Journal of Mathematical Inequalities
Volume 15, Number 2 (2021), 467–476 doi:10.7153/jmi-2021-15-35

SCHUR–CONVEXITY RELATED TO CO–ORDINATED CONVEX FUNCTIONS IN PLANE

N. SAFAEI AND A. BARANI*

(*Communicated by J. Pečarić*)

Abstract. In the paper, we investigate Schur-convexity of some functions which are obtained from the co-ordinated convex functions on a square in plane. A version of celebrated Leibniz's derivative formula for double integrals is also given.

1. Introduction

The notion of Schur-convexity was done first by Issai Schur in 1923. Since then numerous papers have been published in this literature, see for example [3, 5, 6, 9]. Schur-convexity has many important applications in analytic inequality, geometric inequality, combinatorial analysis, numerical analysis, matrix theory, and so on. Let us recall the definition of Schur-convexity.

DEFINITION 1.1. [1] Suppose that I is an interval of real numbers. A function $f : I^n \to \mathbb{R}$, is said to be Schur-convex on I^n if

$$f(x_1,x_2,\ldots,x_n) \leqslant f(y_1,y_2,\ldots,y_n)$$

for all $x = (x_1,x_2,\ldots,x_n)$, $y = (y_1,y_2,\ldots,y_n) \in I^n$ with $x \prec y$, that is

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]} \quad k = 1,2,\ldots,n-1,$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$, denotes the $i-$th largest component in x. A function f is said to be Schur-concave on I if $-f$ is Schur-convex.

Mathematics subject classification (2020): 26A51, 26D15.
Keywords and phrases: Schur-convexity, convex functions on the co-ordinates.
* Corresponding author.

Recall that a $n \times n$ square matrix P is said to be a permutation matrix if each row and column has a single unite entry, and all other entries are zero. Also the function $f : I^n \to \mathbb{R}$ is said to be a symmetric function if $f(Px) = f(x)$, for every permutation matrix P, and for every $x \in I^n$, see [1, 7]. In order to prove our result, we shall need the following theorem which gives a useful characterization of Schur-convexity, see [1].

THEOREM 1.1. *Let $f : I^n \to \mathbb{R}$ be a continuous symmetric function. If f is differentiable on I^n, then f is Schur-convex if and only if*

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geqslant 0,$$

for all $x_i, x_j \in I, i, j = 1, 2, \ldots, n$. The function f is Schur-concave if and only if the reverse inequality holds.

A Schur-convex function need not be convex (consider $f(x,y) := |y-x|^{\frac{1}{2}}$ on $\mathbb{R}^2$) and a convex function need not be Schur-convex (consider $f(x,y) := x + y^2$ on $\mathbb{R}^2$). In [5] Elezović and Pečarić proved a theorem which gives relationship between convexity and Schur-convexity.

THEOREM 1.2. *Let f be a continuous function on an interval $I \subset \mathbb{R}$, and*

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt, & x,y \in I,\ x \neq y, \\ f(x), & x = y \in I. \end{cases}$$

Then $F(x,y)$ is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

Let $I \subset \mathbb{R}$ be an open interval and $f \in C^2(I)$. In [3] Y. Chu et al. proved the following theorem.

THEOREM 1.3. *Let $f : I \to \mathbb{R}$ be a continuous function. The function*

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)dt - f(\frac{x+y}{2}), & x,y \in I,\ x \neq y, \\ 0, & x = y \in I, \end{cases}$$

is Schur-convex (Schur-concave) on I^2 if and only if f is convex (concave) on I.

In [4], S. S. Dragomir defined convex function on the co-ordinates (or co-ordinated convex functions) on the set $[a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$ as follows.

DEFINITION 1.2. A function $f : [a,b] \times [c,d] \to \mathbb{R}$ is said to be convex on the co-ordinates on $[a,b] \times [c,d]$ if for every $y \in [c,d]$ and $x \in [a,b]$, the partial mappings,

$$f_y \colon [a,b] \to \mathbb{R}, \quad f_y(u) = f(u,y),$$

and

$$f_x\colon [c,d] \to \mathbb{R}, \quad f_x(v) = f(x,v),$$

are convex. This means that for every (x,y), $(z,w) \in [a,b] \times [c,d]$ and $t,s \in [0,1]$,

$$\begin{aligned} f(tx+(1-t)z, sy+(1-s)w) \leqslant\ & tsf(x,y) + s(1-t)f(z,y) \\ & + t(1-s)f(x,w) + (1-t)(1-s)f(z,w). \end{aligned}$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex functions which are not convex. The following Hermite-Hadamard type inequality for co-ordinated convex functions was also proved in [4].

THEOREM 1.4. *Suppose that $f : [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on $[a,b] \times [c,d]$. Then,*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leqslant \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\ &\leqslant \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y)dydx \\ &\leqslant \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx \right. \\ &\quad \left. + \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right] \\ &\leqslant \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

The above inequalities are sharp.

We recall the following lemma from [2], which is known as Leibniz's Formula.

LEMMA 1.1. *Suppose that $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ and $\frac{\partial f}{\partial t} : [a,b] \times [c,d] \to \mathbb{R}$ are continuous and $\alpha_1, \alpha_2 : [c,d] \to [a,b]$ are differentiable functions. Then, the function $\varphi : [c,d] \to \mathbb{R}$ defined by*

$$\varphi(t) = \int_{\alpha_1(t)}^{\alpha_2(t)} f(x,t)dx,$$

has a derivative for each $t \in [c,d]$, which is given by

$$\varphi'(t) = f(\alpha_2(t),t)\alpha_2'(t) - f(\alpha_1(t),t)\alpha_1'(t) + \int_{\alpha_1(t)}^{\alpha_2(t)} \frac{\partial f}{\partial t}(x,t)dx.$$

Next, we establish the generalized Leibniz's derivative formula for double integrals which is a corrected version of similar result introduced in [8].

LEMMA 1.2. *Let $F(u,v)=\int_u^v\int_u^v f(x,y)dxdy$, where $f(x,y)$ is continuous on the rectangle $[a,p]\times[a,q]$, $u=u(b)$ and $v=v(b)$ are differentiable with $a\leqslant u(b)\leqslant p$ and $a\leqslant v(b)\leqslant q$. Then,*

$$\begin{aligned}\frac{\partial F}{\partial b}=&\left(\int_u^v f(x,v)dx+\int_u^v f(v,y)dy\right)v'(b)\\&-\left(\int_u^v f(x,u)dx+\int_u^v f(u,y)dy\right)u'(b).\end{aligned}\tag{1}$$

Proof. Since $F(u,v)=\int_u^v\int_u^v f(x,y)dxdy$, by the chain rule for derivation of composite functions, we have

$$\frac{\partial F}{\partial b}=\frac{\partial F}{\partial u}\frac{du}{db}+\frac{\partial F}{\partial v}\frac{dv}{db}\tag{2}$$

Let $H(u,v,y):=\int_u^v f(x,y)dx$, therefore $F(u,v)=\int_u^v H(u,v,y)dy$. By using the Lemma 1.1 we have

$$\begin{aligned}\frac{\partial F}{\partial v}=&H(u,v,v)+\int_u^v\frac{\partial}{\partial v}H(u,v,y)dy\\=&\int_u^v f(x,v)dx+\int_u^v\frac{\partial}{\partial v}\left(\int_u^v f(x,y)dx\right)dy\\=&\int_u^v f(x,v)dx+\int_u^v f(v,y)dy.\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial F}{\partial u}=&-H(u,v,u)+\int_u^v\frac{\partial}{\partial u}H(u,v,y)dy\\=&-\int_u^v f(x,u)dx+\int_u^v\frac{\partial}{\partial u}\left(\int_u^v f(x,y)dx\right)dy\\=&-\int_u^v f(x,u)dx-\int_u^v f(u,y)dy.\end{aligned}$$

By replacing $\frac{\partial F}{\partial u}$ and $\frac{\partial F}{\partial v}$ in (2) we obtain required results in (1). □

2. Main results

In this section we establish new results concerns the improvements of Theorem 1.2 and Theorem 1.3 for co-ordinated convex functions setting. To reach our goal, we need the following two lemmas.

LEMMA 2.1. *Let $D:=[a_1,b_1]\times[a_1,b_1]$ be a square in $\mathbb{R}^2$ with $a_1<b_1$, and the function $f:D\to\mathbb{R}$ is continuous, and has continuous second order partial derivatives on D° (the interior of D). Choose $a,b\in(a_1,b_1)$, with $a<b$, and let $\Delta:=[a,b]\times[a,b]$. Suppose that the function $F:\Delta\to\mathbb{R}$ is defined by*

$$F(x,y):=\begin{cases}\frac{1}{(y-x)^2}\int_x^y\int_x^y f(t,s)dtds, & x\neq y,\quad x,y\in[a,b],\\ f(x,x), & x=y,\quad x,y\in[a,b].\end{cases}$$

Then, for all $t_0 \in [a,b]$,

$$\frac{\partial F}{\partial x}\Big|_{(t_0,t_0)} = \frac{\partial F}{\partial y}\Big|_{(t_0,t_0)} = \frac{\frac{\partial f}{\partial t}(t,s)\Big|_{(t_0,t_0)} + \frac{\partial f}{\partial s}(t,s)\Big|_{(t_0,t_0)} + 2\frac{\partial f}{\partial t}(t,t)\Big|_{t_0}}{6}. \tag{3}$$

Proof. Fix $t_0 \in [a,b]$. By using the L'Hospital's rule, and Lemmas 1.1, 1.2 we see that

$$\begin{aligned}
\frac{\partial F}{\partial x}\Big|_{(t_0,t_0)} &= \lim_{t\to 0} \frac{F(t_0+t,t_0) - F(t_0,t_0)}{t} \\
&= \lim_{t\to 0} \frac{1}{t^3}\left[\int_{t_0}^{t_0+t}\int_{t_0}^{t_0+t} f(u,v)dudv - t^2 f(t_0,t_0)\right] \\
&= \lim_{t\to 0} \frac{1}{3t^2}\left[\int_{t_0}^{t_0+t} f(u,t_0+t)du + \int_{t_0}^{t_0+t} f(t_0+t,v)dv - 2tf(t_0,t_0)\right] \\
&= \lim_{t\to 0} \frac{1}{6t}\left[\int_{t_0}^{t_0+t} \frac{\partial f}{\partial t}(u,t_0+t)du + \int_{t_0}^{t_0+t} \frac{\partial f}{\partial t}(t_0+t,v)dv \right. \\
&\quad \left. + 2f(t_0+t,t_0+t) - 2f(t_0,t_0)\right] \\
&= \lim_{t\to 0} \frac{1}{6}\left[\int_{t_0}^{t_0+t} \frac{\partial^2 f}{\partial t^2}(u,t_0+t)du + \int_{t_0}^{t_0+t} \frac{\partial^2 f}{\partial t^2}(t_0+t,v)dv \right. \\
&\quad \left. + \frac{\partial f}{\partial t}(u,t_0+t)\Big|_{u=t_0+t} + \frac{\partial f}{\partial t}(t_0+t,v)\Big|_{v=t_0+t} + 2\frac{\partial f}{\partial t}(t_0+t,t_0+t)\right] \\
&= \frac{\frac{\partial f}{\partial t}(t,s)\Big|_{(t_0,t_0)} + \frac{\partial f}{\partial s}(t,s)\Big|_{(t_0,t_0)} + 2\frac{\partial f}{\partial t}(t,t)\Big|_{t_0}}{6}.
\end{aligned} \tag{4}$$

By changing the role of x by y in (4), we obtain required results in (3). □

The proof of the following lemma is similar to once in lemma 2.1 hence we omit it.

LEMMA 2.2. *Let* $D := [a_1,b_1]\times[a_1,b_1]$ *be a square in* $\mathbb{R}^2$ *with* $a_1 < b_1$, *and the function* $f : D \to \mathbb{R}$ *is continuous, and has continuous third order partial derivatives on* D°. *Choose* $a,b \in (a_1,b_1)$, *with* $a < b$, *and let* $\Delta := [a,b]\times[a,b]$. *Suppose that the function* $G : \Delta \to \mathbb{R}$ *is defined by*

$$G(x,y) := \begin{cases} \frac{1}{(y-x)^2}\int_x^y\int_x^y f(t,s)dtds - f(\frac{x+y}{2},\frac{x+y}{2}), & x \neq y,\ x,y \in [a,b], \\ 0, & x = y,\ x,y \in [a,b]. \end{cases}$$

Then, for all $t_0 \in [a,b]$,

$$\frac{\partial G}{\partial x}\Big|_{(t_0,t_0)} = \frac{\partial G}{\partial y}\Big|_{(t_0,t_0)} = \frac{\frac{\partial f}{\partial t}(t,s)\Big|_{(t_0,t_0)} + \frac{\partial f}{\partial s}(t,s)\Big|_{(t_0,t_0)} - \frac{\partial f}{\partial t}(t,t)\Big|_{t_0}}{6}.$$

Following result is a improvement of theorem 1.2 in co-ordinated convex functions setting.

THEOREM 2.1. *Let $D := [a_1, b_1] \times [a_1, b_1]$ be a square in $\mathbb{R}^2$ with $a_1 < b_1$, and the function $f : D \to \mathbb{R}$ is continuous, and has continuous second order partial derivatives on D°. Choose $a, b \in (a_1, b_1)$, with $a < b$, and let $\Delta := [a,b] \times [a,b]$. Suppose that f is convex on the co-ordinates on Δ, then the function $F : \Delta \to \mathbb{R}$ defined by*

$$F(x,y) := \begin{cases} \frac{1}{(y-x)^2} \int_x^y \int_x^y f(t,s)dtds, & x \neq y, \quad x,y \in [a,b], \\ f(x,x), & x = y, \quad x,y \in [a,b], \end{cases} \tag{5}$$

is Schur-convex on Δ.

Proof. *Case* 1: If $x, y \in [a,b]$, with $x = y$. Then Lemma 2.1 implies that

$$(y-x)\left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right) = 0.$$

Case 2: If $x, y \in [a,b]$, with $x \neq y$. Then by Lemma 1.2 we have

$$\begin{aligned} \frac{\partial F}{\partial y} =& \frac{-2}{(y-x)^3} \int_x^y \int_x^y f(t,s)dtds \\ &+ \frac{1}{(y-x)^2} \Big(\int_x^y f(t,y)dt + \int_x^y f(y,s)ds \Big), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial F}{\partial x} =& \frac{2}{(y-x)^3} \int_x^y \int_x^y f(t,s)dtds \\ &- \frac{1}{(y-x)^2} \Big(\int_x^y f(t,x)dt + \int_x^y f(x,s)ds \Big). \end{aligned}$$

Thus,

$$\begin{aligned} (y-x)\left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right) =& \frac{-4}{(y-x)^2} \int_x^y \int_x^y f(t,s)dtds \\ &+ \frac{1}{y-x} \left(\int_x^y \big(f(t,x) + f(t,y)\big)dt \right. \\ &\left. + \int_x^y \big(f(x,s) + f(y,s)\big)ds \right). \end{aligned}$$

Then, $(y-x)\big(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\big)$ is nonnegative if

$$\begin{aligned} &\frac{1}{(y-x)^2} \int_x^y \int_x^y f(t,s)dtds \\ \leqslant\ & \frac{1}{4(y-x)} \left(\int_x^y \big(f(t,y) + f(t,x)\big)dt + \big(f(y,s) + f(x,s)\big)ds \right). \end{aligned}$$

The last inequality follows from Theorem 1.4. Therefore, by Theorem 1.1 the function F is Schur-convex. $\square$

A consequence of theorem 1.2 is given in [5] as follows: If $f > 0$ on I and $\frac{f'}{f}$ is convex (concave) then the function

$$F(x,y) = \begin{cases} \frac{\log f(x) - \log f(y)}{x-y}, & x,y \in I,\ x \neq y, \\ \frac{f'(x)}{f(x)}, & x = y \in I, \end{cases} \tag{6}$$

is Schur-convex (Schur-concave) on I^2. The following example show that for the function $F(x,y)$ in (6) the function $F^2(x,y)$ is not Schur-convex and not Schur-concave in general.

EXAMPLE 2.1. Consider the function $f(t) := e^{\frac{1}{3}t^3 - t}$ for $-2 \leqslant t \leqslant 0$. It is easy to see that for the function F in (6), we have $F(x,x) = x^2 - 1$, for every $x \in [-2,0]$, and

$$F(x,y) = \frac{1}{3}(x^2 + y^2 + xy) - 1,$$

for every $x,y \in [-2,0]$, with $x \neq y$. Thus,

$$F(x,y) = \frac{1}{3}(x^2 + y^2 + xy) - 1,$$

for every $x,y \in [-2,0]$. If $x,y \in [-2,0]$, we have

$$\left(\frac{\partial F^2}{\partial y} - \frac{\partial F^2}{\partial x}\right)(y-x) = \frac{2}{3}(y-x)^2\left(\frac{1}{3}(x^2+y^2+xy) - 1\right).$$

Since $-1 \leqslant \frac{1}{3}(x^2+y^2+xy) - 1 \leqslant 3$, then $(\frac{\partial F^2}{\partial y} - \frac{\partial F^2}{\partial x})(y-x)$ has both positve values and negative values. Therefore by Theorem 1.1 the function $F^2(x,y)$ is not Schur-convex and not Schur-concave on $[-2,0] \times [-2,0]$.

In the following corollary we give a condition in which the function $F^2(x,y)$ is Schur-convex.

COROLLARY 2.1. *Let $f > 0$ on interval I and $f \in C^2(I)$. Suppose that the function $g : I^2 \to \mathbb{R}$ defined by $g(t,s) := \frac{f'(t)f'(s)}{f(t)f(s)}$ is convex on the co-ordinates (concave on the co-ordinates) on I^2. Then the function*

$$F^2(x,y) := \begin{cases} \left(\frac{\log f(x) - \log f(y)}{x-y}\right)^2, & x,y \in I,\ x \neq y, \\ \left(\frac{f'(x)}{f(x)}\right)^2, & x = y \in I, \end{cases}$$

is Schur-convex (Schur-concave) on I^2.

Proof. Since the function $g(t,s)$ is convex on the co-ordinates on I^2, the result follows from Theorem 2.1. $\square$

A generalized version of theorem 1.3 for co-ordinated convex functions is established in the next theorem.

THEOREM 2.2. *Let $D := [a_1,b_1]\times[a_1,b_1]$ be a square in $\mathbb{R}^2$ with $a_1 < b_1$, and the function $f : D \to \mathbb{R}$ is continuous, and has continuous third order partial derivatives on D°. Choose $a,b \in (a_1,b_1)$, with $a < b$, and let $\Delta := [a,b]\times[a,b]$. Suppose that f is convex on the co-ordinates on Δ, then the function $G : \Delta \to \mathbb{R}$ defined by*

$$G(x,y) := \begin{cases} \frac{1}{(y-x)^2}\int_x^y\int_x^y f(t,s)dtds - f(\frac{x+y}{2},\frac{x+y}{2}), & x \neq y, \quad x,y \in [a,b], \\ 0, & x = y, \quad x,y \in [a,b], \end{cases} \tag{7}$$

is Schur-convex on Δ.

Proof. *Case* 1: If $x,y \in [a,b]$, with $x = y$. Then Lemma 2.2 implies that

$$(y-x)\left(\frac{\partial G}{\partial y} - \frac{\partial G}{\partial x}\right) = 0$$

Case 2: If $x,y \in [a,b]$, with $x \neq y$. Then by Lemma 1.2 we have

$$(y-x)\left(\frac{\partial G}{\partial y} - \frac{\partial G}{\partial x}\right) \geqslant 0,$$

if

$$\begin{aligned} &\frac{1}{(y-x)^2}\int_x^y\int_x^y f(t,s)dtds \\ \leqslant\ &\frac{1}{4(y-x)}\left(\int_x^y \big(f(t,y)+f(t,x)\big)dt + \big(f(y,s)+f(x,s)\big)ds\right). \end{aligned}$$

The result follows from Theorem 1.1 and Theorem 1.4. $\square$

In the following example we show that the converse of theorem 2.1 is not true.

EXAMPLE 2.2. Consider the non co-ordinates convex function:

$$f(t,s) := t^2 - \frac{1}{2}s^2 \quad ,t,s \in [1,2].$$

It is easy to see that for the function F in (5), $F(x,x) = \frac{1}{2}x^2$, for every $x \in [1,2]$. Moreover for every $x,y \in [1,2]$, with $x \neq y$ we have

$$F(x,y) = \frac{1}{(y-x)^2}\int_x^y\int_x^y (t^2 - \frac{1}{2}s^2)dtds = \frac{1}{6}(x^2+y^2+xy).$$

Thus,

$$F(x,y) = \frac{1}{6}(x^2 + y^2 + xy),$$

for every $x,y \in [1,2]$. Clearly F is symmetric, continuous and differentiable on $[1,2] \times [1,2]$.

If $x,y \in [1,2]$, we have

$$(y-x)\left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right) = \frac{1}{6}(y-x)^2 \geqslant 0.$$

Therefore, by Theorem 1.1 the function F is Schur-convex.

The following remark show that the converse of theorem 1.4 is not valid in general.

REMARK 2.1. It is easy to see that for function f was defined in example 2.2 we have:

$$\begin{aligned}
f\left(\frac{x+y}{2}, \frac{x+y}{2}\right) &\leqslant \frac{1}{2(y-x)}\left[\int_x^y f\left(t, \frac{x+y}{2}\right)dt + \int_x^y f\left(\frac{x+y}{2}, s\right)ds\right] \\
&\leqslant \frac{1}{(y-x)^2}\int_x^y \int_x^y f(t,s)dtds \\
&\leqslant \frac{1}{4(y-x)}\left[\int_x^y (f(t,x)dt + f(t,y))dt\right. \\
&\quad \left. + \int_x^y (f(x,s)ds + f(y,s))ds\right] \\
&\leqslant \frac{f(x,x) + f(x,y) + f(y,x) + f(y,y)}{4},
\end{aligned}$$

for every $x,y \in [1,2]$, with $x \neq y$. This means that each of the inequalities in theorem 1.4 is valid while f is not convex on co-ordintes.

Finally the following example illustrates that the converse of theorem 2.2 is not true in general.

EXAMPLE 2.3. Consider the non co-ordinated convex function:

$$f(t,s) := \frac{1}{2}t^2 - \frac{1}{3}s^2, \quad t,s \in [0,1].$$

It is easy to see that for the function G in (7), $G(x,x) = 0$, for every $x \in [0,1]$. Moreover for every $x,y \in [0,1]$, with $x \neq y$ we have

$$\begin{aligned}
G(x,y) &= \frac{1}{(y-x)^2}\int_x^y \int_x^y \left(\frac{1}{2}t^2 - \frac{1}{3}s^2\right)dtds - \frac{1}{6}\left(\frac{x+y}{2}\right)^2 \\
&= \frac{1}{72}x^2 + \frac{1}{72}y^2 - \frac{1}{36}xy.
\end{aligned}$$

Thus,

$$G(x,y) = \frac{1}{72}x^2 + \frac{1}{72}y^2 - \frac{1}{36}xy,$$

for every $x,y \in [0,1]$. Clearly G is symmetric, continuous and differentiable on $[0,1] \times [0,1]$.

If $x,y \in [0,1]$, we have

$$(y-x)\left(\frac{\partial G}{\partial y} - \frac{\partial G}{\partial x}\right) = \frac{1}{18}(y-x)^2 \geqslant 0.$$

Therefore, by Theorem 1.1 the function G is Schur-convex.

REFERENCES

[1] B. C. ARNOLD, A. W. MARSHAL AND I. OLKIN, *Inequalities: Theory of Majorization and It's Applications*, Springer Series in Statistics, New York-Dordrecht-Heidelberg-London, 2011 (second edition).

[2] R. G. BARTEL, *The Elements of Real Analysis*, Jon Wiley and Sons, 1976 (second edition).

[3] Y. CHU, G. WANG, AND X. ZHANG, *Schur-convexity and Hadamard's inequality*, Math. Inequal. Appl., 13 (4) (2010), 725–731.

[4] S. S. DRAGOMIR, *On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plan*, Taiwan. J. Math., 5 (2001), 775–778.

[5] N. ELEZOVIĆ, J. PEČARIĆ, *A note on Schur-convex functions*, Rocky Mountain J. Math., 30 (3) (2000), 853–856.

[6] I. FRANJIĆ, J. PEČARIĆ, *Schur-convexity and the Simpson formula*, Applied Mathematics Letters, 24 (2011), 1565–1568.

[7] A. W. ROBERTS, D. E. VARBERG, *Convex functions*, Academic Press, New York, 1973.

[8] H.-N. SHI, *Schur-convex functions related to Hadamard-type inequalities*, J. Mat. Inequal., 1 (1) (2007), 127–136.

[9] X. ZHANG. Y. CHU, *Convexity of the integral arithmetic mean of a convex function*, Rocky mountain, J. Math., 40 (3) (2010), 1061–1068.

(Received May 21, 2018)

N. Safaei
Department of Mathematics, Faculty of Science
Lorestan University
6815144316, Khoramabad, Iran
e-mail: nouzarsafaei@yahoo.com

A. Barani
Department of Mathematics, Faculty of Science
Lorestan University
6815144316, Khoramabad, Iran
e-mail: barani.a@lu.ac.ir