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OPTIMAL CONTROL FOR ELLIPTIC HEMIVARIATIONAL INEQUALITIES INVOLVING NONLINEAR WEAKLY CONTINUOUS OPERATORS

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Abstract. We study an optimal control problem governed by elliptic hemivariational inequalities involving nonlinear weakly continuous operators. By exploiting the surjectivity theorem of multivalued weakly upper continuous operators, we present an existence result for a class of subgradient inclusions involving nonlinear weakly continuous operators. Then we obtain the existence of optimal pairs for the optimal control problem. Moreover, we consider a perturbed optimal control problem and obtain the convergence of optimal pairs. This study can be applied to stationary Navier-Stokes problems with multivalued frictional boundary condition.

1. Introduction

Let Y and Z be two separable and reflexive Banach spaces. Let a nonlinear operator $A : Y \to Y^*$, a linear operator $M : Y \to Z$ with its dual operator $M^* : Z^* \to Y^*$, a locally Lipschitz functional $J : Z \to \mathbb{R}$, $f \in Y^*$. We consider the elliptic hemivariational inequality problem of the following form.

PROBLEM 1. Find $y \in Y$ such that

$$\langle Ay, v\rangle_Y + J^\circ(My; Mv) \geqslant \langle f, v\rangle_Y, \qquad \forall v \in Y.$$

It is clear Problem 1 is equivalent to the following subgradient inclusion problem.

PROBLEM 2. Find $y \in Y$ such that

$$Ay + M^*\partial J(My) \ni f.$$

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DEFINITION 3. A function $y \in Y$ is called a solution to Problem 2 if there exists $\xi \in Z^*$ such that

$$\begin{cases} Ay + M^*\xi = f, \\ \xi \in \partial J(My). \end{cases}$$

The theory of hemivariational inequalities has been initiated in early 1980s with the pioneering works of Panagiotopoulos, cf. [29, 30] and the references therein. Elliptic hemivariational inequalities have been studied in e.g. [3, 14, 15, 18, 19, 24, 27, 28, 31] by using methods based on surjectivity results for several classes of monotone operators, in e.g. [4, 37] by using sub-supersolution method, and in e.g. [21, 34, 36, 41] by introducing and applying some concepts of quasimonotonicity and KKM technique. Optimal control problems for elliptic hemivariational inequalities have been studied by [1, 6, 16, 20, 32, 33] and the reference therein.

In this paper we study the elliptic hemivariational inequality with a nonlinear weakly continuous, bounded and coercive operator governing the process and a multivalued term involving the Clarke subgradient of a locally Lipschitz function. It is well known that such inequalities have often an equivalent formulation as operator subgradient inclusions. By exploiting the surjectivity theorem of multivalued weakly upper continuous operators, we first present an existence result for the subgradient inclusions. Then we obtain the existence of optimal pairs for the optimal control problem. Moreover, we consider a perturbed optimal control problem and obtain the convergence of optimal pairs. This study can be applied to stationary Navier-Stokes equations.

It is worth pointing out that there are several novelties of the present paper. First, instead of linearity, monotonicity, pseudomonotonicity, M-condition or strong continuity, used in all aformentioned papers, we suppose the weak continuity, boundedness and coercivity conditions of the operator. Weak continuity and coercivity of the operator were used in [12] to study the existence of solution to elliptic equations and in [40] to study the existence of solution to stationary inclusions. Second, we obtain the existence of optimal pairs to an optimal control problem and a perturbed optimal control problem with weakly continuous operators, and show the convergence of optimal pairs. Third, as an application, we use weakly continuous operators to provide a simple proof of existence of solutions to stationary Navier-Stokes problems with multivalued frictional boundary condition. Results on solvability of stationary and non-stationary hemivariational inequalities for Navier-Stokes can be found in e.g. [10, 11, 23, 24, 25, 26].

The rest of this paper is organized as follows. In the next section, we will briefly recall some definitions and preliminary results. We give an existence result for Problem 2 by using the surjectivity theorem of weakly-weakly u.s.c. multivalued operator. In Section 3, we consider an optimal control problem and obtain its existence of optimal pairs. In Section 4, we show the convergence of optimal pairs to the perturbed optimal control problem. In the last section, we apply our main results to stationary Navier-Stokes problems with multivalued frictional boundary condition.

2. Preliminaries

Let $(X, \|\cdot\|_X)$ be a Banach space. We denote by X^* its dual space, by w-X the space X endowed with the weak topology, and by $\langle\cdot,\cdot\rangle_X$ the duality pairing between X^* and X. We denote by "$\rightarrow$" the strong convergence and by "$\rightharpoonup$" the weak convergence.

DEFINITION 4. ([12]) Let X, Y be two reflexive Banach spaces. An operator $F\colon X \to Y$ is said to be weakly continuous, if for any sequence $\{x_n\}_{n\geqslant 1} \subset X$ with $x_n \rightharpoonup x$ in X, then $Fx_n \rightharpoonup Fx$ in Y.

DEFINITION 5. ([39]) An operator $F\colon X \to X^*$ is said to be

(i) bounded, if there exists a continuous increasing function $\beta\colon [0,+\infty) \to [0,+\infty)$ such that

$$\|Fu\|_{X^*} \leqslant \beta(\|x\|_X) \text{ for all } x \in X.$$

(ii) coercive, if

$$\lim_{\|x\|_X\to\infty} \frac{\langle Fx, x\rangle_X}{\|x\|_X} = +\infty.$$

(ii) coercive with constant c, if

$$\langle Fx, x\rangle_X \geqslant c\|x\|_X^2.$$

DEFINITION 6. ([2, 17]) A multivalued operator $F\colon X \to 2^{X^*}$ with closed values is said to be

(i) upper semicontinuous (u.s.c.), if for every open subset $\mathscr{O} \subset X^*$ the "strong inverse image" of $\mathscr{O}$ under F given by $F^+(\mathscr{O}) = \{x \in X \mid F(x) \subset \mathscr{O}\}$ is open in X.

(ii) weakly-weakly u.s.c., if for every open subset $\mathscr{O} \subset (w\text{-}X^*)$ the set $F^+(\mathscr{O})$ is open in w-X.

(iii) closed, if for any $(x_n, x_n^*) \in Gr(F) = \{(x, x^*) \in X \times X^* \mid x^* \in F(x)\}$ with $x_n \to x$ in X, $x_n^* \to x^*$ in X^*, we have $(x, x^*) \in Gr(F)$.

(iv) weakly-weakly closed, if for any $(x_n, x_n^*) \in Gr(F)$ with $x_n \rightharpoonup x$ in X, $x_n^* \rightharpoonup x^*$ in X^*, we have $(x, x^*) \in Gr(F)$.

(v) coercive, if

$$\lim_{\|x\|_X\to\infty} \frac{\inf_{x^*\in F(x)} \langle x^*, x\rangle_X}{\|x\|_X} = \infty.$$

It is known, see [17] that if $F\colon X \to 2^{X^*}$ is u.s.c. (weakly-weakly u.s.c.) with closed (weakly closed) values, then F is closed (weakly-weakly closed).

THEOREM 7. ([40]) *Let X be a separable reflexive Banach space and $F : X \to 2^{X^*}$ be weakly-weakly u.s.c. and coercive with nonempty, bounded, closed and convex values. Then F is surjective, i.e., for every $f \in X^*$, there is $x \in X$ such that $Fx \ni f$.*

DEFINITION 8. ([5, 29]) Given a locally Lipschitz function $\varphi\colon X\to\mathbb{R}$, we denote by $\varphi^0(x;y)$ the (Clarke) generalized directional derivative of φ at the point $x\in X$ in the direction $y\in X$ defined by

$$\varphi^0(x;y)=\limsup_{\lambda\to 0^+,\,\zeta\to x}\frac{\varphi(\zeta+\lambda y)-\varphi(\zeta)}{\lambda}.$$

The generalized gradient of φ at $x\in X$, denoted by $\partial\varphi(x)$, is a subset of X^* given by

$$\partial\varphi(x)=\{x^*\in X^*\mid \varphi^0(x;y)\geqslant\langle x^*,y\rangle_X\ \text{ for all }\ y\in X\}.$$

At the end of this section, we give an existence result for Problem 2. We will make the following hypotheses on the data of Problem 2.

(H_A) $A:Y\to Y^*$ is bounded, weakly continuous and coercive with constant $\alpha>0$.

(H_J) $J:Z\to\mathbb{R}$ is locally Lipschitz and there exist $c_0,c_1>0$ such that

$$\|\partial J(z)\|_{Z^*}\leqslant c_0+c_1\|z\|_Z,\qquad \forall z\in Z.$$

(H_M) $M:Y\to Z$ is linear, continuous and compact.

(H_0) $\alpha>c_1\|M\|^2$.

THEOREM 9. *Assume that hypotheses $(H_A),(H_J),(H_M),(H_0)$ hold. Then Problem 2 has at least one solution $y\in Y$ and there exists a constant $C>0$ such that*

$$\|y\|_Y\leqslant C(1+\|f\|_{Y^*}).\tag{1}$$

Proof. Define a multivalued operator $F\colon Y\to 2^{Y^*}$ by

$$F(y)=Ay+M^*\partial J(My),\qquad \forall y\in Y.$$

We will use Theorem 7 to prove that F is surjective. It is sufficient to show that F is weakly-weakly u.s.c. and coercive.

It is clear that F has nonempty and convex values. From [5, Proposition 2.1.2] we see that the Clarke subgradient ∂J has weakly compact values in Z^*, hence its values are also closed. Moreover, we use the fact that a compact subset with respect to the weak topology of a Banach space is bounded. Thus, the values of F are bounded subsets in Y^*.

We will show that F is coercive. For $y\in Y$ and $y^*\in F(y)$, we have

$$\begin{aligned}\langle y^*,y\rangle_Y&\geqslant\alpha\|y\|_Y^2-(c_0+c_1\|M\|^2\|y\|_Y)\|y\|_Y\\&=(\alpha-c_1\|M\|^2)\|y\|_Y^2-c_0\|y\|_Y.\end{aligned}$$

Hence F is coercive.

It follows from (H_A), the upper semicontinuity of ∂J and the compactness of M that F is weakly-weakly u.s.c. Therefore, by applying Theorem 7 we deduce that Problem 2 has a solution $y\in Y$. Moreover, (1) follows from the following inequalities

$$(\alpha-c_1\|M\|^2)\|y\|_Y^2-c_0\|y\|_Y\leqslant\langle y^*,y\rangle_Y\leqslant\|f\|_{Y^*}\|y\|_Y,\qquad\forall y^*\in F(y).$$

This completes the proof. $\square$

3. An optimal control problem

In this section, we consider an optimal control problem for Problem 2.

PROBLEM 10. Given a control $u \in U$, find $y \in Y$ such that

$$Ay + M^*\partial J(My) \ni f + Bu,$$

where $B : U \to Y^*$ is a nonlinear control operator with U being a reflexive Banach space.

Denote by $Sol(u)$ the set of all solutions of Problem 10 corresponding to the control u and let

$$V = \{(u,y) \in U \times Y | y \in Sol(u)\}.$$

Consider the following hypothesis on the control operator B.
(H_B) $B : U \to Y^*$ is weakly continuous and there exist $b_0, b_1 \geqslant 0$ such that

$$\|Bu\|_{Y^*} \leqslant b_0 + b_1\|u\|_U, \qquad \forall u \in U.$$

THEOREM 11. *Assume that hypotheses $(H_A), (H_J), (H_M), (H_0), (H_B)$ hold. Then for every $u \in U$, Problem 10 has at least one solution $y \in Sol(u)$ and there exists a constant $C_1 > 0$ such that*

$$\|y\|_Y \leqslant C_1(1 + \|f\|_{Y^*} + \|u\|_U). \tag{2}$$

Moreover, the solution operator $Sol : U \to Y$ is weakly-weakly closed.

Proof. The existence and boundedness follows from 9. We show that the solution operator $Sol : U \to Y$ is weakly-weakly closed. For any sequence $\{u_n\}$ with $u_n \rightharpoonup \overline{u}$ in U and $y_n \in Sol(u_n)$ with $y_n \rightharpoonup \overline{y}$ in Y, there exists $\xi_n \in \partial J(My_n)$ such that

$$Ay_n + M^*\xi_n = f + Bu_n. \tag{3}$$

It comes from (H_J) that the sequence $\{\xi_n\}$ is bounded in Z. Since the space Z^* is reflexive, there is a subsequence of $\{\xi_n\}$, denoted by $\{\xi_n\}$ again, such that $\xi_n \rightharpoonup \overline{\xi}$ for some $\overline{\xi} \in Z^*$. Since the operator M is compact, we deduce that $M^*\xi_n \to M^*\overline{\xi}$ and $\overline{\xi} \in \partial J(M\overline{y})$. On the other hand, $(H_A), (H_B)$ imply that

$$Ay_n \rightharpoonup A\overline{y}, \qquad Bu_n \rightharpoonup B\overline{u}.$$

Letting $n \to \infty$ in (3) we obtain by the assumptions that

$$A\overline{y} + M^*\overline{\xi} = f + B\overline{u},$$

where $\overline{\xi} \in \partial J(M\overline{y})$. Therefore, $\overline{y} \in Sol(\overline{u})$, i.e., $Sol : U \to Y$ is weakly-weakly closed. The proof is complete. $\square$

Next, we study an optimal control problem for Problem 10. Given another Banach space (the so-called observation space) W, a nonlinear operator $\Sigma : Y \to W$, and a target $\Upsilon \in W$, we consider the following cost function:

$$H(u,y) := \|\Sigma(u) - \Upsilon\| + \varepsilon\|y\|^2,$$

where $\varepsilon > 0$. The optimal control problem studied in this work is to solve the following minimization problem.

PROBLEM 12. Find a pair $(u^*,y^*) \in V$ such that

$$H(u^*,y^*) = \min_{(u,y)\in V} H(u,y).$$

Consider the following hypothesis on Σ.

(H_Σ) $\Sigma : Y \to W$ is weakly continuous and there exist $d_0, d_1 \geqslant 0$ such that

$$\|\Sigma(y)\|_W \leqslant d_0 + d_1\|y\|_Y, \qquad \forall y \in Y.$$

THEOREM 13. *Assume that hypotheses* $(H_A),(H_J),(H_M),(H_0),(H_B),(H_\Sigma)$ *hold. Then Problem 12 has a solution* $(\overline{u},\overline{y}) \in V$.

Proof. By applying Theorem 9, for every control $u \in U$, Problem 10 has a solution $y \in Sol(u)$. Let $\{(u_n,y_n)\} \subset V$ be a minimizing sequence. That is,

$$\lim_{n\to\infty} H(u_n,y_n) = \inf\{H(u,y) : (u,y) \in V\},$$

where $u_n \in U$ and $y_n \in Sol(u_n)$. Then we have

$$Ay_n + M^*\xi_n = f + Bu_n,$$

where $\xi_n \in \partial J(My_n)$. As n is large enough, we obtain

$$\varepsilon\|u_n\|^2 \leqslant \|\Sigma(y_n) - \Upsilon\|^2 + \varepsilon\|u_n\|^2 \leqslant \lim_{n\to\infty} H(u_n,y_n) + 1,$$

hence the sequence $\{u_n\}$ is bounded in the Banach space U. Since the space U is reflexive, there is a subsequence of $\{u_n\}$, denoted by $\{u_n\}$ again, such that $u_n \rightharpoonup \overline{u}$ for some $\overline{u} \in U$.

Consequently, we choose $\{y_n\}$ to be a subsequence of solutions of Problem 10 that corresponds to the subsequence of controls $\{u_n\}$. From (2) we know that $\{y_n\}$ remains bounded. Let $\{y_n\}$ be a subsequence converging weakly to $\overline{y}$ for some $\overline{y} \in Y$. From Theorem 9 we know that $\overline{y} \in Sol(\overline{u})$.

Finally, we have

$$\begin{aligned} H(\overline{u},\overline{y}) &= \|\Sigma(\overline{y}) - \Upsilon\|^2 + \varepsilon\|\overline{u}\|^2 \\ &\leqslant \liminf_{n\to\infty} \|\Sigma(y_n) - \Upsilon\|^2 + \liminf_{n\to\infty} \varepsilon\|u_n\|^2 \\ &\leqslant \liminf_{n\to\infty} H(u_n,y_n) \\ &= \inf\{H(u,y) : u \in U, y \in Sol(u)\}, \end{aligned}$$

which shows that $(\overline{u},\overline{y})$ is an optimal pair to Problem 12. This completes the proof. $\square$

4. Convergence of optimal pairs

Given mappings $\Sigma_n : Y \to W$ $(n \in \mathbb{N})$, we consider the following perturbed cost function:

$$H_n(u,y) := \|\Sigma_n(y) - \Upsilon\| + \varepsilon\|u\|^2,$$

where $\varepsilon > 0$. Consider the following perturbed subgradient inclusion:

PROBLEM 14. Find $y \in Y$ such that

$$A_n y + M^* \partial J_n(My) \ni f + B_n u.$$

Denote by $Sol_n(u)$ the set of all solutions of Problem 14 corresponding to the control u and let

$$V_n = \{(u,y) \in U \times Y | y \in Sol_n(u)\}.$$

We are interested in the convergence behavior of the following optimal problem.

PROBLEM 15. Find a pair $(u_n^*, y_n^*) \in V_n$ such that

$$H_n(u_n^*, y_n^*) = \min_{(u,y)\in V_n} H_n(u,y).$$

To obtain the result of this section, we need the following assumptions.

(H_{A_n}) $A_n : V \to V^*$ is weakly continuous and coercive with constant $\alpha_n > 0$, and there exist a constant $\varepsilon_{A_n} > 0$ and a continuous function $\tau_1 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|A_n y - Ay\| \leqslant \varepsilon_{A_n} \tau_1(\|y\|), \qquad \forall y \in Y.$$

(H_{J_n}) There exist a constant $\varepsilon_{J_n} > 0$ and a continuous function $\tau_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|J_n^\circ(z;w) - J^\circ(z;w)\| \leqslant \varepsilon_{J_n} \tau_2(\|z\|), \qquad \forall z, w \in Z.$$

(H_{B_n}) $B_n : Y \to V^*$ is weakly continuous and there exist a constant $\varepsilon_{B_n} > 0$ and a continuous function $\tau_3 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|B_n u - Bu\| \leqslant \varepsilon_{B_n} \tau_3(\|u\|), \qquad \forall u \in U.$$

(H_{Σ_n}) $\Sigma_n : V \to V^*$ is weakly continuous and there exist a constant $\varepsilon_{\Sigma_n} > 0$ and a continuous function $\tau_4 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|\Sigma_n(y) - \Sigma(y)\| \leqslant \varepsilon_{\Sigma_n} \tau_4(\|y\|), \qquad \forall y \in Y.$$

(H_{0_n}) There exists a constant $m_0 > 0$ such that $\alpha_n \geqslant m_0 > c_1\|M\|^2$ and

$$\varepsilon_{A_n},\ \varepsilon_{J_n},\ \varepsilon_{B_n},\ \varepsilon_{\Sigma_n} \to 0 \qquad \text{as } n \to \infty.$$

The following theorem is the main result of this section.

THEOREM 16. *Assume that hypotheses* $(H_A), (H_J), (H_M), (H_B), (H_\Sigma)$ *and* (H_{A_n}), (H_{J_n}), (H_{B_n}), (H_{Σ_n}), (H_{0_n}) *hold. Then for every* $n \in \mathbb{N}$, *Problem* 15 *has a solution* $(y_n, u_n) \in V_n$, *and there exists a subsequence of* $\{(u_n, y_n)\}$ *that converges weakly to a solution of Problem* 12.

Proof. By applying Theorem 13, for every $n \in \mathbb{N}$, Problem 15 has a solution $(u_n^*, y_n^*) \in V$. That is,

$$H_n(u_n^*, y_n^*) = \inf_{(u,y) \in V} H_n(u, y).$$

where $u_n^* \in U$ and $y_n^* \in Sol_n(u_n^*)$. Consequently, we have

$$A_n y_n^* + M^* \xi_n^* = f + B_n u_n^*, \tag{4}$$

where $\xi_n^* \in \partial J_n(My_n^*)$.

We claim that the sequence $\{u_n^*\}$ is bounded in the Banach space U. Arguing by contradiction, assume that $\{u_n^*\}$ is not bounded in U. Then, passing to a subsequence still denoted $\{u_n^*\}$, we have

$$\|u_n^*\|_U \to +\infty \qquad \text{as } n \to +\infty.$$

Since $H_n(u_n^*, y_n^*) \geqslant \varepsilon \|u_n^*\|_U$, by passing to the limit as $n \to +\infty$ we deduce that

$$\lim H_n(u_n^*, y_n^*) \to +\infty \text{ as } n \to +\infty. \tag{5}$$

On the other hand, since (u_n^*, y_n^*) is a solution to Problem 15, we have

$$H_n(u_n^*, y_n^*) \leqslant H_n(u_n, y_n), \qquad \forall u_n \in U, \;\; y_n \in Sol_n(u_n).$$

We now fix an element $u^0 \in U$ and let $y_n^0 \in Sol_n(u^0)$. Then

$$H_n(u_n^*, q_n^*) \leqslant H_n(u^0, y_n^0) = \|\Sigma_n(y_n^0) - \Upsilon\| + \varepsilon \|u^0\|^2. \tag{6}$$

By (H_{Σ_n}) we deduce that there exists $C > 0$ such that

$$\|\Sigma_n(y_n^0) - \Upsilon\|^2 + \varepsilon \|u^0\|^2 \leqslant C, \qquad \text{for sufficient large } n \in \mathbb{N}. \tag{7}$$

Relations (5), (6) and (7) lead to a contradiction, which concludes the claim.

Since the space U is reflexive, there is a subsequence of $\{u_n^*\}$, denoted by $\{u_n^*\}$ again, such that $u_n^* \rightharpoonup \overline{u}$ for some $\overline{u} \in U$.

Next, from (2) we know that $\{y_n^*\}$ remains bounded. Let $\{y_n^*\}$ be a subsequence converging weakly to $\overline{y}$ for some $\overline{y} \in Y$. We will show that $\overline{y} \in Sol(\overline{u})$.

Since

$$A_n y_n^* - A\overline{y} = A_n y_n^* - Ay_n^* + Ay_n^* - A\overline{y}$$

and

$$B_n u_n^* - B\overline{u} = B_n u_n^* - Bu_n^* + Bu_n - B\overline{u},$$

Hypotheses $(H_A), (H_B), (H_{A_n}), (H_{B_n}), (H_{0_n})$ imply that

$$A_n y_n^* \rightharpoonup A\overline{y}, \qquad B_n u_n^* \rightharpoonup B\overline{u}.$$

Moreover, we have

$$\begin{aligned}
\langle -A\overline{y} + B\overline{u} + f, y\rangle_Y &\leqslant \limsup_{n\to\infty} \langle -A_n y_n^* + B_n u_n^* + f, y\rangle_Y \\
&= \limsup_{n\to\infty} \langle M^* \xi_n^*, y\rangle_Y \\
&= \limsup_{n\to\infty} \langle \xi_n^*, My\rangle_Z \\
&\leqslant \limsup_{n\to\infty} J_n^\circ(My_n^*; My) \\
&\leqslant \limsup_{n\to\infty} (J_n^\circ(My_n^*; My) - J^\circ(My_n^*; My) + J^\circ(My_n^*; My)) \\
&\leqslant \limsup_{n\to\infty} \varepsilon_{J_n} \tau_2(\|My_n^*\|) + \limsup_{n\to\infty} J^\circ(My_n^*; My) \\
&\leqslant J^\circ(M\overline{y}; My).
\end{aligned}$$

Then there exits $\overline{\xi} \in \partial J(M\overline{y})$ such that

$$-A\overline{y} + B\overline{u} + f = M^* \overline{\xi},$$

i.e.,

$$A\overline{y} + M^* \overline{\xi} = B\overline{u} + f.$$

Therefore, $\overline{y} \in Sol(\overline{u})$.

Finally, we show that $(\overline{u}, \overline{y})$ is an optimal pair to Problem 12. From Theorem 13 we know that Problem 12 has a solution. Let (u', y') be a solution of Problem 12. We can construct a sequence $\{y_n'\}$ with $y_n' \in Sol_n(u')$ such that $y_n' \rightharpoonup y'$. Then, by (H_{Σ_n}) we have

$$\begin{aligned}
H(\overline{u}, \overline{y}) &= \|\Sigma(\overline{y}) - \Upsilon\| + \varepsilon \|\overline{u}\|^2 \\
&\leqslant \liminf_{n\to\infty} \|\Sigma(y_n^*) - \Upsilon\| + \liminf_{n\to\infty} \varepsilon \|u_n^*\|^2 \\
&= \liminf_{n\to\infty} \|\Sigma(y_n^*) - \Sigma_n(y_n^*) + \Sigma_n(y_n^*) - \Upsilon\| + \liminf_{n\to\infty} \varepsilon \|u_n^*\|^2 \\
&\leqslant \liminf_{n\to\infty} (\|\Sigma(y_n^*) - \Sigma_n(y_n^*)\| + \|\Sigma_n(y_n^*) - \Upsilon\|) + \liminf_{n\to\infty} \varepsilon \|u_n^*\|^2) \\
&\leqslant \lim_{n\to\infty} (\varepsilon_{\Sigma_n} \tau_4(\|y_n^*\|))^2 + \liminf_{n\to\infty} (\|\Sigma_n(y_n^*) - \Upsilon\| + \varepsilon \|u_n^*\|^2) \\
&\leqslant \liminf_{n\to\infty} (\|\Sigma_n(y_n') - \Upsilon\| + \varepsilon \|u'\|^2) \\
&\leqslant \|\Sigma(y') - \Upsilon\| + \varepsilon \|u'\|^2 = H(u', y'),
\end{aligned}$$

which shows that $(\overline{y}, \overline{u})$ is an optimal pair to Problem 12. This completes the proof. □

5. Stationary Navier-Stokes problem

In this section we provide an example to illustrate our main theorem and show the existence of solutions to elliptic hemivariational inequality which arise in the study of Navier-Stokes problems with nonmonotone and multivalued frictional boundary condition.

We introduce the physical setting of the problem and provide both classical and weak formulations of the stationary Navier-Stokes problem. We denote by $\mathbb{S}^d$ the space of $d \times d$ symmetric matrices. The canonical inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$y \cdot v = y_i v_i, \quad \|y\| = (y \cdot y)^{1/2} \quad \text{for all } y = (y_i),\ v = (v_i) \in \mathbb{R}^d,$$

$$C : D = C_{ij} D_{ij}, \quad \|C\| = (C : C)^{1/2} \quad \text{for all } C = (C_{ij}),\ D = (D_{ij}) \in \mathbb{S}^d.$$

Here and in the sequel, the summation over two repeated indices is applied.

Let Ω be a bounded open and connected domain in $\mathbb{R}^d$ with $d = 2,\ 3$. The boundary $\Gamma = \partial\Omega$ is supposed to be Lipschitz continuous and it is composed of two measurable parts $\overline{\Gamma}_D$ and $\overline{\Gamma}_C$, with disjoint relatively open sets Γ_D and Γ_C such that $meas(\Gamma_D) > 0$. We denote by $\nu = (\nu_i)$ the unit outward normal vector on Γ and by $x = (x_i) \in \overline{\Omega}$ the position vector. We are concerned with the following statinary problem which classical formulation reads as follows.

PROBLEM 17. Find a flow velocity field $y = y(x)$ and a pressure $p = p(x)$ such that

$$-\nu_0 \Delta y + (y \cdot \nabla) y + \nabla p = f + u \quad \text{in } \Omega, \tag{8}$$

$$\operatorname{div} y = 0 \quad \text{in } \Omega, \tag{9}$$

$$y = 0 \quad \text{on } \Gamma_D, \tag{10}$$

$$u_\nu = 0 \quad \text{on } \Gamma_C, \tag{11}$$

$$-S_\tau \in \partial j(y_\tau) \quad \text{on } \Gamma_C. \tag{12}$$

We briefly comment on the equations and conditions in Problem 17. The system describes the non-stationary flow of incompressible viscous liquid occupying the volume Ω subjected to a given external volume forces of density $f = f(x)$ and a control force $u = u(x)$. Here $\nu_0 > 0$ denotes a viscosity constant of the fluid, $\nu_0 = 1/\mathrm{Re}$, where Re is the Reynolds number. Equation (8) is the conservation law, where the expression $(y \cdot \nabla) v = \left(\sum_{j=1}^{d} y_j \frac{\partial v_i}{\partial x_j} \right)_{i=1}^{d}$ denotes the nonlinear convective term. The solenoidal (divergence free) condition (9) states that the motion of the fluid is incompressible. The total stress tensor in the fluid is given by $\sigma = -pI + S$ in Q, where I denotes the identity matrix and $S : \Omega \to \mathbb{S}^d$ is the extra (viscous) part of the stress tensor. The symmetric part of the velocity gradient $D : \Omega \to \mathbb{S}^d$ is given by $D(y) = \frac{1}{2}(\nabla y + \nabla y^\top)$. We assume that the extra stress tensor S is related with the symmetric part of the velocity gradient D by means of the constitutive law $S = 2\nu_0 D(y)$ in Q.

As for the boundary conditions, we consider on Γ_D the adherence boundary condition boundary conditions (10) (since the fluid is viscous). On the part Γ_C, we decompose the velocity vector into the normal and tangential parts. We denote by u_ν and y_τ the normal and the tangential components of y on the boundary Γ_C, i.e., $u_\nu = y \cdot \nu$ and $y_\tau = y - u_\nu \nu$. Similarly, for an extra stress tensor field S, we define its normal and tangential components by $S_\nu = (S\nu)\cdot\nu$ and $S_\tau = S\nu - S_\nu \nu$, respectively. We assume that there is no flux condition through Γ_C, so that the normal component of the velocity vanishes on this part of the boundary, cf. (11). The tangential components of the stress tensor S_τ and the velocity y_τ are assumed to satisfy the multivalued friction law (12), where ∂j denotes the Clarke subgradient of a locally Lipschitz function $j\colon \mathbb{R}^d \to \mathbb{R}$. The boundary condition (12) is called the boundary conditions of friction type. We refer to [8, 9, 13, 22, 35] and the references therein for more details on stationary Navier-Stokes problems.

Next, we provide the weak formulation of Problem 17. To this end, we introduce the following spaces

$$
\begin{aligned}
\widetilde{V} &= \{ v \in \mathscr{C}^\infty(\overline{\Omega};\mathbb{R}^d) \mid \operatorname{div} v = 0 \text{ in } \Omega,\ v = 0 \text{ on } \Gamma_D,\ v_\nu = 0 \text{ on } \Gamma_C \},\\
V &= \text{ closure of } \widetilde{V} \text{ in } H^1(\Omega;\mathbb{R}^d)
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
\widetilde{H} &= \{ v \in \mathscr{C}^\infty(\overline{\Omega};\mathbb{R}^d) \mid \operatorname{div} v = 0 \text{ in } \Omega,\ v_\nu = 0 \text{ on } \Gamma_C \},\\
H &= \text{ closure of } \widetilde{H} \text{ in } L^2(\Omega;\mathbb{R}^d).
\end{aligned} \tag{14}
$$

The space V is equipped with the norm $\|v\| = \|v\|_{H^1(\Omega;\mathbb{R}^d)}$ for $v \in V$. On V we introduce also the norm given by $\|v\|_V = \|D(v)\|_{L^2(\Omega;\mathbb{S}^d)}$ for $v \in V$. From the Korn inequality $c_K\|v\|_{H^1(\Omega;\mathbb{R}^d)} \leqslant \|D(v)\|_{L^2(\Omega;\mathbb{S}^d)}$ for $v \in V$ with $c_K > 0$ (cf. e.g. [9, Theorem 4]), it follows that $\|\cdot\|_{H^1(\Omega;\mathbb{R}^d)}$ and $\|\cdot\|_V$ are the equivalent norms on V. Moreover, V is a reflexive separable Banach space, H is a separable Hilbert space, the embedding $V \subset H$ is continuous, compact and V is dense in H. This means that (V,H,V^*) forms an evolution triple of spaces.

Next, we introduce the space $Z = L^2(\Gamma_C;\mathbb{R}^d)$ and the continuous and compact trace operator $\gamma\colon V \to Z$. Its norm is denoted by $\|\gamma\| = \|\gamma\|_{\mathscr{L}(V,U)}$.

In the study of Problem 17, we will assume the following hypotheses.

$$
\begin{cases}
j\colon \mathbb{R}^d \to \mathbb{R} \text{ is such that}\\
\text{(a) } j \text{ is locally Lipschitz.}\\
\text{(b) there exist } \overline{c}_0, \overline{c}_1 > 0 \text{ such that}\\
\qquad \|\partial j(\xi)\|_{\mathbb{R}^d} \leqslant \overline{c}_0 + \overline{c}_1 \|\xi\|_{\mathbb{R}^d} \ \text{ for all } \ \xi \in \mathbb{R}^d.
\end{cases} \tag{15}
$$

Using a standard procedure (cf. e.g. [8, 9, 10, 11]), we obtain the following variational formulation of Problem 17 which takes the form of a hemivariational inequality. Remark that, due to the incompressibility condition, the pressure is not involved in the variational formulation.

PROBLEM 18. *Find a velocity field* $y \in V$ *such that*

$$\nu_0 \int_\Omega \nabla y : \nabla v\, dx + \int_\Omega ((y \cdot \nabla) y) \cdot v\, dx + \int_{\Gamma_C} j^0(y_\tau; v_\tau)\, d\Gamma \geqslant \int_\Omega f \cdot v\, dx + \int_\Omega u \cdot v\, dx \text{ for all } v \in V.$$

We have the following existence result.

THEOREM 19. *Assume that* (15) *holds and* $f, u \in V^*$. *Then Problem* 18 *has a solution* $y \in V$ *if* $\nu_0 > \sqrt{2}\bar{c}_1 \|\gamma\|^2$.

Proof. We apply Theorem 9 to show existence of solution to an elliptic inclusion associated with the hemivariational inequality in Problem (18). Let us define the operator $A\colon V \to V^*$ and the functional $J\colon Z \to \mathbb{R}$ by

$$\langle Ay, v\rangle_V = a(y,v) + b(y,y,v) \text{ for } y, v \in V, \tag{16}$$

$$J(v) = \int_{\Gamma_C} j(v_\tau(x))\, d\Gamma \text{ for } v \in U, \tag{17}$$

where

$$a\colon V \times V \to \mathbb{R}, \qquad a(y,v) = \nu_0 \int_\Omega \nabla y : \nabla v\, dx,$$

$$b\colon V \times V \times V \to \mathbb{R}, \qquad b(y,v,w) = \int_\Omega ((y \cdot \nabla) v) \cdot w\, dx$$

for y, v, $w \in V$. We consider an auxiliary evolutionary inclusion. Find $y \in \mathscr{W}$ such that

$$Ay + \gamma^* \partial J(\gamma y) \ni f + u. \tag{18}$$

We will show that (18) has a solution. To this end, we will check the hypotheses $(H_A), (H_J), (H_M)$ and apply Theorem 9.

First, we check hypothesis (H_J). Using hypothesis (15), by [27, Corollary 4.15 (ii)], it is clear that J is locally Lipschitz. From part (v) of the same corollary, we infer that the inequality

$$\|\partial J(z)\|_{Z^*} \leqslant c_0 + c_1 \|z\|_Z$$

holds for all $z \in Z$ with $c_0 = \sqrt{2 meas(\Gamma_C)}\bar{c}_0, c_1 = \sqrt{2}\bar{c}_1$. This implies (H_J).

Condition (H_M) is obvious since γ is linear, continuous and compact.

Subsequently, we will verify condition (H_A). From [12, Proposition 2.2], it is obvious that a linear continuous operator on a reflexive Banach space is weakly continuous. This means that the operator $A_1\colon V \to V^*$ defined by $\langle A_1 y, v\rangle_V = a(y,v)$ for y, $v \in V$ is weakly continuous. The trilinear form b generates the operator $B_1\colon V \to V^*$

defined by $\langle B_1 y, v\rangle_V = b(y,y,v)$ for y, $v \in V$. From [12, Proposition 2.6], we deduce that B_1 is well defined and weakly continuous. By [12, Proposition 2.1], it follows that the sum $A = A_1 + B_1$ of two weakly continuous operators is a weakly continuous operator.

Now, we pass to boundedness of the operator $A\colon V \to V^*$. From e.g. [8, Lemma 13], it follows that

$$\langle B_1 u, v\rangle_V \leqslant c\|u\|_V\|u\|_V\|v\|_V \quad \text{for all } u,v \in V \text{ with } c > 0. \tag{19}$$

We observe that the continuous embedding

$$H^1(\Omega;\mathbb{R}^d) \subset L^q(\Omega;\mathbb{R}^d) \text{ for } \frac{1}{q} \geqslant \frac{1}{2} - \frac{1}{d},$$

with $q = 4$ holds for $d \leqslant 4$. Hence and by the Hölder inequality and (19), we obtain

$$\langle Au, v\rangle_V \leqslant \nu_0\|u\|_V\|v\|_V + \|u\|_V\|u\|_{L^4(\Omega;\mathbb{R}^d)}\|v\|_{L^4(\Omega;\mathbb{R}^d)} \leqslant \nu_0\|u\|_V\|v\|_V + c\,\|u\|_V^2\|v\|_V$$

and hence

$$\|Au\|_{V^*} = \sup_{\|v\|\leqslant 1} |\langle Au, v\rangle_V| \leqslant \beta(\|u\|_V) \text{ for all } u \in V,$$

where $\beta(r) = \max\{\nu_0, c\}(r + r^2)$ for $r \geqslant 0$. Hence A is bounded.

Now, we pass to coercive property of the operator $A\colon V \to V^*$. Note that

$$\langle B_1 y, y\rangle_V = \int_\Omega ((y\cdot\nabla)y)\cdot y\,dx = \int_\Omega \sum_{i,j=1}^d y_i \frac{\partial y_j}{\partial x_i} y_j\,dx = \int_\Omega \sum_{i,j=1}^d y_i \frac{\partial}{\partial x_i}\frac{y_j^2}{2}\,dx$$

$$= -\frac{1}{2}\int_\Omega \operatorname{div} y \sum_{j=1}^d y_j^2\,dx + \frac{1}{2}\int_{\Gamma_C} u_\nu \sum_{j=1}^d y_j^2\,dx = 0 \tag{20}$$

for all $y \in \widetilde{V}$. Then, exploiting density of $\widetilde{V}$ in V, and we get (20) for all $y \in Y$. Therefore, we have

$$\langle Ay, y\rangle_V = \langle A_1 y, y\rangle_V + \langle B_1 y, y\rangle_V \geqslant \alpha\|y\|_V^2 \text{ for all } y \in V$$

with $\alpha = \nu_0$. Hence (H_A) holds.

We have verified all hypotheses of Theorem 9. Therefore, from this theorem, we deduce that (18) has a solution $y \in \mathscr{W}$. Finally, we note that every solution to (18) is a solution to Problem 18. Indeed, let $y \in \mathscr{W}$ solve (18) and $v \in V$. We obtain

$$\langle Ay + \gamma^*\xi, v\rangle_V = \langle f + u, v\rangle_V \tag{21}$$

with $\xi \in \partial J(\gamma y)$. Using the definition of the Clarke subgradient and [27, Proposition 3.47 (iv)], we have the following inequality

$$\langle \gamma^*\xi, v\rangle_V = \langle \xi, \gamma v\rangle_U \leqslant J^0(\gamma y;\gamma v) \leqslant \int_{\Gamma_C} j^0(y_\tau; v_\tau)\,d\Gamma. \tag{22}$$

Hence, combining (21) and (22), we immediately see that $y \in Y$ is a solution to Problem 18. $\square$

Finally, we consider two optimal control problems for Problem 18.

Let $h_1 \colon V^* \times V \to \mathbb{R}$, $h_2 \colon V^* \times L^2(\Gamma_C;\mathbb{R}^d) \to \mathbb{R}$ be of the forms

$$h_1(u,y) = \|y - y_0\|_V + \varepsilon \|u\|_{V^*}^2,$$

and

$$h_2(u,y) = \|\gamma y - y_1\|_{L^2(\Gamma_C;\mathbb{R}^d)} + \varepsilon \|u\|_{V^*}^2,$$

respectively, where $y_0 \in V, y_1 \in L^2(\Gamma_C;\mathbb{R}^d)$ are fixed elements and $\varepsilon > 0$.

Denote by $S(u)$ the set of all solutions of Problem 18 corresponding to the control u and let

$$\mathscr{V} = \{(u,y) \in V^* \times V | y \in Sol(u)\}.$$

Consider the following two optimal problems.

PROBLEM 20. Find a pair $(u^*, y^*) \in \mathscr{V}$ such that

$$h_1(u^*, y^*) = \min_{(u,y)\in\mathscr{V}} h_1(u,y).$$

PROBLEM 21. Find a pair $(u^*, y^*) \in \mathscr{V}$ such that

$$h_2(u^*, y^*) = \min_{(u,y)\in\mathscr{V}} h_2(u,y).$$

We have the following results for above optimal control problems.

THEOREM 22. *Assume that all the hypotheses of Theorem 19 are satisfied. Then Problem 20 and Problem 21 are solvable.*

Note that h_1 and h_2 are weakly continuous since the identity operator $i : V \to V$ and operator $\gamma \colon V \to L^2(\Gamma_C;\mathbb{R}^d)$ are weakly continuous. Therefore, by applying Theorem 13 we can obtain the existence of optimal solutions to the optimal control problem for Problem 18 with the functionals h_1 and h_2.

Finally, we can also the convergence of optimal solutions to corresponding perturbed problems of Problem 20 and Problem 21, which is analogous to Theorem 16.

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