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APPROXIMATION PROPERTIES OF MODIFIED KANTOROVICH TYPE (p,q)–BERNSTEIN OPERATORS

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Abstract. In the present paper, we construct modified Bernstein-Kantorovich operators by adding a parameter α and using new method and idea based on (p,q)-calculus. We establish the moments and the central moments of the operators. Then, we obtain a Korovkin type approximation theorem and discuss two local approximation theorems using Steklov mean and K-functional in terms of modulus of smoothness. Next, the rate of convergence on continuous function space, differentiable function space and Lipschitz function space are studied. Finally, Voronovskaja type theorem is also investigated.

1. Introduction

As we known, the classical Bernstein-Kantorovich operators were introduced as follows: (see [10])

$$\mathfrak{K}_n(f;x) = \sum_{k=0}^{n} b_{n,k}(x) \int_0^1 f\left(\frac{k+u}{n+1}\right) \mathrm{d}u. \tag{1}$$

where $f \in C[0,1]$, $x \in [0,1]$. In [15], M. A. Özarslan et al. generalized the operators (1) by adding a parameter α as follows:

$$\mathfrak{K}_{n,\alpha}(f;x) = \sum_{k=0}^{n} b_{n,k}(x) \int_0^1 f\left(\frac{k+u^{\alpha}}{n+1}\right) \mathrm{d}u \tag{2}$$

and obtained simultaneous approximation results and order of approximation. In [11], N. I. Mahmudov et al. introduced q-analogue of the operators (1) as follows:

$$K^{*}_{n,q}(f;x) = \sum_{k=0}^{n} b^{q}_{n,k}(x) \int_0^1 f\left(\frac{[k]_q + q^k u}{[n+1]_q}\right) \mathrm{d}_q u.$$

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and studied local and global approximation properties. In [3], A. M. Acu et al. defined q-analogue of the operators (2) as follows:

$$K_{n,q}^{\alpha}(f;x)=\sum_{k=0}^{n} b_{n,k}^{q}(x)\int_0^1 f\left(\frac{[k]_q+u^{\alpha}}{[n+1]_q}\right)\mathrm{d}_q u.$$

and established the shape preserving properties of these operators e.g. monotonicity and convexity. In [14], Mursaleen et al. first used (p,q)-calculus to construct the classical Bernstein operators as follows:

$$B_n^{p,q}(f;x)=\sum_{k=0}^{n} b_{n,k}^{p,q}(x) f\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right),\ x\in[0,1],$$

where $0<q<p\leqslant 1$ and $b_{n,k}^{p,q}(x)=\binom{n}{k}_{p,q} p^{[k(k-1)-n(n-1)]/2}x^k(1-x)_{p,q}^{n-k}$, $k=0,1,\cdots,n$. Later, many Kantorovich-type modifications of (p,q) positive linear operators were constructed and studied, we refer to the articles[1, 2, 7, 12, 13]. There are many books about the research and application about q-calculus and (p,q)-calculus, we mention some of them [4, 8, 9].

Motivated by the above work, we construct modified (p,q)-Bernstein-Kantorovich operators as follows:

DEFINITION 1. Let $f\in C[0,1]$, $0<q<p\leqslant 1$, $\alpha>0$ and $n\in\mathbb{N}$, the modified (p,q)-Bernstein-Kantorovich operators can be defined by

$$\mathfrak{K}_{n,\alpha}^{p,q}(f;x)=\sum_{k=0}^{n} b_{n,k}^{p,q}(x)\int_0^1 f\left(\frac{p^{n-k}[k]_{p,q}+u^{\alpha}}{[n+1]_{p,q}}\right)\mathrm{d}_{p,q}u \tag{3}$$

2. Moment estimation

In this section, we will discuss some auxiliary results about moments estimates and central moments estimates, which are necessary to establish the approximation properties about the operators $\mathfrak{K}_{n,\alpha}^{p,q}$.

LEMMA 1. [6, Lemma 2] *Let* $0<q<p\leqslant 1$, $n=1,2,\cdots$ *and* $x\in[0,1]$, *we have*

(1). $B_n^{p,q}(1;x)=1$, $B_n^{p,q}(u;x)=x$, $B_n^{p,q}(u^2;x)=\frac{q[n-1]_{p,q}}{[n]_{p,q}}x^2+\frac{p^{n-1}}{[n]_{p,q}}x$;

(2). $[n]_{p,q}B_n^{p,q}(u^{m+1};px)=p^n x(1-px)D_{p,q}[B_n^{p,q}(u^m;x)]+[n]_{p,q}pxB_n^{p,q}(u^m;px)$, $m=0,1,2,\cdots,n-1$.

LEMMA 2. *For all* $\alpha>0$, $n\in\mathbb{N}$, $x\in[0,1]$ *and* $0<q<p\leqslant 1$, *we have*

$$\mathfrak{K}_{n,\alpha}^{p,q}(u^m;x)=\frac{1}{[n+1]_{p,q}^m}\sum_{i=0}^{m}\binom{m}{i}\frac{[n]_{p,q}^{m-i}}{[\alpha i+1]_{p,q}}B_n^{p,q}(u^{m-i};x).$$

Proof. Direct computation

$$\begin{aligned}
\mathfrak{K}_{n,\alpha}^{p,q}(u^m;x) &= \sum_{k=0}^{n} b_{n,k}^{p,q}(x) \int_0^1 \left(\frac{p^{n-k}[k]_{p,q} + u^{\alpha}}{[n+1]_{p,q}} \right)^m \mathrm{d}_{p,q}u \\
&= \frac{1}{[n+1]_{p,q}^m} \sum_{k=0}^{n} b_{n,k}^{p,q}(x) \int_0^1 \sum_{i=0}^{m} \binom{m}{i} (p^{n-k}[k]_{p,q})^{m-i} u^{\alpha i} \mathrm{d}_{p,q}u \\
&= \frac{1}{[n+1]_{p,q}^m} \sum_{i=0}^{m} \binom{m}{i} \frac{[n]_{p,q}^{m-i}}{[\alpha i+1]_{p,q}} \sum_{k=0}^{n} b_{n,k}^{p,q}(x) \left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}} \right)^{m-i} \\
&= \frac{1}{[n+1]_{p,q}^m} \sum_{i=0}^{m} \binom{m}{i} \frac{[n]_{p,q}^{m-i}}{[\alpha i+1]_{p,q}} \mathfrak{K}_{n}^{p,q}(u^{m-i};x). \quad \square
\end{aligned}$$

Combining Lemma 1 and Lemma 2, we can easily obtain the following corollaries:

COROLLARY 1. *For all* $\alpha > 0$, $n \in \mathbb{N}$, $x \in [0,1]$ *and* $0 < q < p \leqslant 1$, *we have*
(1). $\mathfrak{K}_{n,\alpha}^{p,q}(1;x) = 1$;
(2). $\mathfrak{K}_{n,\alpha}^{p,q}(u;x) = \frac{[n]_{p,q}}{[n+1]_{p,q}}x + \frac{1}{[n+1]_{p,q}[\alpha+1]_{p,q}}$;
(3). $\mathfrak{K}_{n,\alpha}^{p,q}(u^2;x) = \frac{q[n-1]_{p,q}[n]_{p,q}}{[n+1]_{p,q}^2}x^2 + \left(\frac{p^{n-1}[n]_{p,q}}{[n+1]_{p,q}^2} + \frac{2[n]_{p,q}}{[n+1]_{p,q}^2[\alpha+1]_{p,q}} \right) x + \frac{1}{[n+1]_{p,q}^2[2\alpha+1]_{p,q}}$.

COROLLARY 2. *Using Corollary 1, we can easily get the following explicit formulas about the first and the second central moments:*

$$\mu_{n,1}^{p,q}(x) := \mathfrak{K}_{n,\alpha}^{p,q}(u-x;x) = \left(\frac{[n]_{p,q}}{[n+1]_{p,q}} - 1 \right) x + \frac{1}{[n+1]_{p,q}[\alpha+1]_{p,q}}; \tag{4}$$

$$\begin{aligned}
\mu_{n,2}^{p,q}(x) := \mathfrak{K}_{n,\alpha}^{p,q}\left((u-x)^2;x\right) &= \left(\frac{q[n-1]_{p,q}[n]_{p,q}}{[n+1]_{p,q}^2} - \frac{2[n]_{p,q}}{[n+1]_{p,q}} + 1 \right) x^2 \\
&+ \left(\frac{p^{n-1}[n]_{p,q}}{[n+1]_{p,q}^2} + \frac{2([n]_{p,q} - [n+1]_{p,q})}{[n+1]_{p,q}^2[\alpha+1]_{p,q}} \right) x + \frac{1}{[n+1]_{p,q}^2[2\alpha+1]_{p,q}}.
\end{aligned} \tag{5}$$

COROLLARY 3. *The sequences* (p_n), (q_n) *satisfy* $0 < q_n < p_n \leqslant 1$ *such that* $p_n \to 1$, $q_n \to 1$ *and* $p_n^n \to k_1 \in [0,1]$, $q_n^n \to k_2 \in [0,1]$, $[n]_{p_n,q_n} \to \infty$, $(1-p_n)[n+1]_{p_n,q_n} \to k_3 \in [0,\infty)$ *as* $n \to \infty$, $\alpha > 0$, *then*

$$\lim_{n\to\infty} [n+1]_{p_n,q_n} \mu_{n,1}^{p_n,q_n}(x) = (k_3 - k_2)x + \frac{1}{\alpha+1}; \tag{6}$$

$$\lim_{n\to\infty} [n+1]_{p_n,q_n} \mu_{n,2}^{p_n,q_n}(x) = k_1(x - x^2); \tag{7}$$

$$\lim_{n\to\infty} [n+1]_{p_n,q_n} \mathscr{K}_n^{p_n,q_n}\left((u-x)^4;x\right) = 0. \tag{8}$$

Proof. Using $\frac{[n-i]_{p_n,q_n}}{[n+1]_{p_n,q_n}} = q_n^{-i-1} - p_n^{n+1}(p_nq_n)^{-i-1}\frac{[i+1]_{p_n,q_n}}{[n+1]_{p_n,q_n}}$, we can obtain

$$\begin{aligned}
\lim_{n\to\infty}[n+1]_{p_n,q_n}\mu_{n,1}^{p_n,q_n}(x) &= \lim_{n\to\infty}[n+1]_{p_n,q_n}(q_n^{-1}-1)x - k_1x + \frac{1}{\alpha+1} \\
&= \lim_{n\to\infty}[n+1]_{p_n,q_n}(1-q_n)x - k_1x + \frac{1}{\alpha+1} \\
&= \lim_{n\to\infty}[n+1]_{p_n,q_n}(1-p_n+p_n-q_n)x - k_1x + \frac{1}{\alpha+1} \\
&= (k_3-k_2)x + \frac{1}{\lambda+1}.
\end{aligned}$$

Combining $\lim_{n\to\infty}(1-q_n)[n+1]_{p_n,q_n} = k_3+k_1-k_2$ and

$$\begin{aligned}
&\lim_{n\to\infty}[n+1]_{p_n,q_n}\left(\frac{q_n[n-1]_{p_n,q_n}[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}^2} - \frac{2[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} + 1\right) \\
&= \lim_{n\to\infty}[n+1]_{p_n,q_n}\left(q_n\left(q_n^{-2} - \frac{p_n^{n+1}(p_nq_n)^{-2}[2]_{p_n,q_n}}{[n+1]_{p_n,q_n}}\right)\left(q_n^{-1} - \frac{p_n^{n+1}(p_nq_n)^{-1}}{[n+1]_{p_n,q_n}}\right)\right. \\
&\quad \left. -2\left(q_n^{-1} - \frac{p_n^{n+1}(p_nq_n)^{-1}}{[n+1]_{p_n,q_n}}\right) + 1\right) \\
&= \lim_{n\to\infty}[n+1]_{p_n,q_n}\left(\left(q_n^{-2} - 2q_n^{-1} + 1\right) - \frac{p_n^{n+1}(p_nq_n)^{-2}\left([2]_{p_n,q_n} + p_n - 2p_nq_n\right)}{[n+1]_{p_n,q_n}}\right) \\
&= \lim_{n\to\infty}[n+1]_{p_n,q_n}(1-q_n)^2 - k_1 = -k_1,
\end{aligned}$$

we can have $\lim_{n\to\infty}[n+1]_{p_n,q_n}\mu_{n,2}^{p_n,q_n}(x) = k_1(x-x^2)$. While $n\to\infty$, using Lemma 1 and Lema 2 and supposing $r_n = \frac{q_n}{p_n}$, we can rewrite

$$\begin{aligned}
&B_n^{p_n,q_n}(1;x) = 1,\ B_n^{p_n,q_n}(u;x) = x,\ B_n^{p_n,q_n}(u^2;x) = x^2 - \frac{p_n^{n-1}}{[n]_{p_n,q_n}}x^2 + \frac{p_n^{n-1}}{[n]_{p_n,q_n}}x, \\
&B_n^{p_n,q_n}(u^3;x) = x^3 - (2+r_n)\frac{p_n^{n-1}}{[n]_{p_n,q_n}}x^3 + (2+r_n)\frac{p_n^{n-1}}{[n]_{p_n,q_n}}x^2 + o\left(\frac{1}{[n]_{p_n,q_n}}\right), \\
&B_n^{p_n,q_n}(u^4;x) = x^4 - ([3]_{r_n}+r_n+2)\frac{p_n^{n-1}}{[n]_{p_n,q_n}}x^4 + ([3]_{r_n}+r_n+2)\frac{p_n^{n-1}}{[n]_{p_n,q_n}}x^3 + o\left(\frac{1}{[n]_{p_n,q_n}}\right).
\end{aligned}$$

Combining

$$\left(\frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}\right)^m \sim 1 - mp_n^n\frac{1}{[n+1]_{p_n,q_n}} + o\left(\frac{1}{[n]_{p_n,q_n}}\right) \sim 1 - mp_n^{n-1}\frac{1}{[n]_{p_n,q_n}} + o\left(\frac{1}{[n]_{p_n,q_n}}\right),$$

$m=1,2,3,4$ and Lemma 2, we have

$$\mathfrak{K}_{n,\alpha}^{p,q}(1;x)=1,\ \mathfrak{K}_{n,\alpha}^{p_n,q_n}(u;x)\sim x-\frac{p_n^{n-1}}{[n]_{p_n,q_n}}x+\frac{1}{[n+1]_{p_n,q_n}[\alpha+1]_{p_n,q_n}}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)$$

$$\mathfrak{K}_{n,\alpha}^{p_n,q_n}(u^2;x)\sim x^2-\frac{3p_n^{n-1}}{[n]_{p_n,q_n}}x^2+\frac{p_n^{n-1}}{[n]_{p_n,q_n}}x+\frac{2}{[\alpha+1]_{p_n,q_n}}\frac{x}{[n]_{p_n,q_n}}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)$$

$$\mathfrak{K}_{n,\alpha}^{p_n,q_n}(u^3;x)\sim x^3-\frac{(5+r_n)p_n^{n-1}}{[n]_{p_n,q_n}}x^3+\frac{(2+r_n)p_n^{n-1}}{[n]_{p_n,q_n}}x^2+\frac{3}{[\alpha+1]_{p_n,q_n}}\frac{x^2}{[n]_{p_n,q_n}}+o\left(\frac{1}{[n]_{p_n,q_n}}\right)$$

$$\mathfrak{K}_{n,\alpha}^{p_n,q_n}(u^4;x)\sim x^4-\frac{(6+r_n+[3]_{r_n})p_n^{n-1}}{[n]_{p_n,q_n}}x^4+\frac{(2+r_n+[3]_{r_n})p_n^{n-1}}{[n]_{p_n,q_n}}x^3+\frac{4}{[\alpha+1]_{p_n,q_n}}\frac{x^3}{[n]_{p_n,q_n}}$$
$$+o\left(\frac{1}{[n]_{p_n,q_n}}\right).$$

Hence,

$$\begin{aligned}\mathfrak{K}_{n,\alpha}^{p_n,q_n}\left((t-x)^4;x\right)&=\mathfrak{K}_{n,\alpha}^{p_n,q_n}\left(t^4;x\right)-4\mathfrak{K}_{n,\alpha}^{p_n,q_n}\left(t^3;x\right)x+6\mathfrak{K}_{n,\alpha}^{p_n,q_n}\left(t^2;x\right)x^2\\&\quad-4\mathfrak{K}_{n,\alpha}^{p_n,q_n}(t;x)x^3+x^4\\&\sim\frac{(3r_n-[3]_{r_n})\,p_n^{n-1}}{[n]_{p_n,q_n}}(x^4+x^3)+o\left(\frac{1}{[n]_{p_n,q_n}}\right).\end{aligned}$$

By $\lim_{n\to\infty}r_n=1$, we easily get (8). □

COROLLARY 4. *Let $f\in C[0,1]$ and $x\in[0,1]$, we can get $\|\mathfrak{K}_{n,\lambda}^{p,q}(f;x)\|\leqslant\|f\|$.*

Proof. In view of the definition given by (3) and Corollary 1, for any $x\in[0,1]$, we have

$$\left|\mathfrak{K}_{n,\lambda}^{p,q}(f;x)\right|\leqslant\mathfrak{K}_{n,\lambda}^{p,q}(1;x)\|f\|=\|f\|.$$

Taking supremum over all $x\in[0,1]$, we obtain the required result. □

3. Direct estimates

3.1. Korovkin approximation Theorem

THEOREM 1. *Let (p_n), (q_n) be the sequences defined in Corollary 3, $\alpha>0$. Then for any $f\in C[0,1]$, the sequence $\left\{\mathfrak{K}_{n,\lambda}^{p_n,q_n}(f;x)\right\}$ converges uniformly to f on $[0,1]$.*

Proof. By the classical Korovkin Theorem [5, p. 8, Theorem 3.1], it is sufficient to prove that

$$\lim_{n\to\infty}\left\|\mathfrak{K}_{n,\lambda}^{p_n,q_n}\left(u^i;x\right)-x^i\right\|=0,\ i=0,1,2.$$

By Corollary 1 (1), it is clear that $\lim_{n\to\infty}\left\|\mathfrak{K}_{n,\lambda}^{p_n,q_n}(1;x)-1\right\|=0$. Using $q_n[n]_{p_n,q_n}=[n+1]_{p_n,q_n}-p_n^n$, by Corollary 1 (2), we have

$$\begin{aligned}\left|\mathfrak{K}_{n,\lambda}^{p_n,q_n}(u;x)-x\right| &\leqslant \left|\frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}-1\right|x+\frac{1}{[n+1]_{p_n,q_n}[\alpha+1]_{p_n,q_n}}\\ &\leqslant \left|\frac{1}{q_n}-1\right|+\frac{q_n^{-1}p_n^n}{[n+1]_{p_n,q_n}}+\frac{1}{[n+1]_{p_n,q_n}[\alpha+1]_{p_n,q_n}}\\ &\to 0,\ n\to\infty.\end{aligned}$$

By (6) and (7), we have

$$\begin{aligned}\lim_{n\to\infty}\left|\mathfrak{K}_{n,\lambda}^{p_n,q_n}\left(u^2;x\right)-x^2\right| &= \lim_{n\to\infty}\left|\mathfrak{K}_{n,\lambda}^{p_n,q_n}\left(u^2-x^2;x\right)\right|\\ &\leqslant \lim_{n\to\infty}\left|\mathfrak{K}_{n,\lambda}^{p_n,q_n}\left((u-x)^2;x\right)\right|+2x\lim_{n\to\infty}\left|\mathfrak{K}_{n,\lambda}^{p_n,q_n}(u-x;x)\right|\\ &\leqslant \lim_{n\to\infty}\frac{1}{[n+1]_{p_n,q_n}}\left((k_1+k_2+k_3)x+k_1x^2+\frac{1}{\alpha}\right)=0.\end{aligned}$$

Thus the proof is completed. □

3.2. Local approximation

In this subsection, we will discuss two local approximation theorems about the operators (3) by Steklov mean and K-functional independently. Let $f\in C[0,1]$ and $t>0$, the Steklov mean is defined by

$$f_t(x)=\frac{4}{t^2}\int_0^{\frac{t}{2}}\int_0^{\frac{t}{2}}(2f(x+u+v)-f(x+2u+2v))\mathrm{d}u\mathrm{d}v.$$

By simple computation, it is observed that

(i) $\|f_t-f\|\leqslant\omega_2(f;t)$;

(ii) If f is continuous and $f_t',f''\in C[0,1]$, then $\|f_t'\|\leqslant\frac{5}{t}\omega(f;t)$, $\|f_t''\|\leqslant\frac{9}{t^2}\omega_2(f;t)$, where the first and second order modulus of continuity for $\delta\geqslant 0$ are respectively defined by

$$\omega(f;\delta)=\sup_{x,x+y\in[0,1],0<|y|\leqslant\delta}|f(x+y)-f(x)|$$

and

$$\omega_2(f;\delta)=\sup_{x,x+2y\in[0,1],0<|y|\leqslant\delta}|f(x+2y)+f(x)-2f(x+y)|.$$

THEOREM 2. *Let $f\in C[0,1]$, $0<q<p\leqslant 1$, $\alpha>0$, then for any $n\in\mathbb{N}$ and $x\in[0,1]$, we have*

$$\begin{aligned}\left|\mathfrak{K}_{n,\alpha}^{p,q}(f;x)-f(x)\right| \leqslant{}& 5\sqrt{[n]_{p,q}}\mu_{n,1}^{p,q}(x)\omega\left(f;\frac{1}{\sqrt{[n]_{p,q}}}\right)\\ &+\left(\frac{9}{2}[n]_{p,q}\mu_{n,1}^{p,q}(x)+2\right)\omega_2\left(f;\frac{1}{\sqrt{[n]_{p,q}}}\right).\end{aligned}$$

Proof. For $x \in [0,1]$, $t > 0$ and using definition of the Steklov mean, we can write

$$\left|\mathfrak{K}_{n,\alpha}^{p,q}(f;x) - f(x)\right| \leqslant \mathfrak{K}_{n,\alpha}^{p,q}(|f - f_t|;x) + \left|\mathfrak{K}_{n,\alpha}^{p,q}(f_t(u) - f_t(x);x)\right| + |f_t(x) - f(x)|.$$

By Corollary 4 and property (i) of the Steklov mean, we have

$$\mathfrak{K}_{n,\alpha}^{p,q}(|f - f_t|;x) \leqslant \|\mathfrak{K}_{n,\alpha}^{p,q}(f - f_t)\| \leqslant \|f - f_t\| \leqslant \omega_2(f;t).$$

Also, by Taylor's expansion formula, we have

$$\begin{aligned}\left|\mathfrak{K}_{n,\alpha}^{p,q}(f_t(u) - f_t(x);x)\right| &\leqslant |f_t'(x)|\mu_{n,1}^{p,q}(x) + \frac{1}{2}\|f_t''\|\mu_{n,2}^{p,q}(x) \\ &\leqslant \|f_t'\|\mu_{n,1}^{p,q}(x) + \frac{1}{2}\|f_t''\|\mu_{n,2}^{p,q}(x) \\ &\leqslant \frac{5}{t}\mu_{n,1}^{p,q}(x)\omega(f;h) + \frac{9}{2t^2}\mu_{n,2}^{p,q}(x)\omega_2(f;t).\end{aligned}$$

Hence,

$$\left|\mathfrak{K}_{n,\alpha}^{p,q}(f;x) - f(x)\right| \leqslant \frac{5}{t}\mu_{n,1}^{p,q}(x)\omega(f;t) + \left(\frac{9}{2t^2}\mu_{n,2}^{p,q}(x) + 2\right)\omega_2(f;t),$$

for $x \in [0,1]$, $t > 0$. Setting $t = \frac{1}{\sqrt{[n]_{p,q}}}$, we obtain the desired result. $\square$

The second form to obtain the local approximation is the application of K-functional. The Peetre's K-functional is defined by

$$K(f;\delta) = \inf_{\eta'' \in C[0,1]} \left\{\|f - \eta\| + \delta\|\eta''\|\right\}.$$

By [5, p. 177, Theorem 2.4], there exists an absolute constant $C > 0$ such that

$$K(f;\delta^2) \leqslant C\omega_2(f;\delta),\ \delta > 0. \tag{9}$$

THEOREM 3. *Let $f \in C[0,1]$, $0 < q < p \leqslant 1$, $\alpha > 0$, then for any $n \in \mathbb{N}$ and $x \in [0,1]$, we have*

$$\left|\mathfrak{K}_{n,\alpha}^{p,q}(f;x) - f(x)\right| \leqslant 4C\omega_2\left(f;\sqrt{\left(\mu_{n,1}^{p,q}(x)\right)^2 + \mu_{n,2}^{p,q}(x)}\right) + \omega\left(f;\left|\mu_{n,1}^{p,q}(x)\right|\right).$$

Proof. For $x \in [0,1]$, we define the following new operators $\mathbb{K}_{n,\alpha}^{p,q}$ by

$$\mathbb{K}_{n,\alpha}^{p,q}(f;x) = \mathfrak{K}_{n,\alpha}^{p,q}(f;x) + f(x) - f\left(x + \mu_{n,1}^{p,q}(x)\right).$$

From Corollary 2, we observe that the operators $\mathbb{K}_{n,\alpha}^{p,q}(f;x)$ are linear and reproduce the linear functions. Let $x \in [0,1]$ and $\eta'' \in C[0,1]$. Using the Taylor's expansion formula, we can obtain

$$\eta(u) = \eta(x) + \eta'(x)(u - x) + \int_x^u \eta''(v)(u - v)\mathrm{d}v.$$

Hence,

$$\begin{aligned}\mathbb{K}_{n,\alpha}^{p,q}(\eta;x)-\eta(x) &= \mathbb{K}_{n,\alpha}^{p,q}\left(\int_x^u \eta''(v)(u-v)\mathrm{d}v;x\right)\\ &\leqslant \mathfrak{K}_{n,\alpha}^{p,q}\left(\int_x^u \eta''(v)(u-v)\mathrm{d}v;x\right)\\ &\quad+\left|\int_x^{x+\mu_{n,1}^{p,q}(x)} \eta''(v)\left(x+\mu_{n,1}^{p,q}(x)-v\right)\right|\mathrm{d}v\\ &\leqslant \mathfrak{K}_{n,\alpha}^{p,q}\left((u-x)^2;x\right)\|\eta\|+\left(\mu_{n,1}^{p,q}(x)\right)^2\|\eta''\|\\ &=\left(\left(\mu_{n,1}^{p,q}(x)\right)^2+\mu_{n,2}^{p,q}(x)\right)\|\eta''\|.\end{aligned}$$

By Corollary 4, we easily know $|\mathbb{K}_{n,\alpha}^{p,q}(f;x)|\leqslant 3\|f\|$. We can get

$$\begin{aligned}\left|\mathfrak{K}_{n,\alpha}^{p,q}(f;x)-f(x)\right| &= \left|\mathbb{K}_{n,\alpha}^{p,q}(f;x)+f\left(x+\mu_{n,1}^{p,q}(x)\right)-2f(x)\right|\\ &\leqslant \left|\mathbb{K}_{n,\alpha}^{p,q}(f-\eta;x)-(f-\eta)(x)\right|+\left|\mathbb{K}_{n,\alpha}^{p,q}(\eta;x)-\eta(x)\right|\\ &\quad+\left|f\left(x+\mu_{n,1}^{p,q}(x)\right)-f(x)\right|\\ &\leqslant 4\|f-\eta\|+\left(\left(\mu_{n,1}^{p,q}(x)\right)^2+\mu_{n,2}^{p,q}(x)\right)\|\eta''\|+\omega\left(f;\left|\mu_{n,1}^{p,q}(x)\right|\right).\end{aligned}$$

Taking infinum on the right hand side over all $\eta''\in C[0,1]$ using (9), we obtain the desired assertion. □

3.3. Rate of convergence

First, we give the rate of convergence of the operators $\mathfrak{K}_{n,\alpha}^{p,q}(f;x)$ by means of the modulus of continuity $\omega(f;\delta)$.

THEOREM 4. *If* $f\in C[0,1]$, $0<q<p\leqslant 1$, $\alpha>0$ *and* $n\in\mathbb{N}$, *for any* $x\in[0,1]$, *we have*

$$\left|\mathfrak{K}_{n,\alpha}^{p,q}(f;x)-f(x)\right|\leqslant 2\omega\left(f;\sqrt{\mu_{n,2}^{p,q}(x)}\right).$$

Proof. Using [5, p. 41, (6.5)], for any $f\in C[0,1]$ and $\delta>0$, we have

$$|f(u)-f(x)|\leqslant\omega(f;\delta)\left(1+\frac{|u-x|}{\delta}\right).$$

Using the monotonicity and the linearity of $\mathfrak{K}_{n,\alpha}^{p,q}$ and Cauchy-Schwarz inequality, for any $\delta > 0$, we have

$$\begin{aligned}\left|\mathfrak{K}_{n,\alpha}^{p,q}(f;x)-f(x)\right| &\leqslant \mathfrak{K}_{n,\alpha}^{p,q}\left(|f(u)-f(x)|;x\right) \leqslant \mathfrak{K}_{n,\alpha}^{p,q}\left(\omega\left(f;|u-x|\right);x\right)\\ &\leqslant \omega(f;\delta)\left(1+\frac{\mathfrak{K}_{n,\alpha}^{p,q}(|u-x|;x)}{\delta}\right)\\ &\leqslant \omega(f;\delta)\left(1+\frac{\sqrt{\mathfrak{K}_{n,\alpha}^{p,q}((u-x)^2;x)}}{\delta}\right)\\ &\leqslant \omega(f;\delta)\left(1+\frac{\sqrt{\mu_{n,2}^{p,q}(x)}}{\delta}\right).\end{aligned}$$

Finally, let us choose $\delta = \sqrt{\mu_{n,2}^{p,q}(x)}$. We complete the proof of Theorem 4. □

While $f' \in C[0,1]$, we have the following theorem about rate of convergence.

THEOREM 5. *If $f \in C[0,1]$, $0 < q < p \leqslant 1$, $\alpha > 0$ and $n \in \mathbb{N}$, for any $x \in [0,1]$, we have*

$$\left|\mathfrak{K}_{n,\alpha}^{p,q}(f;x)-f(x)\right| \leqslant \left|\mu_{n,1}^{p,q}(x)\right| |f'(x)| + 2\sqrt{\mu_{n,2}^{p,q}(x)}\,\omega\left(f';\sqrt{\mu_{n,2}^{p,q}(x)}\right).$$

Proof. Applying $\mathfrak{K}_{n,\alpha}^{p,q}$ to both sides of $f(u) = f(x) + f'(x)(u-x) + f(u) - f(x) - f'(x)(u-x)$, we have

$$\begin{aligned}&\left|\mathfrak{K}_{n,\alpha}^{p,q}(f;x)-f(x)\right|\\ \leqslant& |f'(x)|\left|\mathfrak{K}_{n,\alpha}^{p,q}(t-x;x)\right| + \mathfrak{K}_{n,\alpha}^{p,q}\left(\left|f(u)-f(x)-f'(x)(u-x)\right|;x\right)\\ \leqslant& \left|\mu_{n,1}^{p,q}(x)\right| |f'(x)| + \mathfrak{K}_{n,\alpha}^{p,q}\left(|u-x|\left(1+\frac{|u-x|}{\delta}\right);x\right)\omega\left(f';\delta\right)\\ \leqslant& \left|\mu_{n,1}^{p,q}(x)\right| |f'(x)| + \sqrt{\mathfrak{K}_{n,\alpha}^{p,q}\left((u-x)^2;x\right)}\left(1+\frac{\sqrt{\mathfrak{K}_{n,\alpha}^{p,q}\left((u-x)^2;x\right)}}{\delta}\right)\omega\left(f';\delta\right)\end{aligned}$$

with the help of Cauchy-Schwartz inequality and mean value theorem. Taking $\delta = \sqrt{\mathfrak{K}_{n,\alpha}^{p,q}\left((u-x)^2;x\right)}$ and by Corollary 2, we can get the desired result. □

Now, we give the rate of convergence of the operators $\mathfrak{K}_{n,\alpha}^{p,q}(f;x)$ by means of the Lipschitz class $\mathrm{Lip}_M(\gamma)$. A function $f \in C[0,1]$ belongs to $\mathrm{Lip}_M(\gamma)$ $(\gamma \in (0,1])$, if the condition

$$|f(u)-f(x)| \leqslant M|u-x|^{\gamma},\ u,x \in [0,1]$$

is satisfied.

THEOREM 6. *If* $f \in \mathrm{Lip}_M(\gamma)$ $(\gamma \in (0,1])$, *we have*

$$\left|\mathfrak{K}_{n,\alpha}^{p,q}(f;x) - f(x)\right| \leqslant M\left(\mu_{n,2}^{p,q}(x)\right)^{\frac{\gamma}{2}}.$$

Proof. According to the monotonicity and the linearity of the operators and taking into account that $f \in \mathrm{Lip}_M(\gamma)$, we can obtain

$$\left|\mathfrak{K}_{n,\alpha}^{p,q}(f;x) - f(x)\right| \leqslant \mathfrak{K}_{n,\alpha}^{p,q}\left(|f(u)-f(x)|;x\right) \leqslant M\mathfrak{K}_{n,\alpha}^{p,q}\left(|u-x|^{\gamma};x\right).$$

Applying well-known Hölder's inequality with $t_1 = \frac{2}{\gamma}$ and $t_2 = \frac{2}{2-\gamma}$, we can get

$$\begin{aligned}\left|\mathfrak{K}_{n,\alpha}^{p,q}(f;x) - f(x)\right| &\leqslant M\mathfrak{K}_{n,\alpha}^{p,q}\left(|u-x|^{\gamma};x\right)\\ &\leqslant M\left(\mathfrak{K}_{n,\alpha}^{p,q}\left(|u-x|^{t_1\gamma};x\right)\right)^{\frac{1}{t_1}}\left(\mathfrak{K}_{n,\alpha}^{p,q}(1^{t_2};x)\right)^{\frac{1}{t_2}}\\ &= M\left(\mathfrak{K}_{n,\alpha}^{p,q}\left(|u-x|^2;x\right)\right)^{\frac{\gamma}{2}}\\ &= M\left(\mu_{n,2}^{p,q}(x)\right)^{\frac{\gamma}{2}}.\end{aligned}$$

We obtain the required result. □

3.4. Voronovskaja type Theorem

In this subsection, we give a Voronovskaja type asymptotic formula for the operators (3) by means of the second and fourth central moments.

THEOREM 7. *Let* (p_n), (q_n) *be the sequences defined in Corollary 3,* $\alpha > 0$ *and* $f \in C[0,1]$. *Suppose that* $f''(x)$ *exists at a point* $x \in [0,1]$, *then we can obtain*

$$\lim_{n\to\infty}[n+1]_{p_n,q_n}\left|\mathfrak{K}_{n,\alpha}^{p_n,q_n}(f;x) - f(x)\right| = \left((k_3-k_2)x + \frac{1}{\alpha+1}\right)f'(x) + \frac{k_1}{2}x^2 f''(x).$$

Proof. Using Taylor's expansion formula, we can obtain

$$f(u) = f(x) + f'(x)(u-x) + \frac{1}{2}f''(x)(u-x)^2 + r(u,x)(u-x)^2, \tag{10}$$

where $R(t,x)$ is the Peano form of the remainder and $\lim_{u\to x} R(t,x) = 0$. Applying $\mathfrak{K}_{n,\alpha}^{p_n,q_n}$ to the both sides of (10), we have

$$\begin{aligned}&[n+1]_{p_n,q_n}\left(\mathfrak{K}_{n,\alpha}^{p_n,q_n}(f;x) - f(x)\right)\\ &= [n+1]_{p_n,q_n} f'(x)\left(\mathfrak{K}_{n,\alpha}^{p_n,q_n}(u-x;x)\right.\\ &\quad + [n+1]_{p_n,q_n}\frac{f''(x)}{2}\mathfrak{K}_{n,\alpha}^{p_n,q_n}\left((u-x)^2;x\right) + [n+1]_{p_n,q_n}\mathfrak{K}_{n,\alpha}^{p_n,q_n}\left(r(u,x)(u-x)^2;x\right).\end{aligned}$$

By the Schwarz inequality, we have

$$\mathfrak{K}_{n,\alpha}^{p_n,q_n}\left(r(u,x)(u-x)^2;x\right) \leqslant \sqrt{\mathfrak{K}_{n,\alpha}^{p_n,q_n}\left(r^2(u,x);x\right)}\sqrt{\mathfrak{K}_{n,\alpha}^{p_n,q_n}\left((u-x)^4;x\right)}. \tag{11}$$

We observe that $r^2(x,x)=0$ and $r^2(\cdot,x)\in C[0,1]$. Then, it follows from Theorem 1 that

$$\lim_{n\to\infty}\mathfrak{K}_{n,\alpha}^{p_n,q_n}\left(r^2(u,x);x\right)=r^2(x,x)=0. \tag{12}$$

Hence, from (8), (11), (12), we can obtain

$$\lim_{n\to\infty}[n+1]_{p_n,q_n}\mathfrak{K}_{n,\alpha}^{p_n,q_n}\left(r(u,x)(t-x)^2;x\right)=0. \tag{13}$$

Combining (6), (7), (13), we obtain the required result. □

COROLLARY 5. *Let (p_n), (q_n) be the sequences defined in Corollary 3, $\alpha>0$ and $f''\in C[0,1]$, then*

$$\lim_{n\to\infty}[n+1]_{p_n,q_n}\left|\mathfrak{K}_{n,\alpha}^{p_n,q_n}(f;x)-f(x)\right|=\left((k_3-k_2)x+\frac{1}{\alpha+1}\right)f'(x)+\frac{k_1}{2}x^2f''(x).$$

uniformly in $x\in[0,1]$.

REFERENCES

[1] T. ACAR, A. ARAL, S. A. MOHIUDDINE, *On Kantorovich modification of (p,q)-Baskakov operators*, J. Inequal. Appl., **98** (2016).

[2] T. ACAR, A. ARAL, S. A. MOHIUDDINE, *On Kantorovich modification of (p,q)-Bernstein operators*, Iran. J. Sci. Technol. Trans. Sci., **42**, 3 (2018), 1459–1464.

[3] A. M. ACU, P. N. AGRAWAL, D. KUMAR, *Approximation properties of modified q-Bernstein-Kantorovich Operators*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., **68**, 2 (2019), 2170–2197.

[4] A. ARAL, V. GUPTA, R. P. AGARWAL, *Application of q-Calculus in Operator Theory*, Springer, Berlin, (2013).

[5] R. A. DEVORE, G. G. LORENTZ, *Constructive Approximation*, Springer, Berlin, (1993).

[6] V. GUPTA, *(p,q)-Genuine Bernstein Durrmeyer operators*, Boll. Unione Mat. Ital., **9**, 3 (2016), 399–409.

[7] V. GUPTA, *(p,q)-Baskakov-Kantorovich Operators*, Appl. Math. Inf. Sci., **10**, 4 (2016), 1551–1556.

[8] V. GUPTA, R. P. AGARWAL, *Convergence Estimates in Approximation Theory*, Springer, New York, (2014).

[9] V. GUPTA, T. M. RASSIAS, P. N. AGRAWAL, A. M. ACU, *Recent Advances in Constructive Approximation Theory*, Springer, New York, (2018).

[10] G. G. LORENTZ, *Bernstein Polynomials*, Math. Expo. Vol. 8. Univ. of Toronto Press, Toronto, 1953.

[11] N. I. MAHMUDOV, P. SABANCIGIL, *Approximation Theorems for q-Bernstein-Kantorovich Operators*, Filomat, **27**, 4 (2013), 721–730.

[12] M. MURSALEEN, A. ALOTAIBI, ,K. J. ANSARI, *On a Kantorovich of (p,q)-Szász-Mirakjan operators*, J. Funct. Space., (2016) [Article ID 1035253].

[13] M. MURSALEEN, A. NAAZ, A. KHAN, *Improved approximation and error estimations by King type (p,q)-Szász-Mirakjan Kantorovich operators*, Appl. Math. Comput., **348**, (2019), 175–185.

[14] M. MURSALEEN, F. KHAN, A. KHAN, *On* (p,q)*-analogue of Bernstein operators*, Appl. Math. Comput., **266**, (2015), 874-882, Erratum: Appl. Math. Comput., **278**, (2016): 70–71.

[15] M. A. ÖZARSLAN, O. DUMAN, *Smoothness properties of modified Bernstein-Kantorovich operators*, Numer. Func. Anal. Opt., **37**, 1 (2016), 92–105.

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