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# MONOTONE ITERATIVE TECHNIQUE FOR $S$–ASYMPTOTICALLY PERIODIC PROBLEM OF FRACTIONAL EVOLUTION EQUATION WITH FINITE DELAY IN ORDERED BANACH SPACE

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*Abstract.* In this paper, we devote to considering $S$-asymptotically periodic problem of fractional evolution equation with delay in ordered Banach space. Under some weaker assumptions, we construct monotone iterative method in the presence of the lower and upper solutions to the delayed fractional evolution equation, and obtain the existence of maximal and minimal $S$-asymptotically periodic mild solutions. Finally, we present two examples to illustrate the feasibility of our abstract results.

## 1. Introduction

In this paper, we use a monotone iterative technique in the presence of the lower and upper solutions to discuss the existence of the minimal and maximal $S$-asymptotically $\omega$-periodic solutions to the following abstract fractional evolution equation

$$\begin{cases} {}^cD_t^q u(t) + Au(t) = F(t, u(t), u_t), & t \geqslant 0, \\ u(t) = \varphi(t), & t \in [-r, 0], \end{cases} \tag{1.1}$$

in the ordered Banach space $E$, where ${}^cD_t^q$ is the Caputo fractional derivation of order $q \in (0,1)$, $A : D(A) \subset E \to E$ is a closed linear operator, and $-A$ generates a $C_0$-semigroup $T(t)$ $(t \geqslant 0)$ in $E$, $F : \mathbb{R}^+ \times E \times \mathscr{B} \to E$ is a given function which will be specified later, $\varphi \in \mathscr{B}$; $r > 0$ is a constant, and $\mathscr{B} := C([-r,0],E)$ denotes the space of continuous functions from $[-r,0]$ into $E$ provided with the uniform norm topology. For $t \geqslant 0$, $u_t$ denotes the history function defined by $u_t(s) = u(t+s)$ for $s \in [-r,0]$, where $u$ is a continuous function from $[-r,\infty)$ into $E$.

In recent years, fractional calculus has attracted extensive attention from many scholars in different fields, such as, mathematicians, physicists, and so on, see the

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monographs of Podlubny [43], Agrawal [1], Zhou [51, 52] and references therein. Compared with integer-order calculus, the main advantage of fractional-order calculus is that it can accurately describe the memory or genetic characteristics of various new materials, or better describe the process or behavior of real dynamic systems. In particular, many scholars have found that in many practical applications, fractional derivatives of time can more truthfully describe the process and phenomena of things' motion development than integer derivatives. Since fractional evolution equations are abstract models in many practical applications such as engineering and physics, the study of fractional evolution equations has attracted more and more attention of mathematicians. There have been some works on the existence of mild solutions for semilinear fractional evolution equations, see [31, 4, 47, 20, 7, 48, 8, 9, 14, 15, 16] and the references therein.

It is well known that the periodic law of the development or movement of things is a common phenomenon in nature and human activities. However, in real life, many phenomena do not have strict periodicity. In order to better characterize these mathematical models, many scholars have introduced other definitions of generalized periodicity, such as almost periodicity, asymptotic periodicity, asymptotic almost periodicity, pseudo almost periodicity and $S$-asymptotic periodicity. On the other hand, because fractional derivative has genetic or memory properties, the solutions of periodic boundary value problems of fractional differential equations can not be extended periodically to time $t$ in $\mathbb{R}^+$. In particular, Ren et al [44] have proved the nonexistence of nonzero periodic solutions for Caputo type linear fractional evolution equation. Therefore, in view of the existence of many generalized periodic phenomena and the advantages of fractional derivatives in real life, such as memory and heredity, many papers focus on these types of solutions of fractional differential equations. Since $S$-asymptotically periodic functions were first studied in Banach space by Henríquez et al. [27], there are some papers about $S$-asymptotically periodic solutions for fractional evolution equations, one can refer to [18, 17, 42, 44, 34, 33].

As we all know, the monotone iteration technique of upper and lower solutions is an effective and flexible mechanism. By using this method, not only the existence theory of solutions can be obtained, but also the approximate iteration sequence of solutions can be obtained, which provides a reasonable and effective theoretical basis for the approximate solution of computers. In fact, the monotone sequences of the lower and upper approximate solutions converge to the minimal and maximal solutions between the lower and upper solutions. As early as the end of last century, Du and Lakshmikantham [22], Sun and Zhao [46] studied the initial values of ordinary differential equations by means of monotone iteration technique of upper and lower solutions. Later, Li [36] applied lower and upper solutions method to periodic solution problems for semilinear evolution equations without delay in abstract spaces, and obtained the existence of maximal and minimal periodic mild solutions by using the characteristics of positive operators semigroups and the monotone iteration scheme. For the abstract evolution equations, there are more results involving monotone iterative techniques and operator semigroups theory, we can see [13, 12, 11, 10, 32]. However, as far as we know, there are few results for the fractional evolution equations $S$-asymptotically periodic problems with delay by using the method of the lower and upper solutions coupled with the monotone iterative technique.

Recently, in [38] we dealt with the second-order ordinary differential equation periodic problem with delay in Banach spaces. Under the conditions that the nonlinear function satisfies quasi-monotonicity, the existence of the minimum and the maximum periodic solutions are obtained by using the monotone iteration technique of the upper and lower solutions. In [39], with the help of positive operator semigroup theory and monotone iterative technique of lower and upper solutions, we also obtained the existence and uniqueness of periodic mild solutions of the abstract evolution equation under some quasi-monotone conditions.

Motivated by the papers mentioned above, the purpose of this paper is to construct the general principle for lower and upper solutions coupled with the monotone iterative technique for the fractional evolution equations $S$-asymptotically periodic problems with delay, and obtain the existence of maximal and minimal periodic mild solutions, which will fill the research gap in this area.

The paper is organized as follows. In Section 2, we introduce some notions, definitions, and preliminary facts which are used throughout this paper. Under the different assumptions, the existence results of the minimum and the maximum $S$-asymptotically $\omega$-periodic mild solutions of the problem (1.1) are given in Section 3. In Section 4, we give two examples to illustrate our main results in Section 3.

## 2. Preliminaries

Throughout this paper, we assume that $(E, \|\cdot\|)$ is an ordered Banach space, whose positive cone $K = \{u \in E | u \geqslant \theta\}$ is normal with normal constant $N$, $\theta$ is the zero element of $E$.

Assume that $A : D(A) \subset E \to E$ is a closed linear operator and $-A$ generates a $C_0$-semigroup $T(t)\ \ (t \geqslant 0)$ in $E$. Here, we only recall some notions and properties that are essential for us. For a general $C_0$-semigroup $T(t)\ \ (t \geqslant 0)$, there exist $M \geqslant 1$ and $\nu \in \mathbb{R}$ such that (see [41])

$$\|T(t)\| \leqslant Me^{\nu t}, \quad t \geqslant 0. \tag{2.1}$$

Specially, $C_0$-semigroup $T(t)(t \geqslant 0)$ is called to be uniformly bounded,

$$\|T(t)\| \leqslant M, \quad t \geqslant 0. \tag{2.2}$$

Let

$$\nu_0 = \inf\{\nu \in \mathbb{R} | \text{ There exists } M \geqslant 1 \text{ such that } \|T(t)\| \leqslant Me^{\nu t},\ \forall t \geqslant 0\}, \tag{2.3}$$

then $\nu_0$ is called the growth exponent of the $C_0$-semigroup $T(t)\ \ (t \geqslant 0)$. Moreover, if $\nu_0 < 0$, then the $C_0$-semigroup $T(t)\ \ (t \geqslant 0)$ is said to be exponentially stable. Clearly, the exponentially stable $C_0$-semigroup $T(t)\ \ (t \geqslant 0)$ is uniformly bounded.

DEFINITION 2.1. ([6]) If $T(t)x \geqslant \theta$ for each $x \geqslant \theta$ and $t \geqslant 0$, then $C_0$-semigroup $T(t)\ \ (t \geqslant 0)$ on $E$ is said to be positive.

It is not difficult to find that $-(A+LI)$ also generates a $C_0$-semigroup $S(t) = e^{-Lt}T(t)(t \geqslant 0)$ in $E$ for any $L \geqslant 0$. And $S(t)(t \geqslant 0)$ is a positive $C_0$-semigroup if

$T(t)$ $(t \geqslant 0)$ is positive. For more details of the properties of the $C_0$-semigroups and the positive $C_0$-semigroup, we can refer to the monographs [40, 45] and the paper [35].

Now, we recall some basic definitions and properties of the fractional calculus theory which are used in this paper.

DEFINITION 2.2. ([29]) The fractional integral of order $q \in (0,1)$ with the lower limit zero for a function $f$ is defined as

$$I_t^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the gamma function.

DEFINITION 2.3. ([30]) The Caputo derivative of order $q \in (0,1)$ with the lower limit zero for a function $f \in C^1[0,\infty)$ is defined as

$${}^cD_t^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} f'(s)ds, \quad t > 0.$$

REMARK. If $f$ is an abstract function with values in $E$, then the integrals which appear in Definition 2.2 and 2.3 are taken in Bochner's sense.

Define operators $\mathfrak{T}(t)(t \geqslant 0)$ and $\mathfrak{S}(t)(t \geqslant 0)$ in $E$ as following

$$\mathfrak{T}(t) = \int_0^\infty \xi_q(s) T(t^q s) ds, \ \ \mathfrak{S}(t) = q \int_0^\infty s\xi_q(s) T(t^q s) ds, \tag{2.4}$$

where

$$\xi_q(s) = \frac{1}{\pi q} \sum_{n=1}^{\infty} (-s)^{n-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \qquad s \in (0,\infty) \tag{2.5}$$

is a probability density function defined on $(0,\infty)$, which satisfies

$$\xi_q(s) \geqslant 0, s \in (0,\infty), \qquad \int_0^\infty \xi_q(s)ds = 1, \qquad \int_0^\infty s\xi_q(s)ds = \frac{1}{\Gamma(1+q)}.$$

LEMMA 2.4. *The operators $\mathfrak{T}(t)(t \geqslant 0)$ and $\mathfrak{S}(t)(t \geqslant 0)$ defined by (2.4) have the following properties:*

*(i) If $T(t)$ $(t \geqslant 0)$ is a uniformly bounded $C_0$-semigroup, then $\mathfrak{T}(t)$ and $\mathfrak{S}(t)$ are linear and bounded operators for any fixed $t \in \mathbb{R}^+$, i.e,*

$$\|\mathfrak{T}(t)x\| \leqslant M\|x\|, \qquad \|\mathfrak{S}(t)x\| \leqslant \frac{M}{\Gamma(q)}\|x\|, \qquad \forall\, x \in E. \tag{2.6}$$

*(ii) If $T(t)$ $(t \geqslant 0)$ is a $C_0$-semigroup, then $\mathfrak{T}(t)(t \geqslant 0)$ and $\mathfrak{S}(t)(t \geqslant 0)$ are strongly continuous operators, which means that for any $x \in E$ and $0 \leqslant t_1 \leqslant t_2$,*

$$\|\mathfrak{T}(t_2)x - \mathfrak{T}(t_1)x\| \to 0 \text{ and } \|\mathfrak{S}(t_2)x - \mathfrak{S}(t_1)x\| \to 0 \text{ as } t_2 - t_1 \to 0.$$

*(iii) If $T(t)$ is a compact semigroup, then $\mathfrak{T}(t)$ and $\mathfrak{S}(t)$ are compact operators for every $t > 0$.*

*(iv) If $T(t)$ is an equicontinuous semigroup, then $\mathfrak{T}(t)$ and $\mathfrak{S}(t)$ are uniformly continuous for $t > 0$.*

*(v) If $T(t)$ is a positive $C_0$-semigroup, then $\mathfrak{T}(t)$ and $\mathfrak{S}(t)$ are positive operators.*

*Proof.* For the proof of (i)–(iii), one can refer to [50, 23, 48]. We only check (iv) and (v) as follows.

(iv) For any $0 < t_1 < t_2$, it is easy to see

$$\|\mathfrak{T}(t_2) - \mathfrak{T}(t_1)\| \leqslant \int_0^\infty \xi_q(s)\|T(t_2^q s) - T(t_1^q s)\|ds,$$

and

$$\|\mathfrak{S}(t_2) - \mathfrak{S}(t_1)\| \leqslant q\int_0^\infty s\xi_q(s)\|T(t_2^q s) - T(t_1^q s)\|ds.$$

Since $T(t)$ $(t \geqslant 0)$ is an equicontinuous semigroup, and for any $0 < t_1 < t_2$ and $s \geqslant 0$,

$$\|T(t_2^q s) - T(t_1^q s)\| \leqslant 2M,$$

then by the Lebesgue dominated convergence theorem and the properties of the function $\xi_q(s)$, one can deduce that $\mathfrak{T}(t)$ and $\mathfrak{S}(t)$ are uniformly continuous by operator norm for $t > 0$.

(v) From (2.4), the positivity of the semigroup $T(t)$ $(t \geqslant 0)$ and the function $\xi_q(s)$ defined by (2.5), it follows that $\mathfrak{T}(t)(t \geqslant 0)$ and $\mathfrak{S}(t)(t \geqslant 0)$ are also positive.

This completes the proof of Lemma 2.4. □

LEMMA 2.5. ([28]) *Assume that $-A$ generates an exponentially stable $C_0$-semigroup $T(t)(t \geqslant 0)$ in $E$, whose growth exponent denotes $\nu_0 < 0$. Let*

$$m = M\max\{\sup_{t\leqslant 0} E_q(\nu_0 t^q)(1+t)^q, \sup_{t\geqslant 0} E_{q,q}(\nu_0 t^q)(1+t)^{2q}\}, \tag{2.7}$$

*where $E_q$ and $E_{q,q}$ are the Mittag-Leffler functions. Then*

$$\|\mathfrak{T}(t)\| \leqslant \frac{m}{(1+t)^q}, \quad \|\mathfrak{S}(t)\| \leqslant \frac{m}{(1+t)^{2q}}, \ t \in \mathbb{R}^+. \tag{2.8}$$

REMARK. For the definitions and properties of the Mittag-Leffler functions, we can refer to [29] and references therein.

Next, let $C_b(\mathbb{R}^+, E)$ denote the Banach space of all bounded and continuous functions from $\mathbb{R}^+$ to $E$ equipped with the norm $\|u\|_C = \sup_{t\in\mathbb{R}^+}\|u(t)\|$, and let $\mathscr{B} = C([-r,0],E)$ denote the space of continuous functions from $[-r,0]$ into $E$ endowed with the uniform norm $\|\phi\|_{\mathscr{B}} = \sup_{s\in[-r,0]}\|\phi(s)\|$, where $r > 0$ is a constant.

DEFINITION 2.6. ([27]) A function $u \in C_b(\mathbb{R}^+, E)$ is called $S$-asymptotically $\omega$-periodic if there exists $\omega > 0$ such that $\lim\limits_{t\to\infty}\|u(t+\omega) - u(t)\| = 0$. In this case, we say that $\omega$ is an asymptotic periodic of $u$. It is clear that if $\omega$ is an asymptotic period for $u$, then every $k\omega$, $k = 1, 2, \cdots$, is also an asymptotic period of $u$.

Let $SAP_\omega(E)$ represent the subspace of $C_b(\mathbb{R}^+,E)$ consisting of all the $E$-value $S$-asymptotically $\omega$-periodic functions endowed with the uniform convergence norm denoted by $\|\cdot\|_C$. Then $SAP_\omega(E)$ is a Banach space (see [27, Proposition 3.5]). If $u\in SAP_\omega(E)$, then it is not difficult to test and verify that the function $t\to u_t$ belongs to $SAP_\omega(\mathscr{B})$ (see [34, 33]).

For the rest of this paper, we define

$$\Omega := \{u\in C([-r,\infty),E)|\ u|_{[-r,0]}\in\mathscr{B} \text{ and } u|_{\mathbb{R}^+}\in SAP_\omega(E)\}.$$

It is easy to see that $\Omega$ is a Banach space equipped with the norm

$$\|u\|_\Omega = \sup_{t\in[-r,\infty)}\|u(t)\|.$$

Define a positive cone $K_\Omega$ by

$$K_\Omega = \{u\in\Omega|\ u(t)\in K,\ t\in[-r,\infty)\},$$

then $\Omega$ is an ordered Banach spaces with the partial order relation "$\leqslant$" induced by the cone $K_\Omega$, and $K_\Omega$ is normal with the normal constant $N$. Similarly, $\mathscr{B}$ is also an order Banach space whose partial ordering "$\leqslant$" induced by a positive cone $K_B = \{\phi\in\mathscr{B}|\phi(s)\in K, s\in[-r,0]\}$ with the normal constant $N$. For $v,w\in\Omega$ with $v\leqslant w$, we use $[v,w]$ to denote the order interval $\{u|v\leqslant u\leqslant w\}$ in $\Omega$, moreover, $[v(t),w(t)]$ and $[v_t,w_t]$ to denote the order intervals $\{u(t)|v(t)\leqslant u(t)\leqslant w(t), t\in[-r,\infty)\}$ in $E$ and $\{u_t|v_t\leqslant u_t\leqslant w_t, t\geqslant 0\}$ in $\mathscr{B}$, respectively. Set

$$C^q(\mathbb{R}^+,E) = \{u\in C(\mathbb{R}^+,E)|{}^cD_t^q u \text{ exists and } {}^cD_t^q u\in C(\mathbb{R}^+,E)\}.$$

By $E_1$, we denote the Banach space $D(A)$ with the graph norm $\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|$.

DEFINITION 2.7. If a function $v\in\Omega$ with $v|_{\mathbb{R}^+}\in C^q(\mathbb{R}^+,E)\cap C(\mathbb{R}^+,E_1)$, satisfies

$$\begin{cases} {}^cD_t^q v(t) + Av(t) \leqslant F(t,v(t),v_t), & t\geqslant 0,\\ v(t)\leqslant\varphi(t), & t\in[-r,0], \end{cases} \tag{2.9}$$

then $v$ is called a lower $S$-asymptotically $\omega$-periodic solution of Eq. (1.1). If the inequality of (2.9) is inverse, we call it an upper $S$-asymptotically $\omega$-periodic solution of Eq. (1.1).

Now, we give the definition of the mild solution for the equation (1.1) as follows.

DEFINITION 2.8. A function $u:[-r,\infty)\to E$ is said to be a mild solution of the problem (1.1) if $u\in C([-r,\infty),E)$ and satisfies

$$u(t) = \begin{cases} \mathfrak{T}(t)\varphi(0) + \int_0^t (t-s)^{q-1}\mathfrak{S}(t-s)F(s,u(s),u_s)ds, & t\geqslant 0,\\ \varphi(t), & t\in[-r,0]. \end{cases} \tag{2.10}$$

Moreover, if $u\in\Omega$, then $u$ is called $S$-asymptotically $\omega$-periodic mild solution of Eq. (1.1).

Next, we recall some properties of measure of noncompactness which will be used in the proof of our main results. Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For any $D \subset C([a,b],E)$ and $t \in [a,b]$, set $D(t) = \{u(t)|u \in D\} \subset E$. If $D$ is bounded in $C([a,b],E)$, then $D(t)$ is bounded in $E$, and $\alpha(D(t)) \leqslant \alpha(D)$. For more details of the definition and properties of the measure of noncompactness, we refer to the monographs [5, 19, 24].

LEMMA 2.9. ([5, 24, 25]) *Let $E$ be a Banach space and let $D \subset C([a,b],E)$ be bounded and equicontinuous. Then $\alpha(D(t))$ is continuous on $[a,b]$, and*

$$\alpha(D) = \max_{t\in[a,b]} \alpha(D(t)).$$

LEMMA 2.10. ([26]) *Let $E$ be a Banach space, $D = \{u_n\} \subset C([a,b],E)$ be a bounded and countable set. Then $\alpha(D(t))$ is Lebesgue integrable on $[a,b]$, and*

$$\alpha\Big(\Big\{\int_a^b u_n(s)ds\Big\}\Big) \leqslant 2\int_a^b \alpha(D(t))dt.$$

The following lemma is also needed.

LEMMA 2.11. ([49]) *Assume that $f(t)$ is a nonnegative function locally integrable on $0 \leqslant t < \Lambda$(some $\Lambda \leqslant \infty$), $g(t)$ is a nonnegative, nondecreasing continuous bounded function on $0 \leqslant t < \Lambda$, $p,q > 0$. Suppose that $h(t)$ is nonnegative and locally integrable on $0 \leqslant t < \Lambda$ with*

$$h(t) \leqslant f(t) + g(t)\int_0^t (t-s)^{q-1}h(s)ds.$$

*Then,*

$$h(t) \leqslant f(t) + \int_0^t \Big(\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(q))^n}{\Gamma(nq)}(t-s)^{nq-1}f(s)\Big)ds.$$

## 3. Main results

THEOREM 3.1. *Let $E$ be an ordered Banach space, whose positive cone $K$ is a normal cone, let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive and compact semigroup $T(t)$ $(t \geqslant 0)$ in $E$, whose growth exponent denotes by $\nu_0$. Assume that $\omega > 0$ is a constant and the problem (1.1) has lower and upper $S$-asymptotically $\omega$-periodic solutions $v^{(0)}, w^{(0)} \in \Omega$ with $v^{(0)} \leqslant w^{(0)}$. If the nonlinear function $F : \mathbb{R}^+ \times E \times \mathscr{B} \to E$ is continuous and satisfies the following conditions*

*(H1) for any bounded sets $D \subset E$, $\mathscr{D} \subset \mathscr{B}$, the set $\{F(t,x,\phi)|t \geqslant 0, x \in D, \phi \in \mathscr{D}\}$ is bounded, and*

$$\lim_{t\to\infty} \|F(t+\omega,x,\phi) - F(t,x,\phi)\| = 0$$

*for all $x \in E$, $\phi \in \mathscr{B}$,* *(H2) there exists a constant $L \geqslant 0$ such that*

$$F(t,x_2,\phi_2) - F(t,x_1,\phi_1) \geqslant -L(x_2 - x_1)$$

*for any $t \in \mathbb{R}^+$ and $v^{(0)}(t) \leqslant x_1 \leqslant x_2 \leqslant w^{(0)}(t)$, $v_t^{(0)} \leqslant \phi_1 \leqslant \phi_2 \leqslant w_t^{(0)}$, then the problem (1.1) has minimal and maximal $S$-asymptotically $\omega$-periodic mild solutions $\underline{u},\overline{u}$ between $v^{(0)}$ and $w^{(0)}$, which can be obtained by monotone iterative sequences starting from $v^{(0)}$ and $w^{(0)}$.*

*Proof.* Obviously, the problem (1.1) is equal to the following problem

$$\begin{cases} {}^cD_t^q u(t) + Au(t) + Lu(t) = F(t,u(t),u_t) + Lu(t), & t \geqslant 0, \\ u(t) = \varphi(t), & t \in [-r,0], \end{cases} \tag{3.1}$$

where the constant $L$ is decided by the condition (H2).

Let $L > |\nu_0|$ (otherwise replace $L$ with $L + |\nu_0|$), then $-(A+LI)$ generates an exponentially stable $C_0$-semigroup $S(t) = e^{-Lt}T(t)(t \geqslant 0)$ in $E$, whose growth exponent is $\mu_0 := -L + \nu_0 < 0$. Moreover, it is easy to see that $S(t)(t \geqslant 0)$ is a positive and compact $C_0$-semigroup since the semigroup $T(t)$ $(t \geqslant 0)$ is positive and compact. Let $\overline{M} = \sup_{t \geqslant 0} \|S(t)\|$ and

$$\overline{m} = \overline{M} \max\{\sup_{t \leqslant 0} E_q(\mu_0 t^q)(1+t)^q, \sup_{t \geqslant 0} E_{q,q}(\mu_0 t^q)(1+t)^{2q}\}. \tag{3.2}$$

We define two operators $\mathscr{T}(t)(t \geqslant 0)$ and $\mathscr{S}(t)(t \geqslant 0)$ by

$$\mathscr{T}(t)x = \int_0^\infty \xi_q(s)S(t^q s)x ds, \quad \mathscr{S}(t)x = q\int_0^\infty s\xi_q(s)S(t^q s)x ds, \tag{3.3}$$

where $x \in E$ and $\xi_q(s)$ is the function defined by (2.6). Thus, the operators $\mathscr{T}(t)(t \geqslant 0)$ and $\mathscr{S}(t)(t \geqslant 0)$ have the properties (i)–(v) in Lemma 2.4. Moreover, from Lemma 2.5, it follows that for any $t \geqslant 0$ ,

$$\|\mathscr{T}(t)\| \leqslant \frac{\overline{m}}{(1+t)^q}, \quad \|\mathscr{S}(t)\| \leqslant \frac{\overline{m}}{(1+t)^{2q}}. \tag{3.4}$$

For each $u \in [v^{(0)}, w^{(0)}]$, it is easy to see that $u_t \in [v_t^{(0)}, w_t^{(0)}] \subset SAP_\omega(\mathscr{B})$ for any $t \geqslant 0$. Now, we define an operator $\mathscr{Q}$ on $[v^{(0)}, w^{(0)}]$ as following

$$\mathscr{Q}u(t) = \begin{cases} \mathscr{T}(t)\varphi(0) + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s) \cdot \Big(F(s,u(s),u_s) + Lu(s)\Big)ds, & t \geqslant 0, \\ \varphi(t), & t \in [-r,0]. \end{cases} \tag{3.5}$$

By the normality of the cone $K$, the conditions (H1) and (H2), we find that for any $u \in [v^{(0)}, w^{(0)}]$, there exists a constant $M_0$ such that

$$\sup_{t \geqslant 0}\{\|F(t,u(t),u_t)\| + L\|u(t)\|\} \leqslant M_0. \tag{3.6}$$

From (3.3), it follows that

$$\begin{aligned}
&\Big\| \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)\cdot\Big(F(s,u(s),u_s)+Lu(s)\Big)ds\Big\| \\
\leqslant& \int_0^t (t-s)^{q-1}\|\mathscr{S}(t-s)\|\cdot\Big\|F(s,u(s),u_s)+Lu(s)\Big\|ds \\
\leqslant& qM_0\overline{M}\int_0^t\int_0^\infty \sigma\xi_q(\sigma)(t-s)^{q-1}e^{\mu_0(t-s)^q\sigma}d\sigma ds \\
\leqslant& M_0\overline{M}\int_0^\infty \xi_q(\sigma)d\sigma\int_0^\infty e^{\mu_0 s}ds = \frac{M_0\overline{M}}{|\mu_0|},
\end{aligned} \tag{3.7}$$

Thus, one can find that $\mathscr{Q}:[v^{(0)},w^{(0)}]\to C([-r,\infty),E)$ is well defined. Therefore, by Definition 2.8, (3.1) and (3.5), we can assert $u\in[v^{(0)},w^{(0)}]$ is an $S$-asymptotically $\omega$-periodic mild solution of the problem (1.1) if and only if $u$ is a fixed point of the operator $\mathscr{Q}$.

Now, we complete the proof by five steps.

*Step* 1. We prove that $\mathscr{Q}([v^{(0)},w^{(0)}])\subset\Omega$.

For any $u\in[v^{(0)},w^{(0)}]$, it is clear that $\mathscr{Q}u$ is defined on $[-r,\infty)$, and because $\varphi\in\mathscr{B}$, we have $\mathscr{Q}u|_{[-r,0]}\in\mathscr{B}$. Thus it suffices to show that the function

$$f:t\to\mathscr{T}(t)\varphi(0)+\int_0^t (t-s)^{q-1}\mathscr{S}(t-s)\cdot\Big(F(s,u(s),u_s)+Lu(s)\Big)ds\in SAP_\omega(E).$$

Since $u|_{\mathbb{R}^+}\in SAP_\omega(E)$ and $u_t\in SAP_\omega(\mathscr{B})$ for all $t\geqslant 0$, hence, for any $\varepsilon>0$, there exists a constant $t_{\varepsilon,1}>0$ such that $\|u(t+\omega)-u(t)\|\leqslant\varepsilon$ and $\|u_{t+\omega}-u_t\|_{\mathscr{B}}\leqslant\varepsilon$ for every $t\geqslant t_{\varepsilon,1}$. Thus, by the condition (H1), for $t\geqslant t_{\varepsilon,1}$, we have

$$\|F(t,u(t+\omega),u_{t+\omega})+Lu(t+\omega)-F(t,u(t),u_t)-Lu(t)\|\leqslant\frac{|\mu_0|}{2\overline{M}}\varepsilon, \tag{3.8}$$

and we can find a positive constant $t_{\varepsilon,2}$ sufficiently large such that for $t\geqslant t_{\varepsilon,2}$,

$$\|F(t+\omega,u(t+\omega),u_{t+\omega})+Lu(t+\omega)-F(t,u(t+\omega),u_{t+\omega})-Lu(t+\omega)\|\leqslant\frac{|\mu_0|}{2\overline{M}}\varepsilon. \tag{3.9}$$

Then for $t>t_\varepsilon:=\max\{t_{\varepsilon,1},t_{\varepsilon,2}\}$, from (3.5), it follows that

$$\begin{aligned}
&f(t+\omega)-f(t)\\
=&\ \mathscr{T}(t+\omega)\varphi(0)+\int_0^{t+\omega}(t+\omega-s)^{q-1}\mathscr{S}(t+\omega-s)\cdot\Big(F(s,u(s),u_s)+Lu(s)\Big)ds\\
&-\mathscr{T}(t)\varphi(0)-\int_0^t(t-s)^{q-1}\mathscr{S}(t-s)\cdot\Big(F(s,u(s),u_s)+Lu(s)\Big)ds\\
=&\ \mathscr{T}(t+\omega)\varphi(0)-\mathscr{T}(t)\varphi(0)\\
&+\int_0^\omega(t+\omega-s)^{q-1}\mathscr{S}(t+\omega-s)\Big(F(s,u(s),u_s)+Lu(s)\Big)ds
\end{aligned}$$

$$+\int_0^t (t-s)^{q-1}\mathscr{S}(t-s)\cdot\Big(F(s+\omega,u(s+\omega),u_{s+\omega})+Lu(s+\omega)$$
$$-F(s,u(s),u_s)-Lu(s)\Big)ds$$
$$:= I_1(t)+I_2(t)+I_3(t).$$

Then
$$\|f(t+\omega)-f(t)\|\leqslant\|I_1(t)\|+\|I_2(t)\|+\|I_3(t)\|. \tag{3.10}$$

By (3.4) and (3.6), we have

$$\begin{aligned}\|I_1(t)\| &\leqslant \|\mathscr{T}(t+\omega)\varphi(0)\|+\|\mathscr{T}(t)\varphi(0)\|\\ &\leqslant (\|\mathscr{T}(t+\omega)\|+\|\mathscr{T}(t)\|)\cdot\|\varphi\|_{\mathscr{B}}\\ &\leqslant \frac{2\overline{m}\|\varphi\|_{\mathscr{B}}}{(1+t)^q},\end{aligned}$$

$$\begin{aligned}\|I_2(t)\| &\leqslant \int_0^{\omega}(t+\omega-s)^{q-1}\|\mathscr{S}(t+\omega-s)\|\cdot\Big\|F(s,u(s),u_s)+Lu(s)\Big\|ds\\ &\leqslant \int_0^{\omega}(t+\omega-s)^{q-1}\cdot\frac{\overline{m}M_0}{(1+t+\omega-s)^{2q}}ds\\ &\leqslant \frac{\overline{m}M_0((t+\omega)^q-t^q)}{q(1+t)^{2q}}\\ &\leqslant \frac{\overline{m}M_0\omega^q}{q(1+t)^{2q}},\end{aligned}$$

hence, we deduce that $\|I_1\|,\|I_2\|$ tend to 0 as $t\to\infty$. By (3.4) and (3.6)–(3.9), we obtain

$$\begin{aligned}\|I_3(t)\| \leqslant& \int_0^t (t-s)^{q-1}\|\mathscr{S}(t-s)\|\cdot\Big\|F(s+\omega,u(s+\omega),u_{s+\omega})+Lu(s+\omega)\\ &-F(s,u(s+\omega),u_{s+\omega})-Lu(s+\omega)\Big\|ds\\ &+\int_0^t (t-s)^{q-1}\|\mathscr{S}(t-s)\|\cdot\Big\|F(s,u(s+\omega),u_{s+\omega})+Lu(s+\omega)\\ &-F(s,u(s),u_s)-Lu(s)\Big\|ds\\ \leqslant& 4M_0\int_0^{t_\varepsilon}\frac{(t-s)^{q-1}\overline{m}}{(1+t-s)^{2q}}ds+\int_{t_\varepsilon}^t (t-s)^{q-1}\|\mathscr{S}(t-s)\|ds\frac{|\mu_0|\varepsilon}{\overline{M}}\\ \leqslant& 4\overline{m}M_0\int_0^{t_\varepsilon}(t-s)^{-q-1}ds+\int_0^t (t-s)^{q-1}\|\mathscr{S}(t-s)\|ds\frac{|\mu_0|\varepsilon}{\overline{M}}\\ \leqslant& 4\overline{m}M_0\frac{(t-t_\varepsilon)^{-q}-t^{-q}}{q}+\varepsilon,\end{aligned}$$

which implies that $\|I_3(t)\|$ tends to 0 as $t\to\infty$.

Thus, from the above results, we can deduce that

$$t \to T(t)\varphi(0) + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)\cdot\Big(F(s,u(s),u_s) + Lu(s)\Big)ds \in SAP_\omega(E).$$

Combining this with the definition of $\mathscr{Q}$, we can conclude that $\mathscr{Q}u \in \Omega$ for any $u \in [v^{(0)}, w^{(0)}]$, which implies that $\mathscr{Q}([v^{(0)}, w^{(0)}]) \subset \Omega$.

*Step* 2. We show that $\mathscr{Q} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is a monotone increasing operator.

On the one hand, let

$${}^cD_t^q v^{(0)}(t) + Av^{(0)}(t) + Lv^{(0)}(t) := h(t), \quad t \geqslant 0.$$

By Definition 2.7, Definition 2.8, and the positivity of operators $\mathscr{T}(t)$ and $\mathscr{S}(t)$, for $t \geqslant 0$, one can obtain that

$$\begin{aligned} v^{(0)}(t) &= \mathscr{T}(t)v^{(0)}(0) + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)h(s)ds \\ &\leqslant \mathscr{T}(t)\varphi(0) + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)\Big(F(s,v^{(0)}(s),v_s^{(0)}) + Lv^{(0)}(s)\Big)ds \\ &= Qv^{(0)}(t), \end{aligned}$$

and $v^{(0)}(t) \leqslant \varphi(t)$ for $t \in [-r, 0]$, which imply that $v^{(0)} \leqslant \mathscr{Q}v^{(0)}$. Similarly, we can show that $Qw^{(0)} \leqslant w^{(0)}$.

On the other hand for any $u^{(1)}, u^{(2)} \in [v^{(0)}, w^{(0)}]$ with $u^{(1)} \leqslant u^{(2)}$ and $t \geqslant 0$, we have $v^{(0)}(t) \leqslant u^{(1)}(t) \leqslant u^{(2)}(t) \leqslant w^{(0)}(t)$, $v_t^{(0)} \leqslant u_t^{(1)} \leqslant u_t^{(2)} \leqslant w_t^{(0)}$. From the condition (H2), we obtain that

$$F(t,u^{(2)}(t),u_t^{(2)}) + Lu^{(2)}(t) \geqslant F(t,u^{(1)}(t),u_t^{(1)}) + Lu^{(1)}(t).$$

Thus, by means of the positivity of the operator $\mathscr{S}(t)(t \geqslant 0)$, one has

$$\begin{aligned} &\int_0^t (t-s)^{q-1}\mathscr{S}(t-s)\Big(F(s,u^{(2)}(s),u_s^{(2)}) + Lu^{(2)}(s)\Big)ds \\ \geqslant &\int_0^t (t-s)^{q-1}\mathscr{S}(t-s)\Big(F(s,u^{(1)}(s),u_s^{(1)}) + Lu^{(1)}(s)\Big)ds. \end{aligned}$$

Therefor, by (3.5) and the positivity of the operator $\mathscr{T}(t)(t \geqslant 0)$, we can obtain that $\mathscr{Q}u^{(1)} \leqslant \mathscr{Q}u^{(2)}$.

Hence, $\mathscr{Q} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is a monotone increasing operator.

Now, we define two sequences $\{v^{(i)}\}$ and $\{w^{(i)}\}$ in $[v^{(0)}, w^{(0)}]$ by the iterative scheme

$$v^{(i)} = \mathscr{Q}v^{(i-1)}, \quad w^{(i)} = \mathscr{Q}w^{(i-1)}, \qquad i = 1, 2, \cdots. \tag{3.11}$$

Then from the monotonicity of the operator $\mathscr{Q}$, it follows that

$$v^{(0)} \leqslant v^{(1)} \leqslant v^{(2)} \leqslant \cdots \leqslant v^{(i)} \leqslant \cdots \leqslant w^{(i)} \leqslant \cdots \leqslant w^{(2)} \leqslant w^{(1)} \leqslant w^{(0)}. \tag{3.12}$$

*Step* 3. We show that $\{v^{(i)}\},\{w^{(i)}\}\subset[v^{(0)},w^{(0)}]$ are equicontinuous in $[-r,\infty)$.

In fact, for any $u\in[v^{(0)},w^{(0)}]$, by (3.5), we only consider it on $[0,\infty)$. Without loss of generality, we may assume that $0\leqslant t_1<t_2$. By (3.5), one can see

$$\begin{aligned}
&\|\mathscr{Q}u(t_2)-\mathscr{Q}u(t_1)\|\\
=&\ \Big\|\mathscr{T}(t_2)u(0)+\int_0^{t_2}(t_2-s)^{q-1}\mathscr{S}(t_2-s)\cdot\Big(F(s,u(s),u_s)+Lu(s)\Big)ds\\
&-\mathscr{T}(t_1)u(0)-\int_0^{t_1}(t_1-s)^{q-1}\mathscr{S}(t_1-s)\cdot\Big(F(s,u(s),u_s)+Lu(s)\Big)ds\Big\|\\
\leqslant&\ \|\mathscr{T}(t_2)u(0)-\mathscr{T}(t_1)u(0)\|\\
&+\int_0^{t_1}((t_2-s)^{q-1}-(t_1-s)^{q-1})\cdot\|\mathscr{S}(t_2-s)\|\cdot\Big\|F(s,u(s),u_s)+Lu(s)\Big\|ds\\
&+\int_0^{t_1}(t_1-s)^{q-1}\cdot\|\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)\|\cdot\Big\|F(s,u(s),u_s)+Lu(s)\Big\|ds\\
&+\int_{t_1}^{t_2}(t_2-s)^{q-1}\cdot\|\mathscr{S}(t_2-s)\|\cdot\Big\|F(s,u(s),u_s)+Lu(s)\Big\|ds\\
:=&\ J_1+J_2+J_3+J_4.
\end{aligned}$$

Next, we check $\|J_i\|$ tend to 0 independently of $u\in[v^{(0)},w^{(0)}]$ as $t_2-t_1\to0, i=1,2,3,4$. By Lemma 2.4 (ii), it is easy to see that $J_1\to0$ as $t_2-t_1\to0$. By (3.4) and (3.6) , we can obtain

$$\begin{aligned}
J_2=&\int_0^{t_1}((t_2-s)^{q-1}-(t_1-s)^{q-1})\cdot\|\mathscr{S}(t_2-s)\|\cdot\Big\|F(s,u(s),u_s)+Lu(s)\Big\|ds\\
\leqslant&\ \overline{m}M_0\int_0^{t_1}\frac{(t_1-s)^{q-1}-(t_2-s)^{q-1}}{(1+t_2-s)^{2q}}ds\\
\leqslant&\ \frac{\overline{m}M_0}{(1+t_2-t_1)^{2q}}\Big(t_1^q-t_2^q+(t_2-t_1)^q\Big)\\
\leqslant&\ 2\overline{m}M_0(t_2-t_1)^q\\
\to&\ 0\ \text{ as }\ t_2-t_1\to0.
\end{aligned}$$

If $t_1=0$ and $t_2>0$, then it easy to see that $J_3=0$. For $t_1>0$ and $\varepsilon>0$ small enough, by (3.4), (3.6) and Lemma 2.4 (iv), we get that

$$\begin{aligned}
J_3=&\int_0^{t_1}(t_1-s)^{q-1}\cdot\|\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)\|\cdot\Big\|F(s,u(s),u_s)+Lu(s)\Big\|ds\\
\leqslant&\ M_0\int_0^{t_1-\varepsilon}(t_1-s)^{q-1}\cdot\|\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)\|ds\\
&+M_0\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{q-1}\cdot\|\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)\|ds
\end{aligned}$$

$$\begin{aligned}
&\leqslant \sup_{s\in[0,t_1-\varepsilon]} \|\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)\|\cdot M_0\int_0^{t_1-\varepsilon}(t_1-s)^{q-1}ds\\
&\quad +2\overline{m}M_0\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{q-1}ds\\
&\leqslant \sup_{s\in[0,t_1-\varepsilon]} \|\mathscr{S}(t_2-s)-\mathscr{S}(t_1-s)\|\cdot\frac{M_0(t_1^q-\varepsilon^q)}{q}+\frac{2\overline{m}M_0\varepsilon^q}{q}\\
&\to 0 \ \text{ as } \ t_2-t_1\to 0.
\end{aligned}$$

Finally, from (3.4) and (3.6), we have

$$J_4\leqslant \overline{m}M_0\int_{t_1}^{t_2}(t_2-s)^{q-1}ds=\frac{\overline{m}M_0}{q}(t_2-t_1)^q\to 0 \ \text{ as } \ t_2-t_1\to 0.$$

As a result, $\|\mathscr{Q}u(t_2)-\mathscr{Q}u(t_1)\|$ tends to 0 independently of $u\in[v^{(0)},w^{(0)}]$ as $t_2-t_1\to 0$, which means that $\mathscr{Q}:[v^{(0)},w^{(0)}]\to[v^{(0)},w^{(0)}]$ is equicontinuous.

*Step* 4. $\{v^{(i)}\}$ and $\{w^{(i)}\}$ are convergent in $\Omega$.

For any $a\in(0,\infty)$, restrict $\{v^{(i)}\}$ to interval $[-r,a]$, then $\{v^{(i)}\}$ is a bounded set of $C([-r,a],E)$. Let $V=\{v^{(i)}\}$ and $V_0=V\cup\{v^{(0)}\}$, obviously, $V(t)=(\mathscr{Q}V_0)(t)$ for $t\in[-r,a]$.

In view of the fact that $v^{(i)}(t)=\varphi(t)$ for $t\in[-r,0]$, thus, $\{v^{(i)}(t)\}$is relatively compact on $E$ for $t\in[-r,0]$. For $\forall\,\varepsilon\in(0,t)$ and $\forall\,\delta>0$, we define a set $\mathscr{Q}_{\varepsilon,\delta}V_0(t)$ by

$$\mathscr{Q}_{\varepsilon,\delta}V_0(t):=\{\mathscr{Q}_{\varepsilon,\delta}v^{(i)}(t)\mid v^{(i)}\in V_0,\ \ t\in[0,a]\},\tag{3.13}$$

where

$$\begin{aligned}
\mathscr{Q}_{\varepsilon,\delta}v^{(i)}(t) &= \mathscr{T}(t)v^{(i-1)}(0)+q\int_0^{t-\varepsilon}\int_\delta^\infty \tau(t-s)^{q-1}\xi_q(\tau)S((t-s)^q\tau)\\
&\qquad\cdot\Big(F(s,v^{(i-1)}(s),v_s^{(i-1)})+Lv^{(i-1)}(s)\Big)d\tau ds\\
&= \mathscr{T}(t)v^{(i-1)}(0)+qS(\varepsilon^q\delta)\int_0^{t-\varepsilon}\int_\delta^\infty \tau(t-s)^{q-1}\xi_q(\tau)S((t-s)^q\tau-\varepsilon^q\delta)\\
&\qquad\cdot\Big(F(s,v^{(i-1)}(s),v_s^{(i-1)})+Lv^{(i-1)}(s)\Big)d\tau ds.
\end{aligned}$$

Then from the compactness of $\mathscr{T}(t)$ and $S(\varepsilon^q\delta)$, we obtain that the set $\mathscr{Q}_{\varepsilon,\delta}V_0(t)$ is relatively compact in $E$ for $\forall\,\varepsilon\in(0,t)$ and $\forall\,\delta>0$. Moreover, for every $v^{(i)}\in V_0$ and $t\in[0,a]$, from the following inequality

$$\begin{aligned}
&\|\mathscr{Q}v^{(i)}(t)-\mathscr{Q}_{\varepsilon,\delta}v^{(i)}(t)\|\\
\leqslant\ &\Big\|q\int_0^t\int_0^\delta \tau(t-s)^{q-1}\xi_q(\tau)S((t-s)^q\tau)\cdot\Big(F(s,v^{(i-1)}(s),v_s^{(i-1)})\\
&+Lv^{(i-1)}(s)\Big)d\tau ds\Big\|
\end{aligned}$$

$$+\left\|q\int_{t-\varepsilon}^{t}\int_{\delta}^{\infty}\tau(t-s)^{q-1}\xi_q(\tau)S((t-s)^q\tau)\cdot\Big(F(s,v^{(i-1)}(s),v_s^{(i-1)})\right.$$
$$\left.+Lv^{(i-1)}(s)\Big)d\tau ds\right\|$$
$$\leqslant q\overline{M}M_0\int_0^t(t-s)^{q-1}ds\int_0^{\delta}\tau\xi_q(\tau)d\tau+q\overline{M}M_0\int_{t-\varepsilon}^{t}(t-s)^{q-1}ds\int_{\delta}^{\infty}\tau\xi_q(\tau)d\tau$$
$$\leqslant \overline{M}M_0t^q\int_0^{\delta}\tau\xi_q(\tau)d\tau+\frac{\overline{M}M_0}{\Gamma(1+q)}\varepsilon^q$$
$$\to 0 \text{ as } \varepsilon\to 0,\delta\to 0,$$

one can obtain that the set $(\mathscr{Q}V_0)(t)$ is relatively compact, which implies that $\{v^{(i)}(t)\}$ is relatively compact on $E$ for $t\in[0,a]$. Thus, we have proved that $\{v^{(i)}(t)\}$ is relatively compact on $E$ for $t\in[-r,a]$.

Therefore, $\{v^{(i)}\}$ is relatively compact in $C([-r,a],E)$ by the Arzela-Ascoli Theorem, which implies that there is convergent subsequence in $\{v^{(i)}\}$. Combining this with the monotonicity and the normality of the cone, we can easily prove that $\{v^{(i)}\}$ themselves is convergent, i.e., there is $\underline{u}\in C([-r,a],E)$ such that $\lim\limits_{i\to\infty}v^{(i)}(t)=\underline{u}(t)$ for $t\in[-r,a]$. According to the arbitrariness of $a$, one can find that $\underline{u}(t)$ is defined on $[-r,\infty)$. On the other hand, it is easy to see $\lim\limits_{t\to\infty}\|\underline{u}(t+\omega)-\underline{u}(t)\|$. Hence, we can deduce that there is $\underline{u}\in\Omega$ such that $\lim\limits_{i\to\infty}v^{(i)}=\underline{u}$. Similarly, it can be shown that there is $\overline{u}\in\Omega$ such that $\lim\limits_{i\to\infty}w^{(i)}=\overline{u}$.

Taking limit in (3.11), we have

$$\underline{u}=\mathscr{Q}\underline{u},\qquad \overline{u}=\mathscr{Q}\overline{u}. \tag{3.14}$$

Therefore $\underline{u},\overline{u}\in\Omega$ are fixed points of $\mathscr{Q}$, and they are the $S$-asymptotically $\omega$-periodic mild solution of the problem (1.1).

*Step* 5. We prove the minimal and maximal properties of $\underline{u},\overline{u}$.

Assume that $\widetilde{u}$ is a fixed point of $\mathscr{Q}$ with $\widetilde{u}\in[v_0,w_0]$, then for every $t\in[-r,\infty)$, $v^{(0)}(t)\leqslant\widetilde{u}(t)\leqslant w^{(0)}(t)$, and

$$v^{(1)}(t)=(\mathscr{Q}v^{(0)})(t)\leqslant(\mathscr{Q}\widetilde{u})(t)=\widetilde{u}(t)\leqslant(\mathscr{Q}w^{(0)})(t)=w^{(1)}(t), \tag{3.15}$$

namely, $v^{(1)}\leqslant\widetilde{u}\leqslant w^{(1)}$. In general

$$v^{(i)}\leqslant\widetilde{u}\leqslant w^{(i)},\quad i=1,2,\cdots. \tag{3.16}$$

Taking limit in (3.16) as $i\to\infty$, we get $\underline{u}\leqslant\widetilde{u}\leqslant\overline{u}$. Therefore $\underline{u},\overline{u}$ are minimal and maximal $S$-asymptotically $\omega$-periodic mild solutions of the problem (1.1), and $\underline{u},\overline{u}$ can be obtained by the iterative sequences defined in (3.11) starting from $v_0$ and $w_0$ respectively. This completes the proof of Theorem 3.1. □

THEOREM 3.2. *Let $E$ be an ordered Banach space, whose positive cone $K$ is a normal cone; let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive equicontinuous $C_0$-semigroup $T(t)$ $(t \geqslant 0)$ in $E$ whose growth exponent denotes by $\nu_0$. Assume that $\omega > 0$ is a constant and the problem (1.1) has lower and upper $S$-asymptotically $\omega$-periodic solutions $v^{(0)}, w^{(0)} \in \Omega$ with $v^{(0)} \leqslant w^{(0)}$. If the nonlinear function $F : \mathbb{R}^+ \times E \times \mathscr{B} \to E$ is continuous and satisfies the conditions (H1), (H2) and the following condition*

*(H3) There exists a constant $C > 0$ such that for all $t \geqslant 0$ and monotonic sequences $\{u^{(i)}\} \subset [v^{(0)}, w^{(0)}]$,*

$$\alpha(\{F(t,u^{(i)}(t),u_t^{(i)})+Lu^{(i)}(t)\}) \leqslant C(\alpha(\{u^{(i)}(t)\})+\sup_{s\in[-r,0]}\alpha(\{u_t^{(i)}(s)\})),$$

*then the problem (1.1) has minimal and maximal $S$-asymptotically $\omega$-periodic mild solutions $\underline{u},\overline{u}$ between $v^{(0)}$ and $w^{(0)}$, which can be obtained by monotone iterative sequences starting from $v^{(0)}$ and $w^{(0)}$ respectively.*

*Proof.* Let $\mathscr{Q}$ be defined by (3.5). From the proof of Theorem 3.1, we know that $\mathscr{Q} : [v^{(0)}, w^{(0)}] \to [v^{(0)}, w^{(0)}]$ is a continuous increasing operator and $v^{(0)} \leqslant \mathscr{Q}v^{(0)}$, $\mathscr{Q}w^{(0)} \leqslant w^{(0)}$. Hence, the iterative sequences $v^{(i)}$ and $w^{(i)}$ defined by (3.11) satisfy (3.12). By $T(t)$ $(t \geqslant 0)$ is an equicontinuous $C_0$-semigroup, it follows that $S(t)(t \geqslant 0)$ is also an equicontinuous $C_0$-semigroup. From the proof of Theorem 3.1, we obtain that $\{v^{(i)}\}$, $\{w^{(i)}\}$ are bounded and equicontinuous in $t \in [-r,\infty)$.

Next, we show that $\{v^{(i)}\}$, $\{w^{(i)}\}$ are convergent in $\Omega$.

For $\forall\ a > 0$, restrict $\{v^{(i)}\}$ to interval $[-r,a]$, then $\{v^{(i)}\}$ is a bounded set on set $C([-r,a],E)$. Hence $\alpha(\{v^{(i)}\}(t))$ is continuous on $[-r,a]$ and from $v^{(i)}(t) = \mathscr{Q}v^{(i-1)}(t) = \varphi(t)$, $t \in [-r,0]$, it follows that $\alpha(\{v^{(i)}\}(t)) = 0$ for $t \in [-r,0]$. For every $t \in [0,a]$, one can see

$$\sup_{s\in[-r,0]}\alpha(\{v_t^{(i)}(s)\}) = \sup_{s\in[-r,0]}\alpha(\{v^{(i)}(t+s)\}) \leqslant \alpha(\{v^{(i)}(t)\}). \tag{3.17}$$

Therefore, for $t \in [0,a]$, taking condition (H3), Lemma 2.10 and (3.17), we can obtain that

$$\begin{aligned}
&\alpha(\{v^{(i)}(t)\}) = \alpha(\{\mathscr{Q}v^{(i-1)}(t)\})\\
&= \alpha\Big(\Big\{\mathscr{T}(t)\varphi(0)+\int_0^t(t-s)^{q-1}\mathscr{S}(t-s)\cdot(F(s,v^{(i-1)}(s),v_s^{(i-1)})+Lv^{(i-1)}(s))ds\Big\}\Big)\\
&\leqslant 2\int_0^t(t-s)^{q-1}\|\mathscr{S}(t-s)\|\cdot\alpha\Big(\Big\{F(s,v^{(i-1)}(s),v_s^{(i-1)})+Lv^{(i-1)}(s)\Big\}\Big)ds\\
&\leqslant 2C\int_0^t(t-s)^{q-1}\|\mathscr{S}(t-s)\|\cdot\Big(\alpha(v^{(i-1)}(s))+\sup_{\tau\in[-r,0]}\alpha(v_s^{(i-1)}(\tau))\Big)ds\\
&\leqslant 4C\overline{m}\int_0^t(t-s)^{q-1}\alpha(v^{(i)}(s))ds.
\end{aligned}$$

Hence, from Lemma 2.11, it follows that $\alpha(\{v^{(i)}(t)\}) \equiv 0$ on $[0,a]$. Thus, we can deduce that $\alpha(\{v^{(i)}(t)\}) \equiv 0$ on $[-r,a]$.

By uniform boundedness and equicontinuous of $\{v^{(i)}\}$ on $[-r,a]$, $\{v^{(i)}\}$ is relatively compact in $C([-r,a],E)$, hence, there is convergent subsequence in $\{v^{(i)}\}$. Combining this with the monotonicity and the normality of the cone, we can easily prove that $\{v^{(i)}\}$ itself is convergent, which means that there is $\underline{u} \in C([-r,a],E)$ such that $\lim_{i\to\infty} v^{(i)}(t) = \underline{u}(t)$ for $t \in [-r,a]$. According to the arbitrariness of $a$, $\underline{u}(t)$ is defined on $[-r,\infty)$, and $\lim_{t\to\infty} \|\underline{u}(t+\omega) - \underline{u}(t)\|$, which implies that $\underline{u} \in \Omega$. Similarly, we can prove that there exists $\overline{u} \in \Omega$ satisfying $\lim_{i\to\infty} w^{(i)}(t) = \overline{u}(t)$ for $t \in [-r,\infty)$.

Therefore, from the proof of Theorem 3.1, $\underline{u}, \overline{u}$ are the minimal and maximal $S$-asymptotically $\omega$-periodic mild solutions of the problem (1.1), which can be obtained by monotone iterative sequences starting from $v^{(0)}$ and $w^{(0)}$. This completes the proof of Theorem 3.2. $\square$

THEOREM 3.3. *Let $E$ be an ordered and weakly sequentially complete Banach space, whose positive cone $K$ is normal, let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive equicontinuous $C_0$-semigroup $T(t)$ $(t \geqslant 0)$ in $E$ whose growth exponent denotes by $\nu_0$. Assume that $\omega > 0$ is a constant and the problem (1.1) has lower and upper $S$-asymptotically $\omega$-periodic solutions $v^{(0)}, w^{(0)} \in \Omega$ with $v^{(0)} \leqslant w^{(0)}$. If the nonlinear function $F : \mathbb{R}^+ \times E \times \mathscr{B} \to E$ is continuous and satisfies the conditions (H1) and (H2), then the problem (3.1) has minimal and maximal $S$-asymptotically $\omega$-periodic mild solutions $\underline{u}, \overline{u} \in [v_0, w_0]$.*

*Proof.* From the proof of Theorem 3.1, it follows that the iterative sequences $\{v_i\}$ and $\{w_i\}$ defined by (3.11) satisfy (3.12). Hence, for any $t \in [-r,\infty)$, $\{v^{(i)}(t)\}$ and $\{w^{(i)}(t)\}$ are monotone and order-bounded sequences in $E$. Noticing that $E$ is a weakly sequentially complete Banach space, from Theorem 2.2 in [21], one can get that $\{v^{(i)}(t)\}$ and $\{w^{(i)}(t)\}$ are precompact in $E$ for any $t \in [-r,\infty)$. Combining this with the monotonicity (3.12), it follows that $\{v^{(i)}(t)\}$ and $\{w^{(i)}(t)\}$ are uniformly convergent in $E$. Denote

$$\underline{u}(t) = \lim_{i\to\infty} v^{(i)}(t), \qquad \overline{u}(t) = \lim_{i\to\infty} w^{(i)}(t), \ t \in [-r,\infty). \tag{3.18}$$

Obviously, $\{v^{(i)}(t)\}$, $\{w^{(i)}(t)\} \subset \Omega$, and $v^{(0)}(t) \leqslant \underline{u}(t) \leqslant \overline{u}(t) \leqslant w^{(0)}(t) (t \in [-r,\infty))$. Moreover, by (3.5), we have

$$\begin{aligned} v^{(i)}(t) &= \mathscr{Q} v^{(i-1)}(t) \\ &= \begin{cases} \mathscr{T}(t)\varphi(0) + \int_0^t (t-s)^{q-1} \mathscr{S}(t-s) \cdot \Big( F(s, v^{(i-1)}(s), v_s^{(i-1)}) \\ \qquad\qquad + L v^{(i-1)}(s) \Big) ds, \quad t \geqslant 0, \\ \varphi(t), \qquad t \in [-r,0], \end{cases} \end{aligned} \tag{3.19}$$

and

$$w^{(i)}(t) = \mathscr{Q}w^{(i-1)}(t)$$

$$= \begin{cases} \mathscr{T}(t)\varphi(0) + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)\cdot\Big(F(s,w^{(i-1)}(s),w_s^{(i-1)}) \\ \qquad\qquad + Lw^{(i-1)}(s)\Big)ds, \quad t \geqslant 0, \\ \varphi(t), \qquad t \in [-r,0]. \end{cases} \tag{3.20}$$

Taking limit in (3.19) and (3.20) as $i \to \infty$, from the Lebesgue dominated convergence theorem, one can obtain

$$\underline{u}(t) = \begin{cases} \mathscr{T}(t)\varphi(0) + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)\cdot\Big(F(s,\underline{u}(s),\underline{u}_s) + L\underline{u}(s)\Big)ds, \quad t \geqslant 0, \\ \varphi(t), \qquad t \in [-r,0], \end{cases}$$

and

$$\overline{u}(t) = \begin{cases} \mathscr{T}(t)\varphi(0) + \int_0^t (t-s)^{q-1}\mathscr{S}(t-s)\cdot\Big(F(s,\overline{u}(s),\overline{u}_s) + L\overline{u}(s)\Big)ds, \quad t \geqslant 0, \\ \varphi(t), \qquad t \in [-r,0], \end{cases}$$

and it is easy to see that $\underline{u},\overline{u} \in \Omega$.

Hence, similar to the proof of Theorem 3.1, we obtain that the $\underline{u},\overline{u}$ are minimal and maximal $S$-asymptotically $\omega$-periodic mild solutions of the problem (1.1) in $[v^{(0)},w^{(0)}]$. This completes the proof of Theorem 3.3. □

REMARK 1. Analytic semigroup and differentiable semigroup are continuous by operator norm for every $t > 0$ (see [41]). In the application of partial differential equations, such as parabolic equations and strongly damped wave equations, the corresponding solution semigroup is analytic semigroup. Therefore, Theorem 3.2 and Theorem 3.3 in this paper has broad applicability.

In the above works, the key assumption (H2) (the monotone on the third variable of the nonlinear function) is employed. However, we hope that the nonlinear function is quasi-monotonicity. In this case, the results have more extensive application background.

In fact, we find that if the problem (1.1) has lower and upper $S$-asymptotically $\omega$-periodic mild solutions $v^{(0)},w^{(0)} \in \Omega$ with $v^{(0)} \leqslant w^{(0)}$ and

*(H4) there is a sufficiently small constant $L_0 > 0$, such that*

$$u^{(2)}(t) - u^{(1)}(t) \geqslant L_0(u_t^{(2)}(\cdot) - u_t^{(1)}(\cdot)),$$

*for any $t \geqslant 0$ and $u^{(1)},u^{(2)} \in [v^{(0)},w^{(0)}]$ with $u^{(2)} \geqslant u^{(1)}$,*
then the condition (H2) can be replaced by the following condition

*(H5) there are nonnegative constants $L_1,L_2$, such that*

$$F(t,x_2,\phi_2) - F(t,x_1,\phi_1) \geqslant -L_1(x_2-x_1) - L_2(\phi_2(\cdot)-\phi_1(\cdot)),$$

*for all* $t \geqslant 0$, $x_1, x_2 \in E$ *and* $\phi_1, \phi_2 \in \mathscr{B}$ *with* $v^{(0)}(t) \leqslant x_1 \leqslant x_2 \leqslant w^{(0)}(t)$, $v_t^{(0)} \leqslant \phi_1 \leqslant \phi_2 \leqslant w_t^{(0)}$.

In fact, for every $t \geqslant 0$ and $u^{(1)}, u^{(2)} \in [v^{(0)}, w^{(0)}]$ with $u^{(1)} \leqslant u^{(2)}$, one can obtain that $v^{(0)}(t) \leqslant u^{(1)}(t) \leqslant u^{(2)}(t) \leqslant w^{(0)}(t)$, $v_t^{(0)} \leqslant u_t^{(1)} \leqslant u_t^{(2)} \leqslant w_t^{(0)}$. From the conditions (H4) and (H5), it follows that

$$\begin{aligned}
& F(t, u^{(2)}(t), u_t^{(2)}) - F(t, u^{(1)}(t), u_t^{(1)}) \\
\geqslant\ & -L_1(u^{(2)}(t) - u^{(1)}(t)) - L_2(u_t^{(2)}(\cdot) - u_t^{(1)}(\cdot)) \\
\geqslant\ & -L_1(u^{(2)}(t) - u^{(1)}(t)) - \frac{L_2}{L_0}(u^{(2)}(t) - u^{(1)}(t)) \\
=\ & -\Big(L_1 + \frac{L_2}{L_0}\Big)(u^{(2)}(t) - u^{(1)}(t)) \\
:=\ & -L(u^{(2)}(t) - u^{(1)}(t)).
\end{aligned}$$

Hence, we can obtain the following results from Theorem 3.1, Theorem 3.2 and Theorem 3.3.

THEOREM 3.4. *Let $E$ be an ordered Banach space, whose positive cone $K$ is normal cone, let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive and compact semigroup $T(t)$ $(t \geqslant 0)$ in $E$, whose growth exponent denotes $\nu_0$. Assume that $\omega > 0$ is a constant and the problem (1.1) has lower and upper $S$-asymptotically $\omega$-periodic solutions $v^{(0)}, w^{(0)} \in \Omega$ with $v^{(0)} \leqslant w^{(0)}$. If the nonlinear function $F : \mathbb{R}^+ \times E \times \mathscr{B} \to E$ is continuous and the conditions (H1), (H4) and (H5) hold, then the problem (1.1) has minimal and maximal $S$-asymptotically $\omega$-periodic mild solutions $\underline{u}, \overline{u} \in [v^{(0)}, w^{(0)}]$.*

THEOREM 3.5. *Let $E$ be an ordered Banach space, whose positive cone $K$ is normal cone, let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive equicontinuous $C_0$-semigroup $T(t)$ $(t \geqslant 0)$ in $E$, whose growth exponent denotes $\nu_0$. Assume that $\omega > 0$ is a constant and the problem (1.1) has lower and upper $S$-asymptotically $\omega$-periodic solutions $v^{(0)}, w^{(0)} \in \Omega$ with $v^{(0)} \leqslant w^{(0)}$. If the nonlinear function $F : \mathbb{R}^+ \times E \times \mathscr{B} \to E$ is continuous and the conditions (H1), (H3), (H4) and (H5) hold, then the problem (1.1) has minimal and maximal $S$-asymptotically $\omega$-periodic mild solutions $\underline{u}, \overline{u} \in [v^{(0)}, w^{(0)}]$.*

THEOREM 3.6. *Let $E$ be an ordered and weakly sequentially complete Banach space, whose positive cone $K$ is normal let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive equicontinuous $C_0$-semigroup $T(t)$ $(t \geqslant 0)$ in $E$ whose growth exponent denotes $\nu_0$. Assume that $\omega > 0$ is a constant and the problem (1.1) has lower and upper $S$-asymptotically $\omega$-periodic solutions $v^{(0)}, w^{(0)} \in \Omega$ with $v^{(0)} \leqslant w^{(0)}$. If the nonlinear function $F : \mathbb{R}^+ \times E \times \mathscr{B} \to E$ is continuous and the conditions (H1), (H4) and (H5) hold, then the problem (3.1) has minimal and maximal $S$-asymptotically $\omega$-periodic mild solutions $\underline{u}, \overline{u} \in [v_0, w_0]$.*

REMARK 2. Obviously, the condition (H4) is easy to satisfy, and the condition (H5) weakens the condition (H2). Hence, Theorem 3.4–3.5 partially improve Theorem 3.1–3.3.

In the end of this section, we discuss the uniqueness of the $S$-asymptotically $\omega$-periodic mild solution for the problem (1.1) under $T(t)$ $(t \geqslant 0)$ is an equicontinuous semigroup.

THEOREM 3.7. *Let $E$ be an ordered Banach space, whose positive cone $K$ is normal cone with normal constant $N$, let $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generate a positive equicontinuous $C_0$-semigroup $T(t)$ $(t \geqslant 0)$, whose growth exponent denotes $\nu_0$. Assume that $\omega > 0$ is a constant and the problem (1.1) has lower and upper $S$-asymptotically $\omega$-periodic solutions $v^{(0)}, w^{(0)} \in \Omega$ with $v^{(0)} \leqslant w^{(0)}$. If the nonlinear function $F : \mathbb{R}^+ \times E \times \mathscr{B} \to E$ is continuous and the conditions (H1), (H4), (H5) and*

*(H6) there exist constants $C_1, C_2 > 0$ such that*

$$F(t,x_2,\phi_2) - F(t,x_1,\phi_1) \leqslant C_1(x_2 - x_1) + C_2(\phi_2(\cdot) - \phi_1(\cdot))$$

*for every $t \geqslant 0$ and $v^{(0)}(t) \leqslant x_1 \leqslant x_2 \leqslant w^{(0)}(t)$, $v_t^{(0)} \leqslant \phi_1 \leqslant \phi_2 \leqslant w_t^{(0)}$;*
*hold, then the periodic problem (1.1) has a unique $S$-asymptotically $\omega$-periodic mild solution in $[v^{(0)}, w^{(0)}]$, which can be obtained by monotone iterative sequences starting from $v^{(0)}$ or $w^{(0)}$.*

*Proof.* We can find that (H4), (H5) and (H6) imply (H3). In fact, for $t \geqslant 0$, let $\{u^{(n)}\} \subset [v^{(0)}, w^{(0)}]$ be an increasing sequence. For $m,n = 1,2,\cdots$ with $m > n$, by (H5) and (H6), we have

$$\begin{aligned}
\theta &\leqslant F(t,u^{(m)}(t),u_t^{(m)}) - F(t,u^{(n)}(t),u_t^{(n)}) \\
&\quad + L_1(u^{(m)}(t) - u^{(n)}(t)) + L_2(u_t^{(m)}(\cdot) - u_t^{(n)}(\cdot)) \\
&\leqslant F(t,u^{(m)}(t),u_t^{(m)}) - F(t,u^{(n)}(t),u_t^{(n)}) + (L_1 + \frac{L_2}{L_0})(u^{(m)}(t) - u^{(n)}(t)) \\
&\leqslant (L_1 + \frac{L_2}{L_0} + C_1)(u^{(m)}(t) - u^{(n)}(t)) + C_2(u_t^{(m)}(\cdot) - u_t^{(n)}(\cdot)).
\end{aligned}$$

Denote $L = L_1 + \frac{L_2}{L_0}$, by the normality of positive cone $K$, we have

$$\begin{aligned}
&\|F(t,u^{(m)}(t),u_t^{(m)}) - F(t,u^{(n)}(t),u_t^{(n)}) + L(u^{(m)}(t) - u^{(n)}(t))\| \\
&\leqslant (NL + NC_1)\|(u^{(m)}(t) - u^{(n)}(t))\| + NC_2\|(u_t^{(m)}(\cdot) - u_t^{(n)}(\cdot))\|. \qquad (3.21)
\end{aligned}$$

From (3.21) and the definition of measure of noncompactness, we can find that

$$\alpha(\{F(t,u^{(n)}(t),u_t^{(n)}) + Lu^{(n)}(t)\}) \leqslant C\Big(\alpha(\{u^{(n)}(t)\}) + \sup_{s\in[-r,0]} \alpha(\{u_t^{(n)}(s)\})\Big),$$

where $C = \max\{(NL + NC_1), NC_2\}$. Hence, (H3) holds.

Therefore, From Theorem 3.5, the problem (1.1) has minimal and maximal $S$-asymptotically $\omega$-periodic mild solutions $\underline{u}, \overline{u} \in [v^{(0)}, w^{(0)}]$.

Obviously, $\underline{u}(t) = \overline{u}(t) = \varphi(t)$ for $t \in [-r, 0]$. For any $t \geqslant 0$, by (3.5),(3.14) and the condition (H6), one can obtain

$$\begin{aligned}
\theta &\leqslant \overline{u}(t) - \underline{u}(t) = \mathscr{Q}\overline{u}(t) - \mathscr{Q}\underline{u}(t) \\
&= \int_0^t (t-s)^{q-1} \mathscr{S}(t-s) \cdot \Big( F(s, \overline{u}(s), \overline{u}_s) + L\overline{u}(s) \Big) ds \\
&\quad - \int_0^t (t-s)^{q-1} \mathscr{S}(t-s) \cdot \Big( F(s, \underline{u}(s), \underline{u}_s) + L\underline{u}(s) \Big) ds \\
&\leqslant \int_0^t (t-s)^{q-1} \mathscr{S}(t-s) \cdot \Big( (C_1 + L)(\overline{u}(s) - \underline{u}(s)) + C_2(\overline{u}_s(\cdot) - \underline{u}_s(\cdot)) \Big) ds \\
&\leqslant (C_1 + L + \frac{C_2}{L_0}) \int_0^t (t-s)^{q-1} \mathscr{S}(t-s)(\overline{u}(s) - \underline{u}(s)) ds.
\end{aligned}$$

Thus, from the normality of the cone $K$, it follows that for $t \geqslant 0$,

$$\|\overline{u}(s) - \underline{u}(s)\| \leqslant N\overline{M}(C_1 + L + \frac{C_2}{L_0}) \int_0^t (t-s)^{q-1} \|\overline{u}(s) - \underline{u}(s)\| ds. \tag{3.22}$$

From Lemma 2.11, it follows that $\underline{u}(t) = \overline{u}(t)$ for $t \geqslant 0$. Hence, $\underline{u} = \overline{u}$ is the unique $S$-asymptotically $\omega$-periodic mild solution of the problem (1.1) in $[v^{(0)}, w^{(0)}]$. From the proof of Theorem 3.1, the unique $S$-asymptotically $\omega$-periodic mild solution can be obtained by monotone iterative sequences starting from $v^{(0)}$ or $w^{(0)}$. This completes the proof of Theorem 3.7. $\square$

## 4. Application

EXAMPLE 4.1. $S$-asymptotically $\omega$-periodic solutions of fractional parabolic equation with delay in $\mathbb{R}^n (n \geqslant 1)$.

Let $\overline{\Omega} \in \mathbb{R}^n$ be a bounded domain with $C^2$-boundary $\partial\Omega$ for $n \in \mathbb{N}$. We consider the following semilinear fractional parabolic equation boundary value problem with delay

$$\begin{cases} \frac{\partial^q}{\partial t^q} u(\xi, t) + \nabla^2 u(\xi, t) = a(t) f(u(\xi, t), u_t(\xi)), & \xi \in \Omega,\ t \in \mathbb{R}^+, \\ u|_{\partial\Omega} = 0, \\ u(\xi, \tau) = \varphi(\xi, \tau), \quad \xi \in \Omega, \tau \in [-r, 0], \end{cases} \tag{4.1}$$

where $\frac{\partial^q}{\partial t^t}$ is the Caputo fractional partial derivative of order $q \in (0,1)$, $\nabla^2$ is a Laplace operator, $a : \mathbb{R}^+ \to \mathbb{R}$ and $f : \mathbb{R} \times C([-r, 0], L^2(\overline{\Omega})) \to \mathbb{R}$ are continuous functions, $\varphi \in C(\overline{\Omega} \times [-r, 0], L^2(\overline{\Omega}))$, $r > 0$ is a constant.

THEOREM 4.1. *Assume that* $a \in SAP_\omega(\mathbb{R})$. *If the following conditions*

*(A1)* $a(t)f(0,0) \geqslant 0$ *for any* $t \in \mathbb{R}^+$, *and there is a function* $0 \leqslant w = w(x,t) \in C(\Omega \times [-r,\infty)) \cap C^{2,q}(\Omega \times \mathbb{R}^+)$ *satisfying* $\lim_{t\to\infty} w(\cdot,t+\omega) - w(\cdot,t) = 0$, *such that*

$$\begin{cases} \frac{\partial^q}{\partial t^q} w(\xi,t) + A(\xi,D)w(\xi,t) \geqslant a(t)f(w(\xi,t),w_t(\xi)), & \xi \in \Omega, t \in \mathbb{R}^+, \\ w|_{\partial\Omega} = 0, \\ w(\xi,\tau) \geqslant \varphi(\xi,\tau), \quad \xi \in \Omega, \tau \in [-r,0], \end{cases}$$

*(A2) there exists a constant* $l > 0$, *such that for any* $\xi \in \Omega$ ,$t \in \mathbb{R}^+$ *and* $0 \leqslant x_1 \leqslant x_2 \leqslant w(x,t)$, $0 \leqslant \phi_1 \leqslant \phi_2 \leqslant w_t(\xi)$,

$$a(t)f(x_2,z_2) - a(t)f(x_1,z_1) \geqslant -l(x_2 - x_1),$$

*hold, then semilinear fractional delayed parabolic equation boundary value problem (4.1) has minimal and maximal time* $S$*-asymptotically* $\omega$*-periodic solutions* $\underline{u}, \overline{u} \in C([-r,\infty), L^2(\Omega)) \cap SAP_\omega(L^2(\Omega))$ *between* $0$ *and* $w$, *which can be obtained by monotone iterative sequences starting from* $0$ *and* $w$.

*Proof.* In order to write the semilinear fractional parabolic equation boundary value problem with delay (4.1) in the form of the problem (1.1), let $E = L^2(\overline{\Omega})$ with the $L^2$-norm $\|\cdot\|_2$, $K = \{u \in E | u(x) \geqslant 0, a.e.\ x \in \overline{\Omega}\}$, then $E$ is an ordered Banach space, whose positive cone $K$ is a normal regeneration cone.

Define an operator $A : D(A) \subset E \to E$ by:

$$D(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), \quad Au = -\nabla^2 u. \tag{4.2}$$

From [3], we know that $-A$ is a selfadjoint operator in $E$, and generates an exponentially stable analytic semigroup $T(t)$ $(t \geqslant 0)$, which is contractive in $E$. Hence, $\|T(t)\|_2 \leqslant M := 1$ for every $t \geqslant 0$. Furthermore, we assume that $\lambda_1$ is the smallest eigenvalue of operator $A$, and from [2, Theorem 1.16], it follows that $\lambda_1 > 0$. On the other hand, by the maximum principle of elliptic operators, we know that $(\lambda I + A)$ has a positive bounded inverse operator $(\lambda I + A)^{-1}$ for $\lambda > 0$, hence $T(t)$ $(t \geqslant 0)$ is a positive $C_0$-semigroup (see [35]). Since the operator $A$ has compact resolvent in $L^2(\Omega)$, thus, $T(t)$ $(t \geqslant 0)$ is a compact semigroup (see [41]), which implies that the growth exponent of the semigroup $T(t)$ $(t \geqslant 0)$ satisfies $\nu_0 = -\lambda_1$.

For $\xi \in \overline{\Omega}$, we set

$$u(t)(\xi) = u(\xi,t),\ \varphi(\tau)(\xi) = \varphi(\xi,\tau),\ \ \tau \in [-r,0],$$

$$F(t,u(t),u_t)(\xi) = a(t)f(u(\xi,t),u_t(\xi)). \tag{4.3}$$

Then the semilinear fractional parabolic equation boundary value problem with delay (4.1) can be rewritten into the abstract form of the problem (1.1). From the assumptions of the functions $a$ and $f$, we can deduce that $F : \mathbb{R}^+ \times E \times C([-r,0],E) \to E$ defined by (4.3) is a continuous function which satisfying the condition (H1). And from the condition (A1), it follows that $v_0 \equiv 0$ and $w_0 = w(\xi,t) \geqslant 0$ are lower and upper time

$S$-asymptotically $\omega$-periodic mild solutions of the problem (4.1), respectively. Thus, by the condition (A2), one can find that the condition (H2) holds.

Therefore, from Theorem 3.1, we can obtain that the problem (4.1) has minimal and maximal time $S$-asymptotically $\omega$-periodic mild solutions $\underline{u}, \overline{u} \in C([-r,\infty),E) \cap SAP_{\omega}(E)$, which can be obtained by monotone iterative sequences starting from 0 and $w$, respectively. $\square$

EXAMPLE 4.2. Time $S$-asymptotically periodic solutions of fractional order delayed partial differential equation with periodic boundary condition.

We are concerned with the existence of $S$-asymptotically $2\pi$-periodic solutions for the semilinear fractional order delayed partial differential equation with periodic boundary condition

$$\begin{cases} \frac{\partial^q}{\partial t^t} u(\xi,t) + \frac{\partial}{\partial \xi} u(\xi,t) = a(t) f(u(\xi,t), u(\xi,t+\tau)), \ \xi \in \mathbb{R}, t \in \mathbb{R}^+, \tau \in [-r,0], \\ u(\xi+2\pi,t) = u(x,t), \quad \xi \in \mathbb{R}, t \in [-r,\infty), \\ u(\xi,\tau) = \varphi(\xi,\tau), \quad \xi \in \mathbb{R}, \tau \in [-r,0], \end{cases} \tag{4.4}$$

where $\frac{\partial^q}{\partial t^t}$ is the Caputa fractional partial derivative of order $q \in (0,1)$, $a : \mathbb{R}^+ \to \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ are continuous functions, $\varphi \in C(\mathbb{R} \times [-r,0])$, $r > 0$ is a constant.

THEOREM 4.3. *Assume that* $a \in SAP_{2\pi}(\mathbb{R})$, $a(t)f(0,0) \geqslant 0$ *and there is a function* $0 \leqslant w = w(x,t) \in C(\mathbb{R} \times [-r,\infty)) \cap C^{1,q}(\mathbb{R} \times \mathbb{R}^+)$ *satisfying* $\lim_{t\to\infty} w(\cdot,t+\omega) - w(\cdot,t) = 0$, *such that*

$$\begin{cases} \frac{\partial^q}{\partial t^q} w(\xi,t) + \frac{\partial}{\partial x} w(\xi,t) \geqslant a(t) f(w(\xi,t), w(\xi,t+\tau)), \quad x \in \mathbb{R}, t \in \mathbb{R}^+, \tau \in [-r,0], \\ w(\xi+2\pi,t) = w(\xi,t), \quad \xi \in \mathbb{R}, t \in [-r,\infty), \\ w(\xi,\tau) \geqslant \varphi(\xi,\tau), \quad \xi \in \mathbb{R}, \tau \in [-r,0]. \end{cases}$$

*For any* $\xi \in \mathbb{R}, t \in \mathbb{R}^+, \tau \in [-r,0]$, *and* $0 \leqslant u_1(\xi,t) \leqslant u_2(\xi,t) \leqslant w(\xi,t)$, *if the following conditions*

*(A3) there exists a positive constant* $l_1$ *such that*

$$u_2(\xi,t) - u_1(\xi,t) \geqslant l_1(u_2(\xi,t+\tau) - u_1(\xi,t+\tau));$$

*(A4) there exist positive constants* $l_2, l_3$ *such that*

$$\begin{aligned} &a(t)f(u_2(\xi,t), u_2(\xi,t+\tau)) - a(t)f(u_1(\xi,t), u_1(\xi,t+\tau)) \\ \geqslant\ & -l_2(u_2(\xi,t) - u_1(\xi,t)) - l_3(u_2(\xi,t+\tau) - u_1(\xi,t+\tau)); \end{aligned}$$

*(A5) there exist positive constants* $c_1, c_2$ *such that,*

$$\begin{aligned} &a(t)f(u_2(\xi,t), u_2(\xi,t+\tau)) - a(t)f(u_1(\xi,t), u_1(\xi,t+\tau)) \\ \leqslant\ & c_1(u_2(\xi,t) - u_1(\xi,t)) + c_2(u_2(\xi,t+\tau) - u_1(\xi,t+\tau)), \end{aligned}$$

*hold, then the semilinear fractional order delayed partial differential equation with periodic boundary condition (4.4) has a unique time $S$-asymptotically $2\pi$-periodic mild solution $u^* \in C(\mathbb{R}\times[-r,\infty))\cap SAP_{2\pi}(C_{2\pi}(\mathbb{R}))$ between $0$ and $w$.*

*Proof.* Let $C_{2\pi}(\mathbb{R})$ denote the Banach space $\{u\in C(\mathbb{R})|u(\xi+2\pi)=u(\xi),\xi\in\mathbb{R}\}$ endowed the maximum norm $\|u\|_C=\max_{\xi\in[0,2\pi]}\|u(\xi)\|$, $K=\{u\in C_{2\pi}(\mathbb{R})|u(\xi)\geqslant 0,\xi\in\mathbb{R}\}$, then $C_{2\pi}(\mathbb{R})$ is an ordered Banach space, whose positive cone $K$ is a normal cone. Let

$$D(A)=C^1_{2\pi}(\mathbb{R}),\ \ A=\frac{\partial u}{\partial\xi}. \tag{4.5}$$

From [37, Lemma 2.1], if $\lambda\neq 0$, we know that $(\lambda I+A)$ has a bounded inverse operator $(\lambda I+A)^{-1}$ in $C_{2\pi}(\mathbb{R})$ and

$$(\lambda I+A)^{-1}h(\xi)=\int_{\xi-2\pi}^{\xi} r(s-y)h(y)dy,\quad h\in C_{2\pi}(\mathbb{R}), \tag{4.6}$$

where

$$r(\xi)=\frac{e^{-\lambda\xi}}{1-e^{-2\pi\lambda}},\quad \xi\in[0,2\pi].$$

By (4.5), it follows that $(\lambda I+A)^{-1}$ is positive operator for $\lambda>0$, and its norm $\|(\lambda I+A)^{-1}\|\leqslant\frac{1}{\lambda}$. From Hille-Yosida Theorem and exponential formula of semogroup (see [41]), we can obtain that $-A$ generates a contractive and positive $C_0$-semigroup $T(t)$ $(t\geqslant 0)$ in $C_{2\pi}(\mathbb{R})$, whose growth exponent $\nu_0\leqslant 0$. Thus, $-(A+LI)$ $(L:=l_2+\frac{l_3}{l_1})$ generates a contractive and positive $C_0$-semigroup $S(t)=e^{-Lt}T(t)(t\geqslant 0)$ with the growth exponent $\mu_0=-L+\nu_0\leqslant -L$ and $\overline{M}:=\sup_{t\geqslant 0}\|S(t)\|\leqslant 1$.

Set $u(t)(\xi)=u(\xi,t),u(t+\tau)(\xi)=u(\xi,t+\tau)$, and

$$F(t,u(t),u_t(\tau))(\xi)=a(t)f(u(\xi,t),u(\xi,t+\tau)), \tag{4.7}$$

then the periodic problem (4.4) can be reformulated as following

$$\begin{cases} {}^cD_t^q u(t)+Au(t)=F(t,u(t),u(t+\tau)), & t\in\mathbb{R}^+,\tau\in[-r,0],\\ u(\tau)=\varphi(\tau),\ \tau\in[-r,0]. \end{cases} \tag{4.8}$$

From the assumptions of the functions $a$ and $f$, we can deduce that $F:\mathbb{R}^+\times C_{2\pi}(\mathbb{R})\times C_{2\pi}(\mathbb{R})\to C_{2\pi}(\mathbb{R})$ defined by (4.8) is a continuous function which satisfying the condition (H1). And from the assumptions, one can find that $v_0\equiv 0$ and $w_0=w(\xi,t)\geqslant 0$ are lower and upper time $S$-asymptotically $2\pi$-periodic mild solutions of the problem (4.4), respectively. By the conditions (A3-A5), we can deduce that the conditions (H4-H6) hold. Therefore, from Theorem 3.7, we can obtain that the problem (4.4) has a unique time $S$-asymptotically $\omega$-periodic mild solutions $u^*\in C(\mathbb{R}\times[-r,\infty))\cap SAP_{2\pi}(C_{2\pi}(\mathbb{R}))$ between $0$ and $w$. $\square$

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