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POLYA—VINOGRADOV INEQUALITY FOR POLYNOMIAL CHARACTER SUMS OVER FINITE FIELDS

ZHIYONG ZHENG AND ZIWEI HONG

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Abstract. A version of Polya–Vinogradov inequality in function fields appeared in [1, 2, 3, 9] recently. In this paper, we show some new bounds for polynomial character sums by making use of polynomial Gauss sums (see [4, 12]) and a formula from L. Carlitz (see [5]) on exponential sums over function fields. The method is elementary. It is worth mentioning that the proofs given in this paper do not depend on the well-known result form A. Weil on L-function associated to algebraic curves over finite fields.

1. Introduction

$\mathbb{F}_q$ is a finite field with q elements of a finite characteristic p. $K=\mathbb{F}_q[T]$ is the polynomial ring over $\mathbb{F}_q$ and $\mathbb{A}$ is the set of monic polynomials in K. $k=\mathbb{F}_q(T)$ is the rational function field. Given $H\in K$ and $H\neq 0$, the absolute value of H is defined by $|H|=q^{\deg H}$. Let χ be a complex-valued character of the multiplicative group $(K/HK)^*$. χ can be extended to K by setting $\chi(A)=0$, when $(A,H)\neq 1$. Then χ is said to be a Dirichlet character modulo H in K. χ_0 is said to be a principal character, if $\chi_0(H')=1$ for all $(H,H')=1$. If $\chi(a)=1$ for all $a\in\mathbb{F}_q^*$, then χ is said to be an even character. Otherwise, χ is said to be an odd character. In Proposition 2.1 of [9], Hsu used the well-known result from A. Weil on L-function associated to algebraic curves over finite fields and he obtained the following inequality

$$\Big|\sum_{\substack{h\in\mathbb{A},\\ \deg h=d}}\chi(h)\Big|\leqslant\begin{cases}2\sqrt{q}^{-2+\deg f_\chi}, & \text{if } \chi \text{ is even;}\\ \sqrt{q}^{-2+\deg f_\chi}, & \text{if } \chi \text{ is odd;}\end{cases}\tag{1.1}$$

where $d\geqslant 0$ is an integer. f_χ is the conductor of χ when χ is regard as a character of the idele group J_k of k (see [6]).

$$J_k\longrightarrow Gal(k(\Lambda_H)/k)\cong(K/HK)^*\xrightarrow{\chi}S^1.\tag{1.2}$$

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Local class-field theory gives a detailed description of (1.2). Moreover, (5.6) of [6] shows that

$$f_\chi = \begin{cases} H, & \text{if } \chi \text{ is even;} \\ \infty H, & \text{if } \chi \text{ is odd.} \end{cases} \tag{1.3}$$

Combined with (1.1), indeed, Hsu showed the following Polya–Vingradov inequality for polynomial character sums over finite fields

$$|\sum_{\substack{h\in\mathbb{A} \\ \deg h=d}} \chi(h)| \leqslant \begin{cases} \frac{2}{q}|H|^{\frac{1}{2}}, & \text{if } \chi \neq \chi_0 \text{ is even;} \\ \frac{1}{\sqrt{q}}|H|^{\frac{1}{2}}, & \text{if } \chi \neq \chi_0 \text{ is odd.} \end{cases} \tag{1.4}$$

In lemma 2.1 of [7], a more appealing result is given by $(\chi \neq \chi_0)$

$$|\sum_{\substack{h\in\mathbb{A} \\ \deg h=d}} \chi(h)| \leqslant \binom{\deg H-1}{d} q^{\frac{d}{2}}. \tag{1.5}$$

(1.5) is a sharper bound rather than (1.4) when d is a small integer. While (1.4) is better when d is large. Both (1.4) and (1.5) depend on A. Weil's well known result on L-function associated to algebraic curves over finite fields.

In this paper, we will show some new bounds for polynomial character sums by an elementary method. To illustrate our results, we have to introduce the conductor of a Dirichlet character modulo H and the primitive Dirichlet character modulo H (see Definition 3.2 and Definition 3.3 of [12]). Let χ be a Dirichlet character modulo H in K. A polynomial $N \in K$ is called an induced modulus of χ, if $N|H$ and

$$\chi(A) = 1, \text{whenever } A \equiv 1 \pmod{N}. \tag{1.6}$$

An induced modulus N of χ is called the conductor of χ, if N is monic and N has the least degree among all induced moduli of χ. If the degree of the conductor equals to degree of H, then we say χ is a primitive character modulo H in K. Now we stand on a position to state our main result.

THEOREM 1.1. *Let χ be a Dirichlet character modulo H in K, and $\chi \neq \chi_0$. Let N be the conductor of χ, then for all $d > 0$, we have*

$$|\sum_{\substack{h\in K \\ \deg h<d}} \chi(h)| \leqslant \sigma\left(\frac{H}{N}\right)|N|^{\frac{1}{2}} \tag{1.7}$$

and

$$|\sum_{\substack{h\in\mathbb{A} \\ \deg h=d}} \chi(h)| \leqslant \sigma\left(\frac{H}{N}\right)|N|^{\frac{1}{2}}. \tag{1.8}$$

Moreover, if χ is a primitive character, we have

$$|\sum_{\substack{h\in K \\ \deg h=d}} \chi(h)| \leqslant c(d)|H|^{\frac{1}{2}}. \tag{1.9}$$

If χ is an even primitive character, we have ($\deg H > 1$)

$$\sum_{\substack{h\in\mathbb{A}\\ \deg h=\deg H-1}} \chi(h) = 0, \tag{1.10}$$

where $c(d)=1$, if $d=\deg H-1$, and $c(d)=q-1$ if $d \leqslant \deg H-2$. $\sigma(\frac{H}{N})$ equals to the number of monic divisors of $\frac{H}{N}$.

By (1.5) and (1.10), we have the following consequence.

COROLLARY 1.1. *Suppose χ is an even and primitive Dirichlet character modulo H, $\deg H > 1$, then for any integer $0 \leqslant d < \deg H$, we have*

$$|\sum_{\substack{h\in\mathbb{A}\\ \deg h=d}} \chi(H)| \leqslant \binom{\deg H-2}{d} q^{\frac{d}{2}}. \tag{1.11}$$

Obviously, the left-hand sides of (1.7) and (1.9) vanish, if χ is an odd character; hence (1.7) and (1.9) are valid when χ is even characters. Inequalities (1.7) and (1.9) are analogues of Polya-Vinogradov inequality in positive characteristic case, they have a weaker form appeared in [3], Proposition 1.8. in some special cases, (1.8) gives a bound sharper than (1.4) and (1.5). For example, we set H is a power of an irreducible and χ is a non-primitive character modulo H. To prove Theorem 1.1, we will use the separable Gauss sums and a formula from L. Carlitz on the exponential sums over function field (see Theorem 1 of [5]). The proof is very familiar with what we did in [1]. It is worth mentioning that our proof is independent on A. Weil's result on Riemann hypothesis over function fields.

2. Exponential sums and Gauss sums in k_∞

Let $v: k \longrightarrow \mathbb{Z}$ be the valuation function over k with $v(\frac{1}{T})=1$ and $v(0)=\infty$. Let $k_\infty = \mathbb{F}_q((\frac{1}{T}))$ be the Laurent series field, which is the completion of k at the infinite place, ∞, with respect to v. If $\alpha \in k_\infty$, we may express α as a Laurent series as

$$\alpha = \sum_{i=v(\alpha)}^{+\infty} a_i \left(\frac{1}{T}\right)^i, \text{where } a_i \in \mathbb{F}_q. \tag{2.1}$$

Define

$$[\alpha] = \sum_{i=v(\alpha)}^{0} a_i \left(\frac{1}{T}\right)^i, \text{if } v(\alpha) \leqslant 0 \tag{2.2}$$

and $[\alpha]=0$, otherwise. Let $\langle\alpha\rangle = \alpha-[\alpha]$, $[\alpha]$ is called the "integral part" of α and $\langle\alpha\rangle$ is called the "fractional part" of α. The absolute value functions $|\alpha|$ and $||\alpha||$ are given by

$$|\alpha| = q^{-v(\alpha)} \text{ and } ||\alpha|| = |\langle\alpha\rangle| = |\alpha-[\alpha]|. \tag{2.3}$$

It is easy to check that

$$||\alpha+\beta|| \leqslant \max\{||\alpha||,||\beta||\} \text{ and } ||\alpha|| = \inf_{A\in K} |\alpha - A|. \tag{2.4}$$

Thus, $||\alpha||$ is the smallest distance from α to any element of K and $[\alpha]$ is the nearest polynomial to α.

The valuation ring $\mathfrak{P}_0$ and the valuation ideal $\mathfrak{P}$ of k_∞ are given by

$$\mathfrak{P}_0 = \{\alpha \in k_\infty : |\alpha| \leqslant 1\} \text{ and } \mathfrak{P} = \{\alpha \in k_\infty : |\alpha| < 1\}. \tag{2.5}$$

If n is an integer, the fractional ideal $\mathfrak{P}_n$ is given by

$$\mathfrak{P}_n = \left(\frac{1}{T}\right)^n \mathfrak{P}_0 = \{\alpha \in k_\infty : |\alpha| \leqslant q^{-n}\}. \tag{2.6}$$

Obviously, $\mathfrak{P}_1 = \mathfrak{P}$, and

$$\cdots \supset \mathfrak{P}_{-2} \supset \mathfrak{P}_{-1} \supset \mathfrak{P}_0 \supset \mathfrak{P}_1 \supset \mathfrak{P}_2 \supset \cdots.$$

Let k_∞^+ be the additive group of k_∞. An additive character of k_∞ is a group homomorphism $\psi : k_\infty^+ \longrightarrow S^1$, where S^1 is the circle group of complex number. All the additive characters of k_∞ form a group, the product of homomorphisms is defined as follow: $\phi\psi(\alpha) = \phi(\alpha)\psi(\alpha)$ for any additive characters of k_∞ and any $\alpha \in k_\infty$. We denote this group by $\widehat{k_\infty^+}$ and call it the dual group of k_∞^+. As usual, the identity element of $\widehat{k_\infty^+}$ is ψ_0 which holds that $\psi_0(\alpha) = 1$ for any $\alpha \in k_\infty$. If $\psi \neq \psi_0$, there exists an integer n such that ψ is trivial on $\mathfrak{P}_n$, i.e. $\psi(\mathfrak{P}_n) = 1$ (see (2.6) of [6]). Let

$$n(\psi) := \min\{n : \psi(x) = 1 \text{ for all } x \in \mathfrak{P}_n\}, \tag{2.7}$$

which is called the conductor of ψ. Suppose $\alpha \in k_\infty$ and ψ is a non-trivial additive character, we use ψ_α to denote the homomorphism $\psi_\alpha(x) = \psi(\alpha x)$ for all $x \in k_\infty$. Clearly, ψ_α is again an additive character of k_∞ and the conductor of ψ_α is given by (see (2.12) of [6])

$$n(\psi_\alpha) = n(\psi) - v(\alpha). \tag{2.8}$$

Next, we will introduce two characters which play an important role in the left part of this paper. Let $\alpha = \sum\limits_{i=v(\alpha)}^{+\infty} a_i(\frac{1}{T})^i$ be arbitrary element in k_∞ and m arbitrary integer, we define an additive character $\psi^{(m)}$ by

$$\psi^{(m)}(\alpha) = \lambda^{\mathrm{tr}(a_m)}, \tag{2.9}$$

where "tr" is the usual trace map from $\mathbb{F}_q$ to $\mathbb{F}_p$, λ is a fixed primitive p-th root of 1. By the definition of conductor and the form of $\psi^{(m)}$, we claim (2.10) without proof:

$$n(\psi^{(m)}) = m+1. \tag{2.10}$$

In particular, when $m=1$, we define $e(\alpha)=\psi^{(1)}(\alpha)=\lambda^{\mathrm{tr}(a_1)}$. $e(\alpha)$ is called the exponent function in k_∞. It's easy to prove that

$$e(\alpha+H)=e(\alpha)$$

and

$$e(\alpha)=e(\langle\alpha\rangle)$$

for all $\alpha\in k_\infty$ and $H\in K$. By definition, we have

$$n(e)=2. \tag{2.11}$$

In somehow, e is an analogue of the exponential function e^x. On the one hand, e gives an explicit example of an additive character of k_∞. On the other hand, starting from e and the construction of ψ_α, we obtain an explicit description of the dual group $\widehat{k_\infty^+}=\{\psi_\alpha:\psi=e,\alpha \text{ runs through } k_\infty\}$. To study more details about conductor and additive character, readers should consult Chap. I, [11]. With the above notations, a formula from L. Carlitz on exponential sums may be stated as follow.

LEMMA 2.1. *Let $\theta\in k_\infty$ and $M\in K$, $M\neq 0$. Then we have*

$$S(\theta,M)=\sum_{\substack{h\in K\\|h|<|M|}}e(\theta h)=\begin{cases}0, & \textit{if } |M|\geqslant||\theta||^{-1};\\ |M|, & \textit{if } |M|<||\theta||^{-1}.\end{cases} \tag{2.12}$$

This formula firstly appeared in Carlitz [5], Theorem 1.1. He gave this for formula without a proof. Now, we will give a proof by conductors.

Proof. First,we have $S(\theta,M)=S(\langle\theta\rangle,M)$, where $\langle\theta\rangle$ is the fractional part of θ. Then, we note that all polynomial h with $|h|<|M|$ form a subgroup G of k_∞^+. Hence, if we write $\psi_{\langle\theta\rangle}(\alpha)=e(\langle\theta\rangle\alpha)$, by orthogonality of characters, it is sufficient to show that $\psi_{\langle\theta\rangle}$ is a trivial character of G when $|M|<||\theta||^{-1}$ and $\psi_{\langle\theta\rangle}$ is non-trivial on G when $|M|\geqslant||\theta||^{-1}$. By (2.10), we have

$$n(\psi_{\langle\theta\rangle})=2-v(\langle\theta\rangle).$$

If $h\in K$ and $|h|<|M|<||\theta||^{-1}$, then $h\in\mathfrak{P}_{2-v(\langle\theta\rangle)}$ and we have $\psi_{\langle\theta\rangle}(h)=1$. If $|M|\geqslant||\theta||^{-1}$, then $\deg M-1\geqslant v(\langle\theta\rangle)-1$. Taking $h\in K$ with $\deg h=\deg M-1$, then we have $\deg h>v(\langle\theta\rangle)-2$ and $h\notin\mathfrak{P}_{2-v(\langle\theta\rangle)}$. Therefore, by the definition of conductor, there exists at least one such h with $\psi_{\langle\theta\rangle}(h)\neq 1$. It follows that $\psi_{\langle\theta\rangle}$ is non-trivial on G and we thus have

$$S(\langle\theta\rangle,M)=0, \text{ if } |M|\geqslant||\theta||^{-1}.$$

We completed the proof of lemma 2.1. □

Before proving Theorem 1.1, we need to introduce some lemmas about polynomial Gauss sums. The polynomial Gauss sums was first defined by L. Carlitz in [4] and

Hayes in [8]. Let $H \in K$ and $H \neq 0$. χ is a Dirichlet character modulo H. We define a polynomial Gauss sum $G(A,\chi)$ by

$$G(A,\chi) = \sum_{h \bmod H} \chi(h) e\left(\frac{Ah}{H}\right) = \sum_{\substack{h \in K \\ |h|<|H|}} \chi(h) e\left(\frac{Ah}{H}\right), \tag{2.13}$$

where $A \in K$ and h runs through a complete residue system modulo H in K. If $(A,H) = 1$, it is easy to see that

$$G(A,\chi) = \bar{\chi}(A) G(1,\chi). \tag{2.14}$$

If $G(A,\chi) = \bar{\chi}(A)G(1,\chi)$ for all $A \in K$, then $G(A,\chi)$ is called a separable Gauss sum (see Definition 3.1 of [12]).

The proof of our main theorem is based on some lemmas of polynomial Gauss sums. The following lemma is a summary of three statements in [12].

LEMMA 2.2. *Let χ be a Dirichlet character modulo H, then*

1. *χ is primitive if and only if $G(A,\chi)$ is a separable Gauss sum;*
2. *Suppose that χ is a primitive character modulo H, then we have*

$$|G(1,\chi)|^2 = |H|; \tag{2.15}$$

3. *Let N be the conductor of χ, then χ can be expressed as a product*

$$\chi = \chi_0 \delta, \tag{2.16}$$

where χ_0 is the principal Dirichlet character modulo H and δ is a primitive Dirichlet character modulo N.

Proof. The statement (a) is lemma 3.8 of [12]. (b) and (c) are lemma 3.1 and lemma 3.5 of [12], respectively. □

3. Proof of Theorem 1.1

First, we assume χ is a primitive character modulo H. In this case, $\sigma(\frac{H}{N}) = 1$ and $|N| = |H|$. Thus we only prove

$$\Big|\sum_{\substack{h \in K \\ \deg h < d}} \chi(h)\Big| \leqslant |H|^{\frac{1}{2}}, \tag{3.1}$$

$$\Big|\sum_{\substack{h \in K \\ \deg h = d}} \chi(h)\Big| \leqslant c(d)|H|^{\frac{1}{2}} \tag{3.2}$$

and

$$|\sum_{\substack{h\in\mathbb{A}\\ \deg h=d}}\chi(h)|\leqslant|H|^{\frac{1}{2}}. \tag{3.3}$$

Moreover,

$$\sum_{\substack{h\in\mathbb{A}\\ \deg h=\deg H-1}}\chi(h)=0\text{ , if }\chi\text{ is even.} \tag{3.4}$$

Without loss of generality, we may suppose that $d<\deg H$, since if $d=\deg H$, we see that

$$\sum_{\substack{h\in K\\ \deg h<\deg H}}\chi(h)=\sum_{h\bmod H}\chi(h)=0.$$

If $d\geqslant\deg H$, we have (see Proposition 4.3 of [10])

$$\sum_{\substack{h\in\mathbb{A}\\ \deg h=d}}\chi(h)=0. \tag{3.5}$$

Therefore, we may suppose that $0\leqslant d<\deg H$. Let $M=T^d$, by Lemma 2.2 statement (a) and (b), we have

$$\chi(A)=\tau^{-1}(\bar{\chi})G(A,\bar{\chi})\text{ ,and }|\tau(\bar{\chi})|=|H|^{\frac{1}{2}}, \tag{3.6}$$

where $A\in K$ and $\tau(\chi)=G(1,\chi)$. It follows that

$$\begin{aligned}\sum_{\substack{A\in K\\ |A|<|M|}}\chi(A)&=\tau^{-1}(\bar{\chi})\sum_{\substack{h\in K\\ |h|<|H|}}\bar{\chi}(h)\sum_{\substack{A\in K\\ |A|<|M|}}e\left(\frac{Ah}{H}\right)\\ &=\tau^{-1}(\bar{\chi})\left(\sum_{\substack{h\in K\\ |h|<|\frac{H}{M}|}}\bar{\chi}(h)+\sum_{\substack{h\in K\\ |\frac{H}{M}|\leqslant|h|<|H|}}\bar{\chi}(h)\right)\sum_{\substack{A\in K\\ |A|<|M|}}e\left(\frac{Ah}{H}\right).\end{aligned} \tag{3.7}$$

By Lemma 2.1, we have

$$\sum_{\substack{A\in K\\ |A|<|M|}}e\left(\frac{Ah}{H}\right)=\begin{cases}|M|,\text{ if }|h|<|\frac{H}{M}|;\\ 0,\quad\text{if }|h|\geqslant|\frac{H}{M}|.\end{cases} \tag{3.8}$$

It follows that

$$\sum_{\substack{A\in K\\ |A|<|M|}}\chi(A)=\tau^{-1}(\bar{\chi})|M|\sum_{\substack{h\in K\\ |h|<|\frac{H}{M}|}}\bar{\chi}(h).$$

By (3.6), we thus have

$$|\sum_{\substack{A\in K\\ |A|<|M|}}\chi(A)|\leqslant|\tau(\bar{\chi})|^{-1}\cdot|H|=|H|^{\frac{1}{2}}$$

and (3.1) holds. To prove (3.3), we write ($M=T^d$) then

$$\begin{aligned}\sum_{\substack{A\in\mathbb{A}\\ \deg A=d}}\chi(A)&=\sum_{\substack{A\in K\\ |A|<|M|}}\chi(T^d+A)\\ &=\tau^{-1}(\bar{\chi})\sum_{\substack{h\in K\\ |h|<|H|}}\bar{\chi}(h)e\left(\frac{hT^d}{H}\right)\sum_{\substack{A\in K\\ |A|<|M|}}e\left(\frac{Ah}{H}\right)\\ &=\tau^{-1}(\bar{\chi})|M|\sum_{\substack{h\in K\\ |h|<|\frac{H}{M}|}}\bar{\chi}(h)e\left(\frac{hT^d}{H}\right).\end{aligned}\tag{3.9}$$

(3.3) follows immediately. To show (3.2), by Lemma 2.1, it is easily see that

$$\sum_{a\in\mathbb{F}_q}e(\frac{aT^d}{H})=\begin{cases}0, & \text{if } d=\deg H-1;\\ q, & \text{if } d<\deg H-1.\end{cases}\tag{3.10}$$

It follows that

$$\sum_{a\in\mathbb{F}_q^*}e(\frac{aT^d}{H})=\begin{cases}-1, & \text{if } d=\deg H-1;\\ q-1, & \text{if } d<\deg H-1.\end{cases}\tag{3.11}$$

We may write ($M=T^d$)

$$\sum_{\substack{A\in K\\ \deg A=d}}\chi(h)=\sum_{a\in\mathbb{F}_q^*}\sum_{\substack{A\in K\\ |A|<|M|}}\chi(aT^d+A)=c_1(d)\tau^{-1}(\bar{\chi})\sum_{\substack{h\in K\\ |h|<|H|}}\bar{\chi}(h)\sum_{\substack{A\in K\\ |A|<|M|}}e\left(\frac{Ah}{H}\right).\tag{3.12}$$

where $c_1(d)=-1$, if $d=\deg H-1$, and, $c_1(d)=q-1$, if $d<\deg H-1$. By (3.8), we have

$$|\sum_{\substack{A\in K\\ \deg A=d}}\chi(A)|\leqslant c(d)|H|^{\frac{1}{2}},\tag{3.13}$$

where $c(d)=1$, if $d=\deg H-1$, and $c(d)=q-1$ if $d\leqslant\deg H-2$. For (3.4), suppose that χ is an even character and $d=\deg H-1$, then the inner sums on the right-hand side of (3.9) is zero.

At least, we deal with the case that χ is not primitive. Let $\mu(A)$ be the Möbius function over K (see (2.11) of [13]), by (2.12) of [13], we have

$$\sum_{D|H}\mu(D)=\begin{cases}1, & \text{if } \deg H=0;\\ 0, & \text{if } \deg H\geqslant 1;\end{cases}\tag{3.14}$$

the sum taken over all monic divisors of H. Suppose N is the conductor of H, χ_0 is the principal character modulo H and δ is a primitive character modulo N. By Lemma

2.2 statement (c), we have ($M = T^d$)

$$\sum_{\substack{m\in K\\|m|<|M|}} \chi(m) = \sum_{\substack{m\in K,(m,H)=1\\|m|<|M|}} \delta(m)$$
$$= \sum_{\substack{m\in K\\|m|<|M|}} \delta(m) \sum_{\substack{A\in K\\A|(m,H)}} \mu(A)$$
$$= \sum_{A|H} \mu(A) \sum_{\substack{|m|<|M|\\A|m}} \delta(m) \tag{3.15}$$
$$= \sum_{A|H} \mu(A)\delta(A) \sum_{\substack{m\in K\\|m|<|\frac{M}{A}|}} \delta(m).$$

It follows that

$$|\sum_{\substack{m\in K\\|m|<|M|}} \chi(m)| \leqslant |N|^{\frac{1}{2}} \sum_{A|H} |\mu(A)\delta(A)|. \tag{3.16}$$

$|\mu(A)| = 1$ means that A is a square-free divisor of H and $|\delta(A)| = 1$ means $(A,N) = 1$. Thus, $A|H$ implied that $A|\frac{H}{N}$. We obtain

$$\sum_{A|H} |\mu(A)||\delta(A)| \leqslant \sum_{A|\frac{H}{N}} 1 = \sigma\left(\frac{H}{N}\right). \tag{3.17}$$

By (3.15), we have

$$|\sum_{\substack{m\in K\\|m|<|M|}} \chi(m)| \leqslant \sigma\left(\frac{H}{N}\right) |N|^{\frac{1}{2}}. \tag{3.18}$$

This complete the proof (1.7). By the same way, we have (1.8) immediately.

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Zhiyong Zheng
School of Mathematics
Renmin University of China
Beijing, P.R. China
e-mail: zhengzy@ruc.edu.cn

Ziwei Hong
School of Mathematics and Systems Science
Beihang University
Beijing, P.R. China
e-mail: hongziwei@live.com