+3113510400\alpha_1^4-4134049920\alpha_1^2-1366485120)\pi^{12}+(58118860800\alpha_1^6-203416012800\alpha_1^4)\pi^{10}}{490377888000\pi^{18}}$$
$$+\frac{(257383526400\alpha_1^2-159688488960)\pi^{10}+(-1394852659200\alpha_1^6+9365439283200)\pi^8+2789705318400\alpha_1^6\pi^7}{490377888000\pi^{18}}$$
$$+\frac{(-13948526592000\alpha_1^4+19527937228800\alpha_1^2-16871075020800)\pi^7+(167382319104000\alpha_1^4-189699961651200)\pi^6}{490377888000\pi^{18}}$$
$$+\frac{(-334764638208000\alpha_1^4+669529276416000\alpha_1^2+379399923302400)\pi^5}{490377888000\pi^{18}}$$
$$+\frac{(-8034351316992000\alpha_1^2+4017175658496000)\pi^4}{490377888000\pi^{18}}$$
$$+\frac{16068702633984000\alpha_1^2\pi^3-48206107901952000\pi^2+128549621071872000}{490377888000\pi^{18}}\approx -1.344\times 10^{-8},$$

$$\lambda_{1,8}=\frac{(546\alpha_1^6-2625\alpha_1^4+3612\alpha_1^2+108969)\pi^{18}+(-240240\alpha_1^6+1135680\alpha_1^4-1550640\alpha_1^2-13091312)\pi^{16}}{3923023104000\pi^{19}}$$
$$+\frac{(69189120\alpha_1^6-317116800\alpha_1^4+426666240\alpha_1^2+990337920)\pi^{14}+(-11623772160\alpha_1^6+49816166400\alpha_1^4)\pi^{12}}{3923023104000\pi^{19}}$$
$$+\frac{(-64761016320\alpha_1^2-21863761920)\pi^{12}}{3923023104000\pi^{19}}$$
$$+\frac{(929901772800\alpha_1^6-3254656204800\alpha_1^4+3719607091200\alpha_1^2-2555015823360)\pi^{10}}{3923023104000\pi^{19}}$$
$$+\frac{(-22317642547200\alpha_1^6+66952927641600\alpha_1^2+149847028531200)\pi^8}{3923023104000\pi^{19}}$$
$$+\frac{(44635285094400\alpha_1^6-223176425472000\alpha_1^4+312446995660800\alpha_1^2-269937200332800)\pi^7}{3923023104000\pi^{19}}$$
$$+\frac{(2678117105664000\alpha_1^4-5356234211328000\alpha_1^2-3035199386419200)\pi^6}{3923023104000\pi^{19}}$$
$$+\frac{(-5356234211328000\alpha_1^4+10712468422656000\alpha_1^2+6070398772838400)\pi^5+64274810535936000\pi^4}{3923023104000\pi^{19}}$$
$$+\frac{-771297726431232000\pi^2+2056793937149952000}{3923023104000\pi^{19}}\approx 5.200\times 10^{-10},$$

$$\lambda_{1,9}=\frac{(546\alpha_1^6-2730\alpha_1^4+3612\alpha_1^2+78219)\pi^{18}+(-240240\alpha_1^6+1201200\alpha_1^4-1550640\alpha_1^2-7478432)\pi^{16}}{1961511552000\pi^{20}}$$
$$+\frac{(69189120\alpha_1^6-345945600\alpha_1^4+426666240\alpha_1^2+240789120)\pi^{14}+(-11623772160\alpha_1^6+58118860800\alpha_1^4)\pi^{12}}{1961511552000\pi^{20}}$$
$$+\frac{(-64761016320\alpha_1^2+48709140480)\pi^{12}}{1961511552000\pi^{20}}$$
$$+\frac{(929901772800\alpha_1^6-4649508864000\alpha_1^4+3719607091200\alpha_1^2-7204524687360)\pi^{10}}{1961511552000\pi^{20}}$$
$$+\frac{(-22317642547200\alpha_1^6+111588212736000\alpha_1^4+66952927641600\alpha_1^2+373023454003200)\pi^8}{1961511552000\pi^{20}}$$
$$+\frac{(44635285094400\alpha_1^6-223176425472000\alpha_1^4+312446995660800\alpha_1^2-269937200332800)\pi^7}{1961511552000\pi^{20}}$$
$$+\frac{(-5356234211328000\alpha_1^2-9730492150579200)\pi^6}{1961511552000\pi^{20}}$$
$$+\frac{(10712468422656000\alpha_1^2+6070398772838400)\pi^5+112480918437888000\pi^4-771297726431232000\pi^2}{1961511552000\pi^{20}}$$
$$+\frac{2056793937149952000}{1961511552000\pi^{20}}\approx 2.671\times 10^{-10},$$

$$
\begin{aligned}
\lambda_{1,10} = {} & \frac{(-35\alpha_1^6+175\alpha_1^4-245\alpha_1^2-21959)\pi^{20}+(21840\alpha_1^6-109200\alpha_1^4+152880\alpha_1^2+3133520)\pi^{18}-9609600\alpha_1^6\pi^{16}}{39230231040000\pi^{21}} \\
& +\frac{(48048000\alpha_1^4-67267200\alpha_1^2-302107520)\pi^{16}}{39230231040000\pi^{21}} \\
& +\frac{(2767564800\alpha_1^6-13837824000\alpha_1^4+19372953600\alpha_1^2+10938470400)\pi^{14}}{39230231040000\pi^{21}} \\
& +\frac{(-464950886400\alpha_1^6+2324754432000\alpha_1^4-3254656204800\alpha_1^2+1571976806400)\pi^{12}+37196070912000\alpha_1^6\pi^{10}}{39230231040000\pi^{21}} \\
& +\frac{(-185980354560000\alpha_1^4+260372496384000\alpha_1^2-224947666944000)\pi^{10}-892705701888000\alpha_1^6\pi^8}{39230231040000\pi^{21}} \\
& +\frac{(4463528509440000\alpha_1^4-6248939913216000\alpha_1^2+9862272516096000)\pi^8+1785411403776000\alpha_1^6\pi^7}{39230231040000\pi^{21}} \\
& +\frac{(-8927057018880000\alpha_1^4+12497879826432000\alpha_1^2-10797488013312000)\pi^7-26781171056640000\pi^6}{39230231040000\pi^{21}} \\
& +\frac{449923673751552000\pi^4-3085190905724928000\pi^2+8227175748599808000}{39230231040000\pi^{21}} \approx -9.057\times 10^{-12}, \\
\lambda_{1,11} = {} & \frac{(-105\alpha_1^6-735\alpha_1^2-35672)\pi^{20}+(65520\alpha_1^6+458640\alpha_1^2+4118400)\pi^{18}+(-28828800\alpha_1^6-201801600\alpha_1^2)\pi^{16}}{58845346560000\pi^{22}} \\
& +\frac{-296524800\pi^{16}+8302694400\alpha_1^2(\alpha_1^4+7)\pi^{14}+(-1394852659200\alpha_1^6-9763968614400\alpha_1^2+3719607091200)\pi^{12}}{58845346560000\pi^{22}} \\
& +\frac{111588212736000\alpha_1^6+781117489152000\alpha_1^2-557941063680000\pi^{10}-2678117105664000\alpha_1^6\pi^8}{58845346560000\pi^{22}} \\
& +\frac{(-18746819739648000\alpha_1^2+36154580926464000)\pi^8+5356234211328000\alpha_1^2(\alpha_1^4+7)\pi^7-98019086067302400 0\pi^6}{58845346560000\pi^{22}} \\
& +\frac{1349771021254656000\pi^4-9255572717174784000\pi^2+24681527245799424000}{58845346560000\pi^{22}} \approx -3.783\times 10^{-12}, \\
\lambda_{1,12} = {} & \frac{365\pi^{22}-57624\pi^{20}+6652800\pi^{18}-479001600\pi^{16}+6008596070400\pi^{12}-901289410560000\pi^{10}}{47528933760000\pi^{23}} \\
& +\frac{58403553804288000\pi^8-1583385236471808000\pi^6+21803993420267520000\pi^4}{47528933760000\pi^{23}} \\
& +\frac{-149513097738977280000\pi^2+398701593970606080000}{47528933760000\pi^{23}} \approx 1.140\times 10^{-13}, \\
\lambda_{1,13} = {} & \frac{5\pi^{22}-528\pi^{20}+17107200\pi^{16}-4790016000\pi^{14}+703174348800\pi^{12}-54606182400000\pi^{10}}{848730960000\pi^{24}} \\
& +\frac{2263799881728000\pi^8-56549472731136000\pi^6+778714050723840000\pi^4}{848730960000\pi^{24}} \\
& +\frac{-5339753490677760000\pi^2+14239342641807360000}{848730960000\pi^{24}} \approx 3.481\times 10^{-14}, \\
\lambda_{1,14} = {} & \frac{-743\pi^{24}+120120\pi^{22}-12684672\pi^{20}+410983372800\pi^{16}-115075344384000\pi^{14}+16893060555571200\pi^{12}}{10194956291520000\pi^{25}} \\
& +\frac{-1311858925977600000\pi^{10}+54385528358633472000\pi^8-1358544532892811264000\pi^6}{10194956291520000\pi^{25}} \\
& +18707826354589532160000\pi^4+\frac{-128282237860042506240000\pi^2+342085967626780016640000}{10194956291520000\pi^{25}} \\
\approx {} & -1.035\times 10^{-15}, \\
\lambda_{1,15} = {} & \frac{-\pi^{24}+69696\pi^{20}-31363200\pi^{18}+8229703680\pi^{16}-1317254400000\pi^{14}+125318314598400\pi^{12}}{28008121680000\pi^{26}} \\
& +\frac{-7467504655564800\pi^{10}+298821584388096000\pi^8-7464530400509952000\pi^6+102790254695546880000\pi^4}{28008121680000\pi^{26}} \\
& +\frac{-704847460769464320000\pi^2+1879593228718571520000}{28008121680000\pi^{26}} \approx -2.343\times 10^{-16}, \\
\lambda_{1,16} = {} & \frac{5\pi^{26}-728\pi^{24}+50738688\pi^{20}-22832409600\pi^{18}+5991224279040\pi^{16}-958961203200000\pi^{14}}{10194956291520000\pi^{27}} \\
& +\frac{912317330276352000\pi^{12}-54363433892511744000\pi^{10}+2175421134345338880000\pi^8-54341781315712450560000\pi^6}{10194956291520000\pi^{27}}
\end{aligned}
$$

$$+\frac{7483130541835812864000\pi^4-5131289514401700249600000\pi^2+13683438705071200665600000}{10194956291520000\pi^{27}}$$

$$\approx 6.901\times 10^{-18},$$

$$\lambda_{1,17} = \frac{\pi^{12}-528\pi^{10}+190080\pi^8-42577920\pi^6+5109350400\pi^4-245248819200\pi^2+1961990553600}{\pi^{28}}$$

$$\times\frac{\pi^{12}-132\pi^{10}+11880\pi^8-665280\pi^6+19958400\pi^4-239500800\pi^2+958003200}{7002030420000}\approx 1.024\times 10^{-18},$$

$$\lambda_{1,18} = \frac{-\pi^{28}+132496\pi^{24}-87447360\pi^{22}+35993333376\pi^{20}-9885049574400\pi^{18}+1770015717550080\pi^{16}}{463870511264160000\pi^{29}}$$

$$+\frac{-207057159429120000\pi^{14}+16864258243011379200\pi^{12}-989414496843713740800\pi^{10}}{463870511264160000\pi^{29}}$$

$$+\frac{+39592664645085167616000\pi^8-989020419945966600192000\pi^6+13619297586141179412480000\pi^4}{463870511264160000\pi^{29}}$$

$$+\frac{-93389469162110944542720000\pi^2+249038584432295852113920000}{463870511264160000\pi^{29}}\approx -3.405\times 10^{-20},$$

$$\lambda_{1,19} = \frac{\pi^{14}-182\pi^{12}+24024\pi^{10}-2162160\pi^8+121080960\pi^6-3632428800\pi^4+43589145600\pi^2-174356582400}{231935255632080000}$$

$$\times\Big(\frac{\pi^{14}-728\pi^{12}+384384\pi^{10}-138378240\pi^8+30996725760\pi^6-3719607091200\pi^4}{\pi^{30}}$$

$$+\frac{178541140377600\pi^2-1428329123020800}{\pi^{30}}\Big)\approx 1.774\times 10^{-22},$$

$$\lambda_{2,0} = \frac{33\alpha_2^{10}-1386\alpha_2^6+5940\alpha_2^4-7887\alpha_2^2-6940}{59875200}\approx -6.1801\times 10^{-5},$$

$$\lambda_{2,1} = 0,$$

$$\lambda_{2,2} = \frac{-273\alpha_2^{12}+36036\alpha_2^6-165165\alpha_2^4+223860\alpha_2^2+660230}{65383718400}\approx 6.9274\times 10^{-6},$$

$$\lambda_{2,3} = \frac{\pi^{14}-728\pi^{12}+384384\pi^{10}-138378240\pi^8+30996725760\pi^6-3719607091200\pi^4}{43589145600\pi^{15}}$$

$$+\frac{178541140377600\pi^2-1428329123020800}{43589145600\pi^{15}}\approx 7.4474\times 10^{-14},$$

$$\lambda_{2,4} = \frac{(90\alpha_2^{14}-30030\alpha_2^6+141960\alpha_2^4-194460\alpha_2^2-1820149)\pi^{16}-1477260\pi^{14}+270073440\pi^{12}}{3923023104000\pi^{16}}$$

$$+\frac{-36185909760\pi^{10}+3412407398400\pi^8-217597014835200\pi^6+8703880593408000\pi^4}{3923023104000\pi^{16}}$$

$$+\frac{-160687026339840000\pi^2+128549621071872000\pi+514198484287488000}{3923023104000\pi^{16}}\approx -3.71\times 10^{-7},$$

$$\lambda_{2,5} = \frac{\alpha_2^2(\pi^{13}-624\pi^{11}+274560\pi^9-79073280\pi^7+13284311040\pi^5-1062744883200\pi^3)}{3113510400\pi^{14}}$$

$$+\frac{\alpha_2^2(25505877196800\pi-51011754393600)}{3113510400\pi^{14}}\approx 4.19\times 10^{-12},$$

$$\lambda_{2,6} = \frac{(182\alpha_2^6-875\alpha_2^4+1204\alpha_2^2+36323)\pi^{16}+(-240\alpha_2^{14}+489960)\pi^{14}+(174720\alpha_2^{12}-88495680)\pi^{12}}{2615348736000\pi^{16}}$$

$$+\frac{(-92252160\alpha_2^{10}+11393141760)\pi^{10}+(33210777600\alpha_2^8-946507161600)\pi^8-7439214182400\alpha_2^6\pi^6}{2615348736000\pi^{16}}$$

$$+\frac{40915678003200\pi^6+(892705701888000\alpha_2^4-446352850944000)\pi^4-42849873690624000\alpha_2^2\pi^2}{2615348736000\pi^{16}}$$

$$+\frac{85699747381248000\pi-342798989524992000\cos(\frac{\alpha_2\pi}{2})+85699747381248000}{2615348736000\pi^{16}}\approx 1.29\times 10^{-8},$$

$$\lambda_{2,7} = \frac{(\alpha_2^2+\frac{17}{30})(\pi^{13}-624\pi^{11}+274560\pi^9-79073280\pi^7+13284311040\pi^5-1062744883200\pi^3)}{18681062400\pi^{14}}$$

$$+\frac{(\alpha_2^2+\frac{17}{30})(25505877196800\pi-51011754393600)}{18681062400\pi^{14}}\approx 8.1127\times 10^{-13},$$

$$\lambda_{2,8} = \frac{(-35\alpha_2^6+175\alpha_2^4-245\alpha_2^2-21959)\pi^{16}+(-4200\alpha_2^4-1230000)\pi^{14}+(2620800\alpha_2^4+224515200)\pi^{12}}{78460462080000\pi^{16}}$$

$$
\begin{aligned}
&+\frac{(-1153152000\alpha_2^4-29981952000)\pi^{10}+(332107776000\alpha_2^4+2822916096000)\pi^8-55794106368000\alpha_2^4\pi^6}{78460462080000\pi^{16}}\\
&+\frac{-185980354560000\pi^6+(4463528509440000\alpha_2^4+8927057018880000)\pi^4-107124684226560000\alpha_2^4\pi^2}{78460462080000\pi^{16}}\\
&+\frac{-267811710566400000\pi^2+214249368453120000\alpha_2^4\pi+1928244316078080000}{78460462080000\pi^{16}}\approx -3.4669\times 10^{-10},\\
\lambda_{2,9} =\;& \frac{\alpha_2^2(\alpha_2^4+7)(\pi^{13}-624\pi^{11}+274560\pi^9-79073280\pi^7+13284311040\pi^5-1062744883200\pi^3)}{1120863744000\pi^{14}}\\
&+\frac{\alpha_2^2(\alpha_2^4+7)(25505877196800\pi-51011754393600)}{1120863744000\pi^{14}}\approx 2.2605\times 10^{-13},\\
\lambda_{2,10} =\;& \frac{949\pi^{16}+(-2205\alpha_2^4+126861)\pi^{14}+(1375920\alpha_2^4-22185072)\pi^{12}+(-605404800\alpha_2^4+2561150592)\pi^{10}}{247150455552000\pi^{16}}\\
&+\frac{(174356582400\alpha_2^4-137824727040)\pi^8+(-29291905843200\alpha_2^4-4184557977600)\pi^6+2343352467456000\alpha_2^4\pi^4}{247150455552000\pi^{16}}\\
&+\frac{490988136038400\pi^4+(-56240459218944000\alpha_2^4+27584606188339200)\pi^2+112480918437888000\alpha_2^4\pi}{247150455552000\pi^{16}}\\
&+\frac{136048348967731200\pi-742374061690060800}{247150455552000\pi^{16}}\approx 5.8053\times 10^{-12},\\
\lambda_{2,11} =\;& 0,\\
\lambda_{2,12} =\;& \frac{-743\pi^{16}-193050\pi^{14}+36756720\pi^{12}-5708102400\pi^{10}+821966745600\pi^8-115075344384000\pi^6}{20389912583040000\pi^{16}}\\
&+\frac{11737685127168000\pi^4-538552611717120000\pi^2+4275279194554368000}{20389912583040000\pi^{16}}\approx -7.4211\times 10^{-14},\\
\lambda_{2,13} =\;& 0,\\
\lambda_{2,14} =\;& \frac{5\pi^{16}+2244\pi^{14}-480480\pi^{12}+101477376\pi^{10}-22832409600\pi^8+4347290787840\pi^6}{20389912583040000\pi^{16}}\\
&+\frac{-498659825664000\pi^4+23659490805350400\pi^2-188907685340774400}{20389912583040000\pi^{16}}\approx 6.6992\times 10^{-16},\\
\lambda_{2,15} =\;& 0,\\
\lambda_{2,16} =\;& \frac{-\pi^{16}-910\pi^{14}+264992\pi^{12}-87447360\pi^{10}+26758892160\pi^8-5729551027200\pi^6}{927741022528320000\pi^{16}}\\
&+\frac{679612898764800\pi^4-32526220446720000\pi^2+260082831981772800}{927741022528320000\pi^{16}}\approx -4.4474\times 10^{-18},\\
\lambda_{2,17} =\;& 0,\\
\lambda_{2,18} =\;& \frac{\pi^{14}-455\pi^{12}+204204\pi^{10}-70270200\pi^8+15558903360\pi^6}{115967627816040000\pi^{16}}\\
&+\frac{-1861619760000\pi^4+89292364761600\pi^2-714251739801600}{115967627816040000\pi^{16}}\approx 2.1855\times 10^{-20},\\
\lambda_{2,19} =\;& 0,\\
\lambda_{2,20} =\;& -\Big(\frac{\pi^{14}-182\pi^{12}+24024\pi^{10}-2162160\pi^8+121080960\pi^6-3632428800\pi^4}{57983813908020000}\\
&+\frac{43589145600\pi^2-174356582400}{57983813908020000}\Big)\times\Big(\frac{\pi^{14}-728\pi^{12}+384384\pi^{10}-138378240\pi^8+30996725760\pi^6}{\pi^{32}}\\
&+\frac{-3719607091200\pi^4+178541140377600\pi^2-142832912302080}{\pi^{32}}\Big)\approx -7.1919\times 10^{-23},\\
\lambda_{3,0} =\;& \frac{-1365\alpha_3^{12}+135135\alpha_3^8-1081080\alpha_3^6+3588585\alpha_3^4-4466280\alpha_3^2+2523373}{326918592000}\approx 3.257\times 10^{-6},\\
\lambda_{3,1} =\;& \frac{-8207\pi^{14}+1500408\pi^{12}-201032832\pi^{10}+18957818880\pi^8-1208872304640\pi^6+48354892185600\pi^4}{43589145600\pi^{15}},\\
&+\frac{-892705701888000\pi^2+714164561510400\pi+2856658246041600}{43589145600\pi^{15}}\approx -2.384\times 10^{-9},\\
\lambda_{3,2} =\;& \frac{(90\alpha_3^{14}-38610\alpha_3^8+330330\alpha_3^6-1119300\alpha_3^4+1401420\alpha_3^2-1597927)\pi^{16}+180\pi^{14}-131040\pi^{12}+69189120\pi^{10}}{3923023104000\pi^{16}}\\
&+\frac{-24908083200\pi^8+5579410636800\pi^6-669529276416000\pi^4+32137405267968000\pi^2-257099242143744000}{3923023104000\pi^{16}}
\end{aligned}
$$

$$
\begin{aligned}
&\approx -2.904\times 10^{-7},\\
\lambda_{3,3} &= -\frac{(-2\alpha_3^{14}+4083)\pi^{14}+(1456\alpha_3^{12}-737464)\pi^{12}+(-768768\alpha_3^{10}+94942848)\pi^{10}+276756480\alpha_3^{8}\pi^{8}}{43589145600\pi^{15}}\\
&\quad-\frac{-7887559680\pi^{8}+(-61993451520\alpha_3^{6}+340963983360)\pi^{6}+(7439214182400\alpha_3^{4}-3719607091200)\pi^{4}}{43589145600\pi^{15}}\\
&\quad-\frac{-357082280755200\alpha_3^{2}\pi^{2}+714164561510400\pi-2856658246041600\cos(\frac{\alpha_3\pi}{2})+714164561510400}{43589145600\pi^{15}}\\
&\approx 1.181\times 10^{-9},\\
\lambda_{3,4} &= \frac{(6435\alpha_3^{8}-56784\alpha_3^{6}+194460\alpha_3^{4}-244308\alpha_3^{2}+616777)\pi^{15}+30240\alpha_3^{2}\pi^{13}-18869760\alpha_3^{2}\pi^{11}+8302694400\alpha_3^{2}\pi^{9}}{47076277248000\pi^{15}}\\
&\quad+\frac{-2391175987200\alpha_3^{2}\pi^{7}+401717565849600\alpha_3^{2}\pi^{5}-32137405267968000\alpha_3^{2}\pi^{3}+771297726431232000\alpha_3^{2}\pi}{47076277248000\pi^{15}}\\
&\quad+\frac{-1542595452862464000\alpha_3^{2}}{47076277248000\pi^{15}}\approx 1.140\times 10^{-8},\\
\lambda_{3,5} &= \frac{(-7\alpha_3^{4}-2050)\pi^{14}+(4368\alpha_3^{4}+374192)\pi^{12}+(-1921920\alpha_3^{4}-49969920)\pi^{10}+(553512960\alpha_3^{4}+4704860160)\pi^{8}}{261534873600\pi^{15}}\\
&\quad+\frac{(-92990177280\alpha_3^{4}-309967257600)\pi^{6}+(7439214182400\alpha_3^{4}+14878428364800)\pi^{4}-178541140377600\alpha_3^{4}\pi^{2}}{261534873600\pi^{15}}\\
&\quad+\frac{-446352850944000\pi^{2}+357082280755200\alpha_3^{4}\pi+3213740526796800}{261534873600\pi^{15}}\approx -9.958\times 10^{-11},\\
\lambda_{3,6} &= \frac{(-585\alpha_3^{8}+5250\alpha_3^{6}-18060\alpha_3^{4}+22710\alpha_3^{2}-128387)\pi^{15}+(50400\alpha_3^{2}+28560)\pi^{13}+(-31449600\alpha_3^{2}-17821440)\pi^{11}}{470762772480000\pi^{15}}\\
&\quad+\frac{(13837824000\alpha_3^{2}+7841433600)\pi^{9}+(-3985293312000\alpha_3^{2}-2258332876800)\pi^{7}+669529276416000\alpha_3^{2}\pi^{5}}{470762772480000\pi^{15}}\\
&\quad+\frac{379399923302400\pi^{5}+(-53562342113280000\alpha_3^{2}-30351993864192000)\pi^{3}+1285496210718720000\alpha_3^{2}\pi}{470762772480000\pi^{15}}\\
&\quad+\frac{728447852740608000\pi-2570992421437440000\alpha_3^{2}-1456895705481216000}{470762772480000\pi^{15}}\approx -2.566\times 10^{-10},\\
\lambda_{3,7} &= \frac{(-105\alpha_3^{4}+6041)\pi^{14}+(65520\alpha_3^{4}-1056432)\pi^{12}+(-28828800\alpha_3^{4}+121959552)\pi^{10}+8302694400\alpha_3^{4}\pi^{8}}{23538138624000\pi^{15}}\\
&\quad+\frac{-6563082240\pi^{8}+(-1394852659200\alpha_3^{4}-199264665600)\pi^{6}+(111588212736000\alpha_3^{4}+23380387430400)\pi^{4}}{23538138624000\pi^{15}}\\
&\quad+\frac{(-2678117105664000\alpha_3^{4}+1313552675635200)\pi^{2}+(5356234211328000\alpha_3^{4}+6478492807987200)\pi}{23538138624000\pi^{15}}\\
&\quad-\frac{35351145794764800}{23538138624000\pi^{15}}\approx 3.246\times 10^{-12},\\
\lambda_{3,8} &= \frac{105\alpha_3^{8}\pi^{15}+3430\alpha_3^{4}\pi^{15}+50009\pi^{15}+(-980\pi^{15}+23520\pi^{13}-14676480\pi^{11}+6457651200\pi^{9}-1859803545600\pi^{7})\alpha_3^{6}}{13181357629440000\pi^{15}}\\
&\quad+\frac{(+312446995660800\pi^{5}-24995759652864000\pi^{3}+599898231668736000\pi-1199796463337472000)\alpha_3^{6}}{13181357629440000\pi^{15}}\\
&\quad+\frac{(-4340\pi^{15}+164640\pi^{13}-102735360\pi^{11}+45203558400\pi^{9}-13018624819200\pi^{7}+2187128969625600\pi^{5})\alpha_3^{2}}{13181357629440000\pi^{15}}\\
&\quad+\frac{(-174970317570048000\pi^{3}+4199287621681152000\pi-8398575243362304000)\alpha_3^{2}}{13181357629440000\pi^{15}}\approx 3.720\times 10^{-12},\\
\lambda_{3,9} &= \frac{(-21\alpha_3^{8}-686\alpha_3^{4}-6240)\pi^{14}+(13104\alpha_3^{8}+428064\alpha_3^{4}+1188096)\pi^{12}}{1318135762944000\pi^{15}}\\
&\quad+\frac{(-5765760\alpha_3^{8}-188348160\alpha_3^{4}-184504320)\pi^{10}+(1660538880\alpha_3^{8}+54244270080\alpha_3^{4}+26568622080)\pi^{8}}{1318135762944000\pi^{15}}\\
&\quad+\frac{(-278970531840\alpha_3^{8}-9113037373440\alpha_3^{4}-3719607091200)\pi^{6}}{1318135762944000\pi^{15}}\\
&\quad+\frac{(22317642547200\alpha_3^{8}+729042989875200\alpha_3^{4}+379399923302400)\pi^{4}}{1318135762944000\pi^{15}}\\
&\quad+\frac{(-535623421132800\alpha_3^{8}-17497031757004800\alpha_3^{4})\pi^{2}}{1318135762944000\pi^{15}}
\end{aligned}
$$

$$+\frac{-17407761186816000\pi^2+1071246842265600\alpha_3^4(\alpha_3^4+\frac{98}{3})\pi+138190842652262400}{1318135762944000\pi^{15}}\approx -6.522\times 10^{-14},$$

$$\begin{aligned}
\lambda_{3,10} &= \frac{-11888\pi^{15}+(97020\alpha_3^6+429660\alpha_3^2+59829)\pi^{13}+(-60540480\alpha_3^6-268107840\alpha_3^2-37333296)\pi^{11}}{326238601328640000\pi^{15}}\\
&+\frac{(26637811200\alpha_3^6+117967449600\alpha_3^2+16426650240)\pi^9+(-7671689625600\alpha_3^6-33974625484800\alpha_3^2)\pi^7}{326238601328640000\pi^{15}}\\
&+\frac{-4730875269120\pi^7+(1288843857100800\alpha_3^6+5707737081446400\alpha_3^2+794787045212160)\pi^5}{326238601328640000\pi^{15}}\\
&+\frac{(-103107508568064000\alpha_3^6-456618966515712000\alpha_3^2-63582963616972800)\pi^3+2474580205633536000\alpha_3^6\pi}{326238601328640000\pi^{15}}\\
&+\frac{(10958855196377088000\alpha_3^2+1525991126807347200)\pi-49491604112670720000\alpha_3^6}{326238601328640000\pi^{15}}\\
&+\frac{-219177103927541760000\alpha_3^2-30519822536146944000}{326238601328640000\pi^{15}}\approx -3.196\times 10^{-14},\\
\lambda_{3,11} &= \frac{17\pi^{14}-3640\pi^{12}+768768\pi^{10}-172972800\pi^8+32934021120\pi^6-3777725952000\pi^4}{308938069440000\pi^{15}}\\
&+\frac{179238566707200\pi^2-1431118828339200}{308938069440000\pi^{15}}\approx 6.671\times 10^{-16},\\
\lambda_{3,12} &= \frac{1}{4077982516608000}\approx 2.452\times 10^{-16},\\
\lambda_{3,13} &= \frac{-5\pi^{14}+1456\pi^{12}-480480\pi^{10}+147026880\pi^8-31481049600\pi^6+3734136806400\pi^4}{10194956291520000\pi^{15}}\\
&+\frac{-178715496960000\pi^2+1429026549350400}{10194956291520000\pi^{15}}\approx -5.292\times 10^{-18},\\
\lambda_{3,14} &= -\frac{1}{927741022528320000}\approx -1.077\times 10^{-18},\\
\lambda_{3,15} &= \frac{\pi^{14}-455\pi^{12}+204204\pi^{10}-70270200\pi^8+15558903360\pi^6-1861619760000\pi^4}{231935255632080000\pi^{15}}\\
&+\frac{89292364761600\pi^2-714251739801600}{231935255632080000\pi^{15}}\approx 3.433\times 10^{-20},\\
\lambda_{3,16} &= -\frac{\pi^{14}-182\pi^{12}+24024\pi^{10}-2162160\pi^8+121080960\pi^6-3632428800\pi^4+43589145600\pi^2-174356582400}{231935255632080000}\\
&\times(\frac{\pi^{14}-728\pi^{12}+384384\pi^{10}-138378240\pi^8+30996725760\pi^6-3719607091200\pi^4}{\pi^{30}}\\
&+\frac{178541140377600\pi^2-1428329123020800}{\pi^{30}})\approx -1.774\times 10^{-22},\\
\lambda_{4,0} &= \frac{33\alpha_4^{10}-495\alpha_4^8+3234\alpha_4^6-10230\alpha_4^4+12573\alpha_4^2-4091}{29937600\pi}\approx -4.825\times 10^{-6},\\
\lambda_{4,1} &= \frac{33\alpha_4^{10}-495\alpha_4^8+3234\alpha_4^6-10230\alpha_4^4+12573\alpha_4^2-4091}{14968800\pi^2}\approx -3.072\times 10^{-6},\\
\lambda_{4,2} &= \frac{720720\alpha_4^{10}-10810800\alpha_4^8+70630560\alpha_4^6-223423200\alpha_4^4+274594320\alpha_4^2-89347440}{163459296000\pi^3}\\
&+\frac{-1365\alpha_4^{12}+135135\alpha_4^8-1081080\alpha_4^6+3588585\alpha_4^4-4466280\alpha_4^2+2523373}{163459296000\pi}\approx 4.341\times 10^{-7},\\
\lambda_{4,3} &= \frac{(-5460\alpha_4^{12}+540540\alpha_4^8-4324320\alpha_4^6+14354340\alpha_4^4-17865120\alpha_4^2+10093507)\pi^{14}}{326918592000\pi^{16}}\\
&+\frac{(2882880\alpha_4^{10}-43243200\alpha_4^8+282522240\alpha_4^6-893692800\alpha_4^4+1098377280\alpha_4^2-357400680)\pi^{12}+5765760\pi^{10}}{326918592000\pi^{16}}\\
&+\frac{-2075673600\pi^8+464950886400\pi^6-55794106368000\pi^4+2678117105664000\pi^2-21424936845312000}{326918592000\pi^{16}}\\
&\approx 2.764\times 10^{-7},\\
\lambda_{4,4} &= \frac{(90\alpha_4^{14}-38610\alpha_4^8+330330\alpha_4^6-1119300\alpha_4^4+1401420\alpha_4^2-1597927)\pi^{16}}{1961511552000\pi^{17}}\\
&+\frac{(-65520\alpha_4^{12}+6486480\alpha_4^8-51891840\alpha_4^6+172252080\alpha_4^4-214381440\alpha_4^2+119644824)\pi^{14}}{1961511552000\pi^{17}}
\end{aligned}$$

$$+\frac{(34594560\alpha_4^{10}-518918400\alpha_4^8+3390266880\alpha_4^6-10724313600\alpha_4^4+13180527360\alpha_4^2-4018734720)\pi^{12}}{1961511552000\pi^{17}}$$

$$+\frac{-36116720640\pi^{10}+3387499315200\pi^8-212017604198400\pi^6+8034351316992000\pi^4-128549621071872000\pi^2}{1961511552000\pi^{17}}$$

$$+\frac{128549621071872000\pi+257099242143744000}{1961511552000\pi^{17}}\approx -1.820\times 10^{-8},$$

$$\lambda_{4,5}=\frac{(90\alpha_4^{14}-38610\alpha_4^8+330330\alpha_4^6-1119300\alpha_4^4+1402050\alpha_4^2-1597927)\pi^{16}-65520\alpha_4^{12}\pi^{14}}{980755776000\pi^{18}}$$

$$+\frac{(6486480\alpha_4^8-51891840\alpha_4^6+172252080\alpha_4^4-214774560\alpha_4^2+119644824)\pi^{14}+(34594560\alpha_4^{10}-518918400\alpha_4^8)\pi^{12}}{980755776000\pi^{18}}$$

$$+\frac{(3390266880\alpha_4^6-10724313600\alpha_4^4+13353500160\alpha_4^2-4018734720)\pi^{12}+(-49816166400\alpha_4^2-36116720640)\pi^{10}}{980755776000\pi^{18}}$$

$$+\frac{(8369115955200\alpha_4^2+3387499315200)\pi^8+(-669529276416000\alpha_4^2-212017604198400)\pi^6}{980755776000\pi^{18}}$$

$$+\frac{(16068702633984000\alpha_4^2+8034351316992000)\pi^4-32137405267968000\alpha_4^2\pi^3-128549621071872000\pi^2}{980755776000\pi^{18}}$$

$$+\frac{128549621071872000\pi+257099242143744000}{980755776000\pi^{18}}\approx -1.158\times 10^{-8},$$

$$\lambda_{4,6}=\frac{(6435\alpha_4^8-56784\alpha_4^6+194460\alpha_4^4-244308\alpha_4^2+616777)\pi^{18}+(-1853280\alpha_4^8+15855840\alpha_4^6-53726400\alpha_4^4)\pi^{16}}{23538138624000\pi^{19}}$$

$$+\frac{(67298400\alpha_4^2-67881216)\pi^{16}+(311351040\alpha_4^8-2490808320\alpha_4^6+8268099840\alpha_4^4-10309178880\alpha_4^2+4150029312)\pi^{14}}{23538138624000\pi^{19}}$$

$$+\frac{(-24908083200\alpha_4^8+162732810240\alpha_4^6-514767052800\alpha_4^4+640968007680\alpha_4^2+12177285120)\pi^{12}}{23538138624000\pi^{19}}$$

$$+\frac{(597793996800\alpha_4^8-2391175987200\alpha_4^2-18770731499520)\pi^{10}+(-1195587993600\alpha_4^8+11158821273600\alpha_4^6)\pi^9}{23538138624000\pi^{19}}$$

$$+\frac{(-39055874457600\alpha_4^4+49417637068800\alpha_4^2+6881273118720)\pi^9}{23538138624000\pi^{19}}$$

$$+\frac{(-133905855283200\alpha_4^6+401717565849600\alpha_4^2+899082171187200)\pi^8}{23538138624000\pi^{19}}$$

$$+\frac{(267811710566400\alpha_4^6-1339058552832000\alpha_4^4+1874681973964800\alpha_4^2-1619623201996800)\pi^7}{23538138624000\pi^{19}}$$

$$+\frac{(16068702633984000\alpha_4^4-32137405267968000\alpha_4^2-18211196318515200)\pi^6-32137405267968000\alpha_4^4\pi^5}{23538138624000\pi^{19}}$$

$$+\frac{(64274810535936000\alpha_4^2+36422392637030400)\pi^5+38564886321561600\pi^4}{23538138624000\pi^{19}}$$

$$+\frac{-462778635858739200\pi^2+1234076362289971200}{23538138624000\pi^{19}}\approx 4.838\times 10^{-10},$$

$$\lambda_{4,7}=\frac{(6435\alpha_4^8-56784\alpha_4^6+194460\alpha_4^4-243048\alpha_4^2+617491)\pi^{18}}{11769069312000\pi^{20}}$$

$$+\frac{(-1853280\alpha_4^8+15855840\alpha_4^6-53726400\alpha_4^4+66512160\alpha_4^2-68326752)\pi^{16}}{11769069312000\pi^{20}}$$

$$+\frac{(311351040\alpha_4^8-2490808320\alpha_4^6+8268099840\alpha_4^4-9963233280\alpha_4^2+4346065152)\pi^{14}}{11769069312000\pi^{20}}$$

$$+\frac{(-24908083200\alpha_4^8 162732810240\alpha_4^6-514767052800\alpha_4^4+541335674880\alpha_4^2-44281036800)\pi^{12}}{11769069312000\pi^{20}}$$

$$+\frac{(597793996800\alpha_4^8+14347055923200\alpha_4^2-9285733416960)\pi^{10}}{11769069312000\pi^{20}}$$

$$+\frac{(-1195587993600\alpha_4^8+11158821273600\alpha_4^6-39055874457600\alpha_4^4+49417637068800\alpha_4^2+6881273118720)\pi^9}{11769069312000\pi^{20}}$$

$$+\frac{(-133905855283200\alpha_4^6-937340986982400\alpha_4^2+140282324582400)\pi^8}{11769069312000\pi^{20}}$$

$$+\frac{(267811710566400\alpha_4^6-1339058552832000\alpha_4^4+1874681973964800\alpha_4^2-1619623201996800)\pi^7}{11769069312000\pi^{20}}$$

$$+\frac{16068702633984000\alpha_4^4\pi^6-32137405267968000\alpha_4^4\pi^5}{11769069312000\pi^{20}}$$

$$+\frac{385648863215616000\pi^4-4627786358587392000\pi^2+12340763622899712000}{11769069312000\pi^{20}}\approx 3.083\times 10^{-10},$$

$$\lambda_{4,8} = \frac{(-585\alpha_4^8+5250\alpha_4^6-18060\alpha_4^4+22710\alpha_4^2-128387)\pi^{20}}{235381386240000\pi^{21}}$$
$$+\frac{(257400\alpha_4^8-2271360\alpha_4^6+7753200\alpha_4^4-9721920\alpha_4^2+17319640)\pi^{18}}{235381386240000\pi^{21}}$$
$$+\frac{(-74131200\alpha_4^8+634233600\alpha_4^6-2133331200\alpha_4^4+2660486400\alpha_4^2-1385978880)\pi^{16}}{235381386240000\pi^{21}}$$
$$+\frac{(12454041600\alpha_4^8-99632332800\alpha_4^6+323805081600\alpha_4^4-398529331200\alpha_4^2-6049105920)\pi^{14}}{235381386240000\pi^{21}}$$
$$+\frac{(-996323328000\alpha_4^8+6509312409600\alpha_4^6-18598035456000\alpha_4^4+21653426995200\alpha_4^2+15166255104000)\pi^{12}}{235381386240000\pi^{21}}$$
$$+\frac{(23911759872000\alpha_4^8-334764638208000\alpha_4^4+573882236928000\alpha_4^2-1487311464038400)\pi^{10}}{235381386240000\pi^{21}}$$
$$+\frac{(-47823519744000\alpha_4^8+446352850944000\alpha_4^6-1562234978304000\alpha_4^4+1976705482752000\alpha_4^2+275250924748800)\pi^9}{235381386240000\pi^{21}}$$
$$+\frac{(-5356234211328000\alpha_4^6+26781171056640000\alpha_4^4-37493639479296000\alpha_4^2+59173635096576000)\pi^8}{235381386240000\pi^{21}}$$
$$+\frac{(10712468422656000\alpha_4^6-53562342113280000\alpha_4^4+74987278958592000\alpha_4^2-64784928079872000)\pi^7}{235381386240000\pi^{21}}$$
$$+\frac{-160687026339840000\pi^6-2699542042509312000\pi^4-18511145434349568000\pi^2+49363054491598848000}{235381386240000\pi^{21}}$$
$$\approx -8.830\times 10^{-12},$$

$$\lambda_{4,9} = \frac{(-585\alpha_4^8+5460\alpha_4^6-18060\alpha_4^4+24180\alpha_4^2-128387)\pi^{20}}{117690693120000\pi^{22}}$$
$$+\frac{(257400\alpha_4^8-2402400\alpha_4^6+7753200\alpha_4^4-10639200\alpha_4^2+17319640)\pi^{18}}{117690693120000\pi^{22}}$$
$$+\frac{(-74131200\alpha_4^8+691891200\alpha_4^6-2133331200\alpha_4^4+3064089600\alpha_4^2-1385978880)\pi^{16}}{117690693120000\pi^{22}}$$
$$+\frac{(12454041600\alpha_4^8-116237721600\alpha_4^6+323805081600\alpha_4^4-514767052800\alpha_4^2-6049105920)\pi^{14}}{117690693120000\pi^{22}}$$
$$+\frac{(-996323328000\alpha_4^8+9299017728000\alpha_4^6-18598035456000\alpha_4^4+41181364224000\alpha_4^2+15166255104000)\pi^{12}}{117690693120000\pi^{22}}$$
$$+\frac{(23911759872000\alpha_4^8-223176425472000\alpha_4^6-334764638208000\alpha_4^4-988352741376000\alpha_4^2-1487311464038400)\pi^{10}}{117690693120000\pi^{22}}$$
$$+\frac{(-47823519744000\alpha_4^8+446352850944000\alpha_4^6-1562234978304000\alpha_4^4+1976705482752000\alpha_4^2+275250924748800)\pi^9}{117690693120000\pi^{22}}$$
$$+\frac{(26781171056640000\alpha_4^4+59173635096576000)\pi^8-53562342113280000\alpha_4^4\pi^7}{117690693120000\pi^{22}}$$
$$+\frac{-64784928079872000\pi^7-160687026339840000\pi^6+2699542042509312000\pi^4-18511145434349568000\pi^2}{117690693120000\pi^{22}}$$
$$+\frac{49363054491598848000}{117690693120000\pi^{22}}\approx -5.580\times 10^{-12},$$

$$\lambda_{4,10} = \frac{(105\alpha_4^8-980\alpha_4^6+3430\alpha_4^4-4340\alpha_4^2+50009)\pi^{22}+(-65520\alpha_4^8+611520\alpha_4^6-2140320\alpha_4^4+2708160\alpha_4^2-7613424)\pi^{20}}{6590678814720000\pi^{23}}$$
$$+\frac{(28828800\alpha_4^8-269068800\alpha_4^6+941740800\alpha_4^4-1191590400\alpha_4^2+756595840)\pi^{18}}{6590678814720000\pi^{23}}$$
$$+\frac{(-8302694400\alpha_4^8+77491814400\alpha_4^6-271221350400\alpha_4^4+343178035200\alpha_4^2-18634936320)\pi^{16}}{6590678814720000\pi^{23}}$$
$$+\frac{(1394852659200\alpha_4^8-13018624819200\alpha_4^6+45565186867200\alpha_4^4-57653909913600\alpha_4^2-8028151971840)\pi^{14}}{6590678814720000\pi^{23}}$$
$$+\frac{(-111588212736000\alpha_4^8+1041489985536000\alpha_4^6-3645214949376000\alpha_4^4+4612312793088000\alpha_4^2)\pi^{12}}{6590678814720000\pi^{23}}$$
$$+\frac{1475444146176000\pi^{12}+(2678117105664000\alpha_4^8-24995759652864000\alpha_4^6)\pi^{10}}{6590678814720000\pi^{23}}$$

$$
\begin{aligned}
&+\frac{(87485158785024000\alpha_4^4-110695507034112000\alpha_4^2-140392850050252800)\pi^{10}}{6590678814720000\pi^{23}}\\
&+\frac{(-5356234211328000\alpha_4^8+49991519305728000\alpha_4^6-174970317570048000\alpha_4^4+221391014068224000\alpha_4^2)\pi^9}{6590678814720000\pi^{23}}\\
&+\frac{30828103571865600\pi^9+809862612752793600 0\pi^8-21956275279075737600 0\pi^6+302348708761042944000 0\pi^4}{6590678814720000\pi^{23}}\\
&+\frac{-2073248288647151616000 0\pi^2+5528662103059070976000 0}{6590678814720000\pi^{23}}\approx 1.129\times 10^{-13},\\
\lambda_{4,11} =\ &\frac{(315\alpha_4^8+10290\alpha_4^4+151840)\pi^{22}+(-196560\alpha_4^8-6420960\alpha_4^4-23971584)\pi^{20}+(86486400\alpha_4^8+2825222400\alpha_4^4)\pi^{18}}{9886018222080000\pi^{24}}\\
&+\frac{2767564800\pi^{18}+(-24908083200\alpha_4^8-813664051200\alpha_4^4-199264665600)\pi^{16}+(4184557977600(\alpha_4^4+\frac{98}{3}))\alpha_4^4\pi^{14}}{9886018222080000\pi^{24}}\\
&+\frac{(-334764638208000\alpha_4^8-10935644848128000\alpha_4^4+2499575965286400)\pi^{12}+8034351316992000\alpha_4^8\pi^{10}}{9886018222080000\pi^{24}}\\
&+\frac{(262455476355072000\alpha_4^4-374936394792960000)\pi^{10}-(16068702633984000(\alpha_4^4+\frac{98}{3}))\alpha_4^4\pi^9}{9886018222080000\pi^{24}}\\
&+\frac{2429587838258380800 0\pi^8-65868825837227212800 0\pi^6+907046126283128832000 0\pi^4}{9886018222080000\pi^{24}}\\
&+\frac{-6219744865941454848000 0\pi^2+1658598630917721292800 00}{9886018222080000\pi^{24}}\approx 7.723\times 10^{-14},\\
\lambda_{4,12} =\ &\frac{-743\pi^{24}+120120\pi^{22}-12684672\pi^{20}+410983372800\pi^{16}-115075344384000\pi^{14}+16893060555571200\pi^{12}}{10194956291520000\pi^{25}}\\
&+\frac{-1311858925977600000\pi^{10}+54385528358633472000\pi^8-1358544532892811264000\pi^6}{10194956291520000\pi^{25}}\\
&+\frac{18707826354589532160000\pi^4-128282237860042506240000\pi^2+342085967626780016640000}{10194956291520000\pi^{25}}\\
\approx\ &-1.035\times 10^{-15},\\
\lambda_{4,13} =\ &\frac{-743\pi^{24}+120120\pi^{22}-12684672\pi^{20}+410983372800\pi^{16}-115075344384000\pi^{14}+16893060555571200\pi^{12}}{5097478145760000\pi^{26}}\\
&+\frac{-1311858925977600000\pi^{10}+54385528358633472000\pi^8-1358544532892811264000\pi^6}{5097478145760000\pi^{26}}\\
&+\frac{+18707826354589532160000\pi^4-128282237860042506240000\pi^2+342085967626780016640000}{5097478145760000\pi^{26}}\\
\approx\ &-6.590\times 10^{-16},\\
\lambda_{4,14} =\ &\frac{5\pi^{26}-728\pi^{24}+50738688\pi^{20}-22832409600\pi^{18}+5991224279040\pi^{16}-958961203200000\pi^{14}}{10194956291520000\pi^{27}}\\
&+\frac{91231733027635200\pi^{12}-5436343389251174400\pi^{10}+217542113434533888000\pi^8-5434178131571245056000\pi^6}{10194956291520000\pi^{27}}\\
&+\frac{74831305418358128640000\pi^4-513128951440170024960000\pi^2+1368343870507120066560000}{10194956291520000\pi^{27}}\\
\approx\ &6.901\times 10^{-18},\\
\lambda_{4,15} =\ &\frac{5\pi^{26}-728\pi^{24}+50738688\pi^{20}-22832409600\pi^{18}+5991224279040\pi^{16}-958961203200000\pi^{14}}{5097478145760000\pi^{28}}\\
&+\frac{91231733027635200\pi^{12}-5436343389251174400\pi^{10}+217542113434533888000\pi^8-5434178131571245056000\pi^6}{5097478145760000\pi^{28}}\\
&+\frac{74831305418358128640000\pi^4-513128951440170024960000\pi^2+1368343870507120066560000}{5097478145760000\pi^{28}}\\
\approx\ &4.393\times 10^{-18},\\
\lambda_{4,16} =\ &\frac{-\pi^{28}+132496\pi^{24}-87447360\pi^{22}+35993333376\pi^{20}-9885049574400\pi^{18}+1770015717550080\pi^{16}}{463870511264160000\pi^{29}}\\
&+\frac{-207057159429120000\pi^{14}+16864258243011379200\pi^{12}-989414496843713740800\pi^{10}}{463870511264160000\pi^{29}}\\
&+\frac{39592664645085167616000\pi^8-989020419945966600192000\pi^6+13619297586141179412480000\pi^4}{463870511264160000\pi^{29}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{-9338946916211094454272000\pi^2+249038584432295852113920000}{463870511264160000\pi^{29}} \approx -3.405\times 10^{-20},\\
\lambda_{4,17} &= \frac{-\pi^{28}+132496\pi^{24}-87447360\pi^{22}+35993333376\pi^{20}-9885049574400\pi^{18}+1770015717550080\pi^{16}}{231935255632080000\pi^{30}}\\
&+\frac{-207057159429120000\pi^{14}+16864258243011379200\pi^{12}-989414496843713740800\pi^{10}}{231935255632080000\pi^{30}}\\
&+\frac{39592664645085167616000\pi^{8}-989020419945966600192000\pi^{6}+13619297586141179412480000\pi^{4}}{231935255632080000\pi^{30}}\\
&+\frac{-9338946916211094454272000\pi^2+249038584432295852113920000}{231935255632080000\pi^{30}} \approx -2.167\times 10^{-20},\\
\lambda_{4,18} &= \frac{\pi^{14}-182\pi^{12}+24024\pi^{10}-2162160\pi^{8}+121080960\pi^{6}-3632428800\pi^{4}+43589145600\pi^{2}-174356582400}{\pi^{31}}\\
&\times\Big(\frac{\pi^{14}-728\pi^{12}+384384\pi^{10}-138378240\pi^{8}+30996725760\pi^{6}-3719607091200\pi^{4}+178541140377600\pi^{2}}{115967627816040000}\\
&-\frac{1428329123020800}{115967627816040000}\Big) \approx 1.129\times 10^{-22},\\
\lambda_{4,19} &= \frac{\pi^{14}-182\pi^{12}+24024\pi^{10}-2162160\pi^{8}+121080960\pi^{6}-3632428800\pi^{4}+43589145600\pi^{2}-174356582400}{57983813908020000}\\
&\times\Big(\frac{\pi^{14}-728\pi^{12}+384384\pi^{10}-138378240\pi^{8}+30996725760\pi^{6}-3719607091200\pi^{4}}{\pi^{32}}\\
&\times\frac{178541140377600\pi^{2}-1428329123020800}{57983813908020000\pi^{32}}\Big) \approx 7.191\times 10^{-23},
\end{aligned}
$$

we have the following lemma.

LEMMA 4. *For $\forall x \in (0,\pi/2)$, we have that*

$$
\begin{cases}
Q_1(x) = \sum\limits_{i=0}^{19}(\lambda_{1,i}B_{19,i}(x)) > 0,\\
Q_2(x) = \sum\limits_{i=0}^{20}(\lambda_{2,i}B_{20,i}(x)) < 0,\\
Q_3(x) = \sum\limits_{i=0}^{16}(\lambda_{3,i}B_{16,i}(x)) > 0,\\
Q_4(x) = \sum\limits_{i=0}^{19}(\lambda_{4,i}B_{19,i}(x)) < 0,
\end{cases}
\tag{18}
$$

Proof. For $\forall x \in (0,\pi/2)$, note that $\lambda_{1,4i} > 0$, $\lambda_{1,4i+1} > 0$, $\lambda_{1,4i+2} < 0$ and $\lambda_{1,4i+3} < 0$, $i = 0,1,2,3$, $\lambda_{1,16} > 0$, $\lambda_{1,17} > 0$, $\lambda_{1,19} > 0$ and $\lambda_{1,18} < 0$, it can be verified that

$$
\begin{aligned}
Q_1(x) \geqslant\ & \sum_{i=0}^{3}\Big(\lambda_{1,4i}+\lambda_{1,4i+1}x+\lambda_{1,4i+2}\big(\tfrac{\pi}{2}\big)^2+\lambda_{1,4i+3}\big(\tfrac{\pi}{2}\big)^2x\Big)x^{4i}\\
&+\Big(\lambda_{1,17}+\lambda_{1,18}\big(\tfrac{\pi}{2}\big)\Big)x^{17} \approx 0.00008+0.00005x+(5.5\times 10^{-7}x^4\\
&+3.4\times 10^{-7}x^5)+(4.9\times 10^{-10}x^8+2.57\times 10^{-10}x^9)\\
&+(1.1\times 10^{-13}x^{12}+3.4\times 10^{-14}x^{13})+9.7\times 10^{-19}x^{17} > 0.
\end{aligned}
\tag{19}
$$

Similarly, note that $\lambda_{2,i} > 0$, $i = 2,3,5,6,7,9,10,14,18$, $\lambda_{2,j} = 0$, $j = 1,11,13,15,17$, 19, and $\lambda_{2,4l} < 0$, $l = 0, 1, \cdots, 4$, it can be verified that

$$
\begin{aligned}
Q_2(x) \leqslant & \sum_{i=0}^{2} (\lambda_{2,4i} + \lambda_{2,4i+1}\tfrac{\pi}{2} + \lambda_{2,4i+2}(\tfrac{\pi}{2})^2 + \lambda_{2,4i+3}(\tfrac{\pi}{2})^3)x^{4i} + (\lambda_{1,12} \\
& + \lambda_{1,14}(\tfrac{\pi}{2})^2)x^{12} + (\lambda_{1,16} + \lambda_{1,18}(\tfrac{\pi}{2})^2)x^{16} \approx -4.4 \times 10^{-5} - 3.3 \times 10^{-7} x^4 \\
& -3.32 \times 10^{-10} x^8 - 7.25 \times 10^{-14} x^{12} - 4.39 \times 10^{-18} x^{16} < 0.
\end{aligned}
\tag{20}
$$

Again, note that $\lambda_{3,4i+1} < 0$ and $\lambda_{3,4i+2} < 0$, $i = 0, 1, 2, 3$, $\lambda_{3,4j+3} > 0$ and $\lambda_{3,4j+4} > 0$, $j = 0, 1, 2$, $\lambda_{3,15} > 0$ and $\lambda_{3,16} < 0$, it can be verified that

$$
\begin{aligned}
Q_3(x) \geqslant & \sum_{i=0}^{2} (\lambda_{3,4i+3} + \lambda_{3,4i+4}x + \lambda_{3,4i+5}(\tfrac{\pi}{2})^2 + \lambda_{3,4i+6}(\tfrac{\pi}{2})^2 x)x^{4i+3} \\
& + (\lambda_{3,0} + \lambda_{3,1}(\tfrac{\pi}{2}) + \lambda_{3,2}(\tfrac{\pi}{2})^2) + (\lambda_{3,15} + \lambda_{3,16}(\tfrac{\pi}{2}))x^{15} \\
\approx & (9.3 \times 10^{-10} + 1.0 \times 10^{-8} x)x^3 + (3.0 + 3.6x) \times 10^{-12} x^7 \\
& + (6.5 + 2.4x) \times 10^{-16} x^{11} + 2.5 \times 10^{-6} + 3.4 \times 10^{-20} x^{15} > 0,
\end{aligned}
\tag{21}
$$

Finally, note that $\lambda_{4,4i+1} < 0$, $\lambda_{4,4i+2} < 0$, $\lambda_{4,4i+3} > 0$ and $\lambda_{3,4i+4} > 0$, $i = 0, 1, \cdots, 4$, it can be verified that

$$
\begin{aligned}
Q_4(x) \leqslant & \sum_{i=0}^{4} (\lambda_{4,4i} + \lambda_{4,4i+1}x + \lambda_{4,4i+2}(\tfrac{\pi}{2})^2 + \lambda_{4,4i+3}(\tfrac{\pi}{2})^2 x)x^{4i} \\
\approx & -3.7 \times 10^{-6} - 2.3 \times 10^{-6} x - (1.7 \times 10^{-8} + 1.0 \times 10^{-8} x)x^4 \\
& -(8.5 \times 10^{-12} + 5.3 \times 10^{-12} x)x^8 - (1.0 \times 10^{-15} + 6.4 \times 10^{-16} x)x^{12} \\
& -(3.3 \times 10^{-20} + 2.1 \times 10^{-20} x)x^{16} < 0.
\end{aligned}
\tag{22}
$$

Combining Eq. (19), Eq. (20), Eq. (21) with Eq. (22), we obtain Eq. (18), and the proof is completed.

Let

$$
\begin{aligned}
\phi_0(x,\alpha) &= \cos(x) - \cos(x)\tfrac{1+\cos(2x)}{2} - 2\cos(\alpha x)x^2, \\
\phi_1(x,\alpha) &= \tfrac{x(105\alpha^4 x^6 + 127x^6 + 630\alpha^4 x^4 + 7560x^2 + 7560)}{7560}, \\
\phi_2(x,\alpha) &= -\tfrac{21x^3(7\alpha^2 x^4 + x^4\alpha^6 + 60\alpha^2 x^2 + 34x^2 + 360\alpha^2)}{7560}, \\
\phi_3(x,\alpha) &= \tfrac{x((98\alpha^4 + 3\alpha^8)x^8 + (840\alpha^4 + 1016)x^6 + 5040\alpha^4 x^4 + 60480x^2 + 60480)}{60480}, \\
\phi_4(x,\alpha) &= -\tfrac{x^3((259 + 1860\alpha^2 + 420\alpha^6)x^6 + (17640\alpha^2 + 2520\alpha^6)x^4 + (85680 + 151200\alpha^2)x^2 + 907200\alpha^2)}{907200}.
\end{aligned}
$$

For $x \in (0, \frac{\pi}{2})$, $\alpha \in (0,2)$, combining with Lemma 3, it can be verified that

$$R_i(x,\alpha) = E_i(x,\alpha)\cdot x^2 = \phi_0(x,\alpha)+\phi_{2i-1}(x,\alpha)\sin(x)+\phi_{2i}(x,\alpha)\sin(x),$$

$$\begin{aligned} R_i(x,\alpha) &\leqslant \bar{C}_4(x,1)-\underline{C}_4(x,1)\tfrac{1+\underline{C}_4(x,2)}{2}-2\underline{C}_4(x,\alpha)x^2 \\ &\quad +\phi_{2i-1}(x,\alpha)\bar{S}_3(x,1)+\phi_{2i}(x,\alpha)\underline{S}_3(x,1)=\bar{R}_i(x,\alpha), \\ R_i(x,\alpha) &\geqslant \underline{C}_4(x,1)-\bar{C}_4(x,1)\tfrac{1+\bar{C}_4(x,2)}{2}-2\bar{C}_4(x,\alpha)x^2 \\ &\quad +\phi_{2i-1}(x,\alpha)\underline{S}_3(x,1)+\phi_{2i}(x,\alpha)\bar{S}_3(x,1)=\underline{R}_i(x,\alpha), \end{aligned} \tag{23}$$

where $i=1,2$, $\underline{R}_i(x,\alpha)$ and $\bar{R}_i(x,\alpha)$ are polynomials in x and α. Combining Eq. (23) with Lemma 4, for $x \in (0, \frac{\pi}{2})$, by using the Maple software, it can be verified that

$$\begin{aligned} R_1(x,\alpha_1) &\geqslant \underline{R}_1(x,\alpha_1) = x^{10}(\tfrac{\pi}{2}-x)\sum_{i=0}^{19}(\lambda_{1,i}B_{19,i}(x))>0, \\ R_1(x,\alpha_2) &\leqslant \bar{R}_1(x,\alpha_2) = x^{12}\sum_{i=0}^{20}(\lambda_{2,i}B_{20,i}(x))<0, \\ R_2(x,\alpha_3) &\geqslant \underline{R}_2(x,\alpha_3) = x^{14}\sum_{i=0}^{16}(\lambda_{3,i}B_{16,i}(x))>0, \\ R_2(x,\alpha_4) &\leqslant \bar{R}_2(x,\alpha_4) = x^{12}(\tfrac{\pi}{2}-x)\sum_{i=0}^{19}(\lambda_{4,i}B_{19,i}(x))<0. \end{aligned} \tag{24}$$

From Eq. (24), we obtain Eq. (11), and the proof of Theorem 1 is completed. □

3. Applications and discussions

By using suitable values of α in Eq. (9), one can recover some previous results. Let $\alpha = 1$, one obtains that

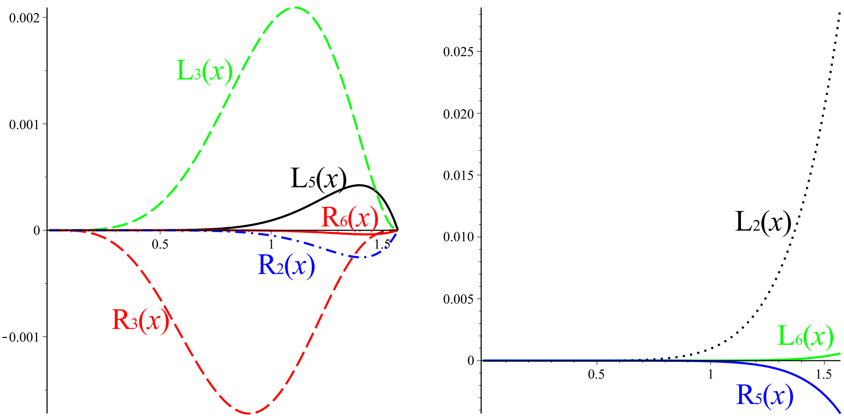
$$\frac{G_1(x,1)}{\cos(x)} = 2+(\tfrac{8}{45}-a(x))x^3\tan x,$$

$$\frac{G_2(x,1)}{\cos(x)} = 2+(\tfrac{8}{45}-b(x))x^3\tan x,$$

and recovers the left and right sides of Eq. (4), which is a result in [24, 19, 25]. Fig. 4 shows the error plots of $(f(x)-L_i(t))\cdot\cos(x)$ and $(f(x)-R_i(t))\cdot\cos(x)$ in Eq. (2) and Eq. (11) from different methods. It shows that the results from Eq. (11) in Theorem 1 can achieve much better approximation effect than those of the results in Eq. (2).

As for future work, there is still plenty of room for further development. Firstly, one can recover or improve more other results by using more other bounding functions. For example, the bounding function

$$\begin{aligned} G_{new}(x,\alpha) = {} & 2\cos(\alpha x)+((\alpha^2-1)x+(\tfrac{-\alpha^4}{12}+\tfrac{\alpha^2}{6}+\tfrac{17}{180})x^3 \\ & +(\tfrac{\alpha^6}{360}-\tfrac{\alpha^4}{72}+\tfrac{7\alpha^2}{360}-\tfrac{127}{7560})x^5 \\ & +(\tfrac{-\alpha^8}{20160}+\tfrac{\alpha^6}{2160}-\tfrac{7\alpha^4}{4320}+\tfrac{31\alpha^2}{15120}+\tfrac{37}{129600})x^7)\sin(x) \end{aligned}$$



(a) (b)

Figure 4: *Error plots of* $(f(x)-L_i(t))\cdot\cos(x)$ *and* $(f(x)-R_i(t))\cdot\cos(x)$ *in Eq. (2) and Eq. (11),* $i=2,3,\cdots,6$.

can be used for recovering $R_2(x)$ in Eq. (2). Secondly, more parameters can be introduced for much better approximation effect. Finally, in principle, the idea can be extended to more other inequalities for much better approximation effect.

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