Journal of Mathematical Inequalities
Volume 15, Number 2 (2021), 615–628
doi:10.7153/jmi-2021-15-45

MONOTONICITY, CONVEXITY AND BOUNDS INVOLVING THE BETA AND RAMANUJAN R–FUNCTIONS

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(*Communicated by G. Nemes*)

Abstract. In the article, we provide several new asymptotical sharp bounds for the functions involving the Beta function and Ramanujan R-functions via the monotonicity and convexity properties of certain combinations defined in terms of polynomials, Beta and Ramanujan R-functions.

1. Introduction

Let $x,y>0$. Then the Ramanujan R-function $R(x,y)$ and Beta function $B(x,y)$ are defined by

$$R(x,y)=-2\gamma-\psi(x)-\psi(y)$$

and

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

respectively, where $\gamma=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)=0.5772\cdots$ is the Euler-Mascheroni constant, and

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt,\quad \psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}.$$

In particular, if $y=1-x$, then we denote

$$R(x)=R(x,1-x)=-2\gamma-\psi(x)-\psi(1-x) \tag{1.1}$$

and

$$B(x)=B(x,1-x)=\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin(\pi x)}. \tag{1.2}$$

From (1.1) and (1.2) we clearly see that both the functions $R(x)$ and $B(x)$ are symmetry with respect to $x=1/2$. Therefore, we only need to assume that $x\in(0,1/2]$ in what follows. It is easy to know that $R(1/2)=\log 16$ by (1.1).

Mathematics subject classification (2020): 33C05, 26D20.
Keywords and phrases: Ramanujan R-function, beta function, monotonicity, convexity.
This research was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11701176, 11626101, 11601485, 11401531), Natural Science Foundation of the Jiangsu Higher Education Institutions of China (No. 17KJD110004) and The "Blue Projec" of Universities in Jiangsu province.
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Let $a,b,c\in\mathbb{R}$ with $c\neq 0,-1,-2,\cdots$. Then the Gaussian hypergeometric function $F(a,b;c;x)$ [1, 2, 6, 7, 8, 9] is defined by

$$F(a,b;c;x)={}_2F_1(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!}\quad(-1<x<1),\tag{1.3}$$

where (a,n) denotes the shifted factorial function $(a,n)\equiv a(a+1)\cdots(a+n-1)$ for $n\in\mathbb{N}$, and $(a,0)=1$ for $a\neq 0$. It is well known that $F(a,b;c;x)$ has wide applications in mathematics and physics, and many elementary and special functions are the particular or limiting cases of the Gaussian hypergeometric function. In particular, if $c=a+b$, then $F(a,b;c;x)$ is said to be zero-balanced. As the special case of the Gaussian hypergeometric function, the generalized elliptic integral $\mathscr{K}_a(r)$ [3, 5] of the first kind can be expressed by

$$\mathscr{K}_a(r)=\frac{\pi}{2}F\left(a,1-a;1;r^2\right),\quad \mathscr{K}_a(0^+)=\frac{\pi}{2},\quad \mathscr{K}_a(1^-)=\infty\tag{1.4}$$

for $r\in(0,1)$ and $a\in(0,1/2]$.

The Ramanujan R-function and Beta function are closely related to the Guassian hypergeometric function $F(a,b;c;x)$ and the generalized elliptic integral $\mathscr{K}_a(r)$ of the first kind. For example, $F(a,b;a+b;x)$ satisfies the asymptotic formula [4]

$$B(a,b)F(a,b;a+b;x)+\log(1-x)=R(a,b)+O((1-x)\log(1-x))\quad(x\to 1),$$

and $\mathscr{K}_a(r)$ has the sharp asymptotical inequalities [20]

$$\pi\left\{1+\left[\frac{B(x)}{R(x)}-1\right](1-r^2)\right\}<\frac{B(x)\mathscr{K}_a(r)}{\log(e^{R(x)/2}/\sqrt{1-r^2})}<\pi\left[1+a(1-a)(1-r^2)\right]$$

and

$$\frac{\pi}{R(x)+[B(x)-R(x)]\,r^2}<\frac{\mathscr{K}_a(r)}{\log(e^{R(x)/2}/\sqrt{1-r^2})}<\frac{\pi}{B(a)\left[1-a(1-a)+a(1-a)r^2\right]}$$

for all $a\in(0,1/2]$ and $r\in(0,1)$. More properties for $B(x)$ and $R(x)$ can be found in the literature [2, 4, 10, 13, 14, 15, 16, 17, 18, 20, 23], in which they used to study the generalized η_k-distortion function $\eta_k^a(t)$ and the generalized λ-distortion function $\lambda(a,K)=\eta_K^a(1)$.

Recently, the properties and bounds for $B(x)$ and $R(x)$ have attracted the attention of many researchers [17, 23]. Qiu, Ma, and Huang [18] found the power series expansions of the function $R(x)-B(x)$ at $x=0$ and $x=1/2$, and proved that

$$2(2c_{2n}+\alpha_{2n-1})x^{2n+1}\leqslant B(x)-R(x)+2\sum_{k=1}^{n}c_kx^k\leqslant(2c_{2n}+\alpha_{2n-1})x^{2n}$$

and

$$(2c_{2n+1}+\alpha_{2n})x^{2n+1}\leqslant B(x)-R(x)+2\sum_{k=1}^{2n+1}c_kx^k\leqslant 2(2c_{2n+1}+\alpha_{2n})x^{2(n+1)},$$

where

$$c_n = \left\{(-1)^n + [1+(-1)^{n+1}]2^{-n-1}\right\}\zeta(n+1), \quad \alpha_n = 2^{n+1}(\pi - \log 16 + 2\sum_{k=1}^{n} 2^{-k}c_k)$$

and

$$\zeta(s) = \sum_{k=1}^{\infty}\frac{1}{k^s} \quad (Re\ s > 1) \tag{1.5}$$

is the Riemann zeta function.

In [13], Huang, Qiu, and Ma discussed the monotonicity and convexity properties of the functions $x(1-x)B(x)$ and $R(x)-[1-x(1-x)]B(x)$, and discovered new bounds for the complete integral $\mathscr{K}_a(r)$ of the first kind.

The main purpose of the article is to provide new monotonicity and convexity properties involving the Ramanujan R-function $R(x)$ and Beta function $B(x)$.

2. Lemmas and definition

In order to prove our main results, we need two lemmas and one definition which we present in this section.

Let $n \in \mathbb{N}$. Then the the special sums of reciprocal powers $\lambda(n+1)$, $\eta(n)$ and $\beta(n)$ [1] are defined by

$$\lambda(n+1) = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^{n+1}}, \quad \eta(n) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k^n}, \quad \beta(n) = \sum_{k=0}^{\infty}(-1)^k\frac{1}{(2k+1)^n}. \tag{2.1}$$

It follows from [1, 23.2.20] that

$$\lambda(n+1) = \left(1-2^{-n-1}\right)\zeta(n+1), \quad \eta(n) = \left(1-2^{1-n}\right)\zeta(n). \tag{2.2}$$

LEMMA 2.1. *The following two conclusions can be found in the literature* [13]*:*
(1) If $x \in (0,1/2]$*, then one has*

$$B(x) = \frac{1}{x} + \sum_{n=1}^{\infty}[1-(-1)^n]\eta(n+1)x^n = 4\sum_{n=0}^{\infty}\beta(2n+1)(1-2x)^{2n} \tag{2.3}$$

and

$$R(x) = \frac{1}{x} + \sum_{n=1}^{\infty}[1+(-1)^n]\zeta(n+1)x^n = \log 16 + 4\sum_{n=1}^{\infty}\lambda(2n+1)(1-2x)^{2n}. \tag{2.4}$$

(2) The function $\lambda(n)$ *is strictly decreasing for* $n \in \mathbb{N}\setminus\{1\}$ *with* $\lambda(2) = \pi^2/8$ *and* $\lambda(n) \to 1$ *as* $n \to +\infty$*, and the function* $\beta(n)$ *is strictly increasing for* $n \in \mathbb{N}$ *with* $\beta(1) = \pi/4$ *and* $\beta(n) \to 1$ *as* $n \to +\infty$.

LEMMA 2.2. (See [2]) *Let* $-\infty < a < b < \infty$, $f, g : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *such that* $g'(x) \neq 0$ *on* (a,b). *If* $f'(x)/g'(x)$ *is increasing (decreasing) on* (a,b), *then so are the functions*

$$[f(x) - f(a)]/[g(x) - g(a)] \quad \textit{and} \quad [f(x) - f(b)]/[g(x) - g(b)].$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

DEFINITION 2.1. (See [11], [12], [22]) A real-valued function f is said to be strictly completely monotonic on an interval $I \subseteq \mathbb{R}$ if $(-1)^n f^{(n)}(x) > 0$ for all $x \in I$ and $n = 0, 1, 2, 3 \cdots$. If $(-1)^n f^{(n)}(x) \geqslant 0$ for all $x \in I$ and $n = 0, 1, 2, 3 \cdots$, then f is said to be completely monotonic on I.

3. Main results

For the convenience of narration, we denote

$$f(x) = x(1-x)B(x)$$

and

$$g(x) = R(x) - [1 - x(1-x)]B(x)$$

throughout this section.

THEOREM 3.1. *Both the functions* $f(x)$ *and* $g(x)$ *are completely monotonic on* $(0, 1/2]$.

Proof. It follows from (2.3) that

$$\begin{aligned} f(x) &= x(1-x)B(x) \\ &= \left[1 - (1-2x)^2\right] \sum_{n=0}^{\infty} \beta(2n+1)(1-2x)^{2n} \\ &= \frac{\pi}{4} + \sum_{n=1}^{\infty} [\beta(2n+1) - \beta(2n-1)](1-2x)^{2n} \end{aligned}$$

and

$$\beta(2n+1) - \beta(2n-1) > 0.$$

Elaborated computations lead to

$$f^{(k)}(x) = \sum_{n=[k/2]}^{\infty} (-1)^k \frac{2^k (2n)!}{(2n-k)!} [\beta(2n+1) - \beta(2n-1)](1-2x)^{2n-k}.$$

Therefore, $(-1)^k f^{(k)}(x) \geqslant 0$ for all $x \in (0, 1/2]$ and $k = 0, 1, 2, 3 \cdots$ and $f(x)$ is completely monotonic on $(0, 1/2]$. It is easy to verify that $f(x)$ is decreasing, $f^{(2n)}(x)$ is decreasing and $f^{(2n+1)}(x)$ is strictly increasing on $(0, 1/2]$ for $n \in \mathbb{N}$.

It follows from 2.1(2) that

$$\begin{aligned} g(x) &= R(x) - [1-x(1-x)]B(x) \\ &= \log 16 + 4\sum_{n=1}^{\infty}\lambda(2n+1)(1-2x)^{2n} - 4[1-x(1-x)]\sum_{n=0}^{\infty}\beta(2n+1)(1-2x)^{2n} \\ &= \log 16 + 4\sum_{n=1}^{\infty}\lambda(2n+1)(1-2x)^{2n} - [(1-2x)^2+3]\sum_{n=0}^{\infty}\beta(2n+1)(1-2x)^{2n} \\ &= \log 16 - \frac{3\pi}{4} + \sum_{n=1}^{\infty}[4\lambda(2n+1)-\beta(2n-1)-3\beta(2n+1)](1-2x)^{2n}, \end{aligned} \tag{3.1}$$

$$4\lambda(2n+1)-\beta(2n-1)-3\beta(2n+1) > 4[\lambda(2n+1)-\beta(2n-1)] > 0$$

and

$$g^{(k)}(x) = \sum_{n=[k/2]}^{\infty}(-1)^k\frac{2^k(2n)!}{(2n-k)!}[4\lambda(2n+1)-\beta(2n-1)-3\beta(2n+1)](1-2x)^{2n-k}.$$

Therefore, $(-1)^k g^{(k)}(x) \geqslant 0$ for all $x \in (0,1/2]$ and $k = 0,1,2,3\cdots$, and $g(x)$ is completely monotonic on $(0,1/2]$. It is easy to check that $g(x)$ is decreasing, $g^{(2n)}(x)$ is decreasing and $g^{(2n+1)}(x)$ is strictly increasing on $(0,1/2]$ for $n \in \mathbb{N}$. $\square$

Let

$$\begin{cases} A_1(1) = -1, A_1(2) = 2\eta(2), \\ A_1(2k-1) = -2\eta(2k-2), \quad A_1(2k) = 2\eta(2k) & (k \geqslant 2), \\ A_2(0) = \beta(1) = \frac{\pi}{4}, \quad A_2(k) = \beta(2k+1)-\beta(2k-1) & (k \geqslant 1). \end{cases} \tag{3.2}$$

THEOREM 3.2. *Let $n \in \mathbb{N}$. Then the following statements are true:*
(1) The function $H_n^1(x)$ defined by

$$H_n^1(x) = \frac{f(x) - P_n^1(x)}{x^{2n+1}}$$

with $P_n^1(x) = 1 + \sum_{k=1}^{2n} A_1(k)x^k$ is strictly increasing and concave from $(0,1/2]$ onto $(-2\eta(2n), H_n^1(1/2)]$. In particular, the double inequality

$$\begin{aligned} 2\left[H_n^1\left(\frac{1}{2}\right) + 2\eta(2n)\right]x^{2n+2} &\leqslant x(1-x)B(x) - P_n^1(x) + 2\eta(2n)x^{2n+1} \\ &\leqslant \left[H_n^1\left(\frac{1}{2}\right) + 2\eta(2n)\right]x^{2n+1} \end{aligned} \tag{3.3}$$

holds for all $n \in \mathbb{N}$ and $x \in (0,1/2]$, and each inequality of (3.3) becomes equality if and only if $x = 1/2$.

(2) The function $H_n^2(x)$ defined by

$$H_n^2(x) = \frac{f(x) - P_n^2(x)}{x^{2n}}$$

with $P_n^2(x) = 1 + \sum_{k=1}^{2n-1} A_1(k)x^k$ is strictly decreasing and convex from $(0,1/2]$ onto $[H_n^2(1/2), 2\eta(2n))$. In particular, the two-sided inequality

$$\begin{aligned}\left[H_n^2\left(\frac{1}{2}\right) - 2\eta(2n)\right]x^{2n} &\leqslant x(1-x)B(x) - P_n^2(x) - 2\eta(2n)x^{2n} \\ &\leqslant 2\left[H_n^2\left(\frac{1}{2}\right) - 2\eta(2n)\right]x^{2n+1} \qquad (3.4)\end{aligned}$$

takes place for all $n \in \mathbb{N}$ and $x \in (0,1/2]$, and each inequality of (3.4) reduces to equality if and only if $x = 1/2$.

(3) The function $I_n(x)$ defined by

$$I_n(x) = \frac{f(x) - P_n^3(x)}{(1-2x)^{2n+2}},$$

with $P_n^3(x) = \sum_{k=0}^{n} A_2(k)(1-2x)^{2k}$ is strictly decreasing and convex from $(0,1/2)$ onto $(A_2(n+1), I_n(0^+))$. In particular, the double inequality

$$\begin{aligned}0 &\leqslant x(1-x)B(x) - P_n^3(x) - A_2(n+1)(1-2x)^{2n+2} \\ &\leqslant \left(I_n(0^+) - A_2(n+1)\right)(1-2x)^{2n+3} \qquad (3.5)\end{aligned}$$

is valid for all $n \in \mathbb{N}$ and $x \in (0,1/2)$, and each inequality of (3.5) becomes equality if and only if $x = 1/2$.

Proof. (1) Let $h_1(x) = f(x) - P_n^1(x)$ and $h_2(x) = x^{2n+1}$. Then $H_n^1(x) = h_1(x)/h_2(x)$, $h_1^{(m)}(0^+) = h_2^{(m)}(0^+) = 0$ for all $m \in N \cup \{0\}$ with $0 \leqslant m \leqslant 2n$, and

$$\frac{h_1^{(2n+1)}(x)}{h_2^{(2n+1)}(x)} = \frac{f^{(2n+1)}(x)}{(2n+1)!}.$$

From Lemma 2.2 we know that $H_n^1(x)$ has the same monotonicity with the function $f^{(2n+1)}(x)$ if $f^{(2n+1)}(x)$ is monotonic. Therefore, it follows from Theorem 3.1 that $H_n^1(x)$ is increasing on $(0,1/2]$.

Elaborated computations lead to

$$\begin{aligned}(H_n^1(x))' &= \left(\frac{h_1(x)}{h_2(x)}\right)' = \frac{x\left(f(x) - P_n^1(x)\right)' - (2n+1)\left(f(x) - P_n^1(x)\right)}{x^{2n+2}} \\ &= \frac{\sum_{k=n}^{\infty}\left[(2n-2k+2)\eta(2k)x^{2k+1} + (2k-2n-1)\eta(2k+2)x^{2k+2}\right]}{2x^{2n+2}}.\end{aligned}$$

Let $h_3(x) = x^{2n+2}$ and

$$h_4(x) = \sum_{k=n}^{\infty} \left[(2n-2k+2)\eta(2k)x^{2k+1} + (2k-2n-1)\eta(2k+2)x^{2k+2} \right].$$

Then we clearly see that $h_3^{(m)}(0^+) = h_4^{(m)}(0^+) = 0$ for all $m \in N \cup \{0\}$ with $0 \leqslant m \leqslant 2n$, and

$$\frac{h_3^{(2n+1)}(x)}{h_4^{(2n+1)}(x)} = \frac{f^{(2n+2)}(x)}{(2n+2)!}.$$

From Lemma 2.2 we know that $(H_n^1(x))'$ has the same monotonicity with the function $f^{(2n)}(x)$ if $f^{(2n)}(x)$ is monotonic. It follows from Theorem 3.1 that the desired monotonicity of the function $(H_n^1(x))'$ is obtained and the desired concavity of the function $H_n^1(x)$ is proved.

Note that $H_n^1(0^+) = -2\eta(2n)$. Therefore, inequalities (3.3) follows from the monotonicity and concavity of the function $H_n^1(x)$.

(2) Let $h_5(x) = f(x) - P_n^2(x)$ and $h_6(x) = x^{2n}$. Then $H_n^2(x) = h_5(x)/h_6(x)$, $h_5^{(m)}(0^+) = h_6^{(m)}(0^+) = 0$ for all $m \in N \cup \{0\}$ with $0 \leqslant m \leqslant 2n-1$, and

$$\frac{h_5^{(2n)}(x)}{h_6^{(2n)}(x)} = \frac{f^{(2n)}(x)}{(2n)!}.$$

From Lemma 2.2 we know that $H_n^2(x)$ has the same monotonicity with the function $f^{(2n)}(x)$. Making use of Theorem 3.1, we know that $H_n^2(x)$ is decreasing on $(0,1/2]$.

Simple computations give

$$(H_n^2(x))' = \frac{x\left(f(x) - P_n^2(x)\right)' - (2n+1)\left(f(x) - P_n^2(x)\right)}{x^{2n-1}}$$
$$= \frac{\sum_{k=n}^{\infty} \left[(2k-2n)\eta(2k)x^{2k} + (2n-2k-1)\eta(2k)x^{2k+1} \right]}{2x^{2n-1}}.$$

Let $h_7(x) = x^{2n-1}$ and

$$h_8(x) = \sum_{k=n}^{\infty} \left[(2k-2n)\eta(2k)x^{2k} + (2n-2k-1)\eta(2k)x^{2k+1} \right].$$

Then $h_7^{(m)}(0^+) = h_8^{(m)}(0^+) = 0$ for all $m \in N \cup \{0\}$ with $0 \leqslant m \leqslant 2n-1$, and

$$\frac{h_7^{(2n)}(x)}{h_8^{(2n)}(x)} = \frac{f^{(2n-1)}(x)}{(2n-1)!}.$$

According to Theorem 3.1 and Lemma 2.2, we can get the desired convexity of $H_n^2(x)$. Using the monotonicity and concavity of $H_n^2(x)$, we obtain inequality (3.4).

(3) It follows from (2.3) that

$$I_n(x) = \frac{f(x) - P_n^3(x)}{(1-2x)^{2n+2}} = \sum_{k=0}^{\infty} A_2(k+n+1)(1-2x)^{2k}.$$

Lemma 2.1(2) leads to the conclusion that $A_2(k+n+1) > 0$ for all $k,n \in \mathbb{N}$. By the monotonicity of $(1-2x)^{2k}$, we can know that $I_n(x)$ is decreasing on $(0,1/2]$. Simple computations lead to $I_n''(x) > 0$, which implies that $I_n(x)$ is convex. Therefore, inequality (3.5) follows from the monotonicity and convexity of the function $I_n(x)$. □

Let $n = 1$. Then inequality (3.3) leads to Corollary 3.3 immediately.

COROLLARY 3.3. *The double inequality*

$$\begin{aligned} 1 - x + \frac{\pi^2}{6}x^2 - \frac{\pi^2}{6}x^3 + 2\left(2\pi - 4 - \frac{\pi^2}{6}\right)x^4 \leqslant x(1-x)B(x) \\ \leqslant 1 - x + \frac{\pi^2}{6}x^2 + \left(2\pi - 4 - \frac{\pi^2}{3}\right)x^3 \end{aligned} \tag{3.6}$$

holds for all $x \in (0,1/2]$.

REMARK 3.4. Corollary 3.3 provide new lower and upper bounds for $x(1-x)B(x)$ in term of cubic and quartic polynomials, respectively.

Let $n = 2$. Then inequality (3.4) becomes Corollary 3.5.

COROLLARY 3.5. *The two-sided inequality*

$$\begin{aligned} 1 - x + \frac{\pi^2}{6}x^2 - \frac{\pi^2}{6}x^3 + \left(4\pi - 8 - \frac{\pi^2}{3}\right)x^4 \leqslant x(1-x)B(x) \\ \leqslant 1 - x + \frac{\pi^2}{6}x^2 - \frac{\pi^2}{6}x^3 + \frac{7\pi^4}{360}x^4 + 2\left(4\pi - 8 - \frac{\pi^2}{3} - \frac{7\pi^4}{360}\right)x^5 \end{aligned} \tag{3.7}$$

takes place for $x \in (0,1/2]$.

REMARK 3.6. Inequality (3.7) provide new lower and upper bounds for $x(1-x)B(x)$ in term of quartic and quintic polynomials, respectively.

Let $n = 1$. Then inequality (3.5) reduces to Corollary 3.7.

COROLLARY 3.7. *The double inequality*

$$\begin{aligned} P_1^3(x) + A_2(2)(1-2x)^4 \leqslant x(1-x)B(x) \\ \leqslant P_1^3(x) + A_2(2)(1-2x)^4 + \left(I_1(0^+) - A_2(2)\right)(1-2x)^5 \end{aligned} \tag{3.8}$$

is valid for $x \in (0,1/2]$, *where*

$$A_2(2) = \frac{5\pi^5}{1536} - \frac{\pi^3}{32} = 0.02722\cdots, \quad I_1(0^+) = -\frac{\pi^2}{32} + 1 = 0.69157\cdots,$$

$$P_1^3(x) = \frac{\pi}{4} + \frac{\pi}{32}\left(\pi^2 - 8\right)(1-2x)^2,$$

and each inequality of (3.8) becomes equality if and only if $x = 1/2$.

REMARK 3.8. Inequality (3.8) provide new asymptotic sharp lower and upper bounds for the function $x(1-x)B(x)$ by the polynomial function of $(1-2x)$.

Next, we present several new properties the function $g(x) = R(x) - [1 - x(1-x)]B(x)$. Let

$$\begin{cases} B_1(0) = 1, \ B_1(1) = -\left(1 + \frac{\pi^2}{6}\right) = -2.6450\cdots, \\ B_1(2k) = 2\zeta(2k+1) + 2\eta(2k), \ B_1(2k+1) = -[2\eta(2k+2) + 2\eta(2k)] \quad (k \geqslant 1), \\ B_2(0) = 4\log 2 - \frac{3\pi}{4} = 0.4164\cdots, \\ B_2(k) = 4\lambda(2k+3) - \beta(2k+1) - 3\beta(2k+3) \quad (k \geqslant 1). \end{cases} \tag{3.9}$$

THEOREM 3.9. *Let* $n \in \mathbb{N}$. *Then the following statements are true:*
(1) Then function $G_n^1(x)$ *defined by*

$$G_n^1(x) = \frac{g(x) - R_n^1(x)}{x^{2n}}$$

with $R_n^1(x) = \sum_{k=0}^{2n-1} B_1(k)x^k$ *is strictly decreasing and convex from* $(0,1/2]$ *into* $[G_n^1(1/2), B_1(2n))$. *In particular, the double inequality*

$$\left[G_n^1\left(\frac{1}{2}\right) - B_1(2n)\right]x^{2n} \leqslant g(x) - R_n^1(x) - B_1(2n)x^{2n} \leqslant 2\left[G_n^1\left(\frac{1}{2}\right) - B_1(2n)\right]x^{2n+1} \tag{3.10}$$

holds for all $n \in \mathbb{N}$ *and* $x \in (0,1/2]$, *and each inequality of (3.10) becomes equality if and only if* $x = 1/2$.

(2) The function $G_n^2(x)$ *defined by*

$$G_n^2(x) = \frac{g(x) - R_n^2(x)}{x^{2n+1}}$$

with $R_n^2(x) = \sum_{k=0}^{2n} B_1(k)x^k$ *is strictly increasing and concave from* $(0,1/2]$ *into* $(B_1(2n+1), G_n^2(1/2)]$. *In particular, the two-sided inequality*

$$2\left[G_n^2\left(\frac{1}{2}\right) - B_1(2n+1)\right]x^{2n+2} \leqslant g(x) - R_n^2(x) - B_1(2n+1)x^{2n+1} \leqslant \left[G_n^2\left(\frac{1}{2}\right) - B_1(2n+1)\right]x^{2n+1} \tag{3.11}$$

is valid for all $n \in \mathbb{N}$ *and* $x \in (0,1/2]$, *and each inequality of (3.11) becomes equality if and only if* $x = 1/2$.

(3) The function $K_n(x)$ *defined by*

$$K_n(x) = \frac{g(x) - R_n^3(x)}{(1-2x)^{2n}},$$

with $R_n^3(x) = \sum_{k=0}^{n-1} B_2(k)(1-2x)^{2k}$ *is strictly decreasing and convex from* $(0,1/2)$ *onto* $(B_2(n), K_n(0^+))$. *In particular, the double inequality*

$$\begin{aligned} 0 &\leqslant g(x) - R_n^3(x) - B_2(n)(1-2x)^{2n} \\ &\leqslant \left[K_n\left(0^+\right) - B_2(n)\right](1-2x)^{2n+1} \end{aligned} \tag{3.12}$$

takes place for all $n \in \mathbb{N}$ *and* $x \in (0,1/2]$, *and each inequality of (3.12) becomes equality if and only if* $x = 1/2$.

Proof. (1) Let $g_1(x) = g(x) - R_n^1(x)$ and $g_2(x) = x^{2n}$. Then $G_n^1(x) = g_1(x)/g_2(x)$, $g_1^{(m)}(0^+) = g_2^{(m)}(0^+) = 0$ for all $m \in N \cup \{0\}$ with $0 \leqslant m \leqslant 2n+1$, and

$$\frac{g_1^{(2n)}(x)}{g_2^{(2n)}(x)} = \frac{g^{(2n)}(x)}{(2n)!}.$$

From Lemma 2.2 we know that the function $G_n^1(x)$ has the same monotonicity with the function $g^{(2n)}(x)$ if $g^{(2n)}(x)$ is monotonic. Therefore, $G_n^1(x)$ is decreasing on $(0,1/2]$ follows from Theorem 3.1.

Elaborated computations lead to

$$(G_n^1(x))' = \left(\frac{g_1(x)}{g_2(x)}\right)' = \frac{x\left(g(x) - R_n^1(x)\right)' - 2n\left(g(x) - R_n^1(x)\right)}{x^{2n+1}} = \frac{g_3(x)}{g_4(x)},$$

where

$$g_3(x) = x\left(g(x) - R_n^1(x)\right)' - 2n\left(g(x) - R_n^1(x)\right), \quad g_4(x) = x^{2n+1}.$$

Making use of (3.1), $g_3(x)$ can be rewritten as

$$g_3(x) = \sum_{k=2n}^{\infty} kB_1(k)x^{k-1}.$$

It is easy to check that $g_3^{(m)}(0^+) = g_4^{(m)}(0^+) = 0$ for all $m \in N \cup \{0\}$ with $0 \leqslant m \leqslant 2n+2$, and

$$\frac{g_3^{(2n)}(x)}{g_4^{(2n)}(x)} = \frac{g^{(2n+1)}(x)}{(2n+1)!}.$$

It follows from Lemma 2.2 that the function $(G_n^1(x))'$ has the same monotonicity with the function $g^{(2n+1)}(x)$ if $g^{(2n+1)}(x)$ is monotonic. Therefore, Theorem 3.1 leads to the conclusion that $(G_n^1(x))'$ is strictly increasing on $(0,1/2]$ and we obtain the desired convexity of $G_n^1(x)$. Note that $G_n^1(0^+) = B_1(2n)$. Hence, inequality (3.10) follows from the monotonicity and convexity of $G_n^1(x)$.

(2) Let $g_5(x) = g(x) - R_n^2(x)$ and $g_6(x) = x^{2n+1}$. Then $G_n^2(x) = g_5(x)/g_6(x)$, $g_5^{(m)}(0^+) = g_6^{(m)}(0^+) = 0$ for all $m \in N \cup \{0\}$ with $0 \leqslant m \leqslant 2n$, and

$$\frac{g_5^{(2n+1)}(x)}{g_6^{(2n+1)}(x)} = \frac{g^{(2n+1)}(x)}{(2n+1)!}.$$

Therefore, $G_n^2(x)$ is strictly increasing on $(0,1/2]$ follows easily from Lemma 2.2 and Theorem 3.1.

Simple computations lead to

$$(G_n^2(x))' = \left(\frac{g_5(x)}{g_6(x)}\right)' = \frac{x\left(g(x) - R_n^2(x)\right)' - (2n+1)\left(g(x) - R_n^2(x)\right)}{x^{2n+2}} = \frac{g_7(x)}{g_8(x)},$$

where

$$g_7(x) = x\left(g(x) - R_n^2(x)\right)' - (2n+1)\left(g(x) - R_n^2(x)\right), g_8(x) = x^{2n+2}.$$

From (3.1) we clearly see that $g_7(x)$ can be rewritten as

$$g_7(x) = \sum_{k=2n+1}^{\infty} kB_1(k)x^{k-1}.$$

It is easy to check that $g_7^{(m)}(0^+) = g_8^{(m)}(0^+) = 0$ for all $m \in N \cup \{0\}$ with $0 \leqslant m \leqslant 2n$, and

$$\frac{g_7^{(2n+1)}(x)}{g_8^{(2n+1)}(x)} = \frac{g^{(2n+2)}(x)}{(2n+2)!}.$$

Therefore, $(G_n^2(x))'$ is strictly decreasing on $(0,1/2]$ follows from Lemma 2.2 and Theorem 3.1. Note that $G_n^2(0^+) = B_1(2n+1)$. Hence, inequality (3.11) can be derived from the monotonicity and concavity of the function $G_n^2(x)$.

(3) It follows from (3.1) that

$$g(x) - R_n^3(x) = \sum_{k=0}^{\infty} [4\lambda(2k+2n+1) - \beta(2k+2n-1) - 3\beta(2k+2n+1)](1-2x)^{2k+2n}$$

and

$$K_n(x) = \sum_{k=0}^{\infty} [4\lambda(2k+2n+1) - \beta(2k+2n-1) - 3\beta(2k+2n+1)](1-2x)^{2k}.$$

According to the monotonicity of $\beta(n)$ and $(1-2x)^{2k}$ we know that $K_n(x)$ is decreasing on $(0,1/2]$.

Simple computations show that $K_n''(x) > 0$, which implies that $K_n(x)$ is convex. Note that $K_n(1/2^-) = B_2(n)$. Therefore, inequality (3.12) can be obtained by using the monotonicity and convexity of the function $K_n(x)$. □

Let $n = 2$. Then inequality (3.10) leads to Corollary 3.10 immediately.

COROLLARY 3.10. *The double inequality*

$$R_2^1(x)+B_1(4)x^4+\left[G_2^1\left(\frac{1}{2}\right)-B_1(4)\right]x^4 \leqslant R(x)-[1-x(1-x)]B(x) \leqslant R_2^1(x)+B_1(4)x^4+2\left[G_2^1\left(\frac{1}{2}\right)-B_1(4)\right]x^5$$

holds for all $x\in(0,1/2]$, *and each inequality of (3.13) becomes equality if and only if* $x=1/2$, *where*

$$B_1(4)=2\zeta(5)+\frac{7\pi^4}{360}=3.9678\cdots,$$

$$G_2^1\left(\frac{1}{2}\right)=64\ln(2)-12\pi-8+\frac{2\pi^2}{3}-8\zeta(3)+\frac{\pi^2(7\pi^2+60)}{180}=2.7044\cdots,$$

$$R_2^1(x)=1-\left(\frac{\pi^2}{6}+1\right)x+\left(2\zeta(3)+\frac{\pi^2}{6}\right)x^2-\frac{\pi^2\left(7\pi^2+60\right)}{360}x^3.$$

Let $n=1$. Then inequality (3.11) leads to Corollary 3.11.

COROLLARY 3.11. *The double inequality*

$$R_1^2(x)+B_1(3)x^3+2\left[G_1^2\left(\frac{1}{2}\right)-B_1(3)\right]x^4 \leqslant R(x)-[1-x(1-x)]B(x) \leqslant R_1^2(x)+B_1(3)x^3+\left[G_1^2\left(\frac{1}{2}\right)-B_1(3)\right]x^3 \tag{3.13}$$

holds for all $x\in(0,1/2]$, *and each inequality of (3.13) becomes equality if and only if* $x=1/2$, *where*

$$B_1(3)=-\frac{7\pi^4}{360}-\frac{\pi^2}{6}=-3.5390\cdots,$$

$$G_1^2\left(\frac{1}{2}\right)=32\ln(2)-6\pi-4+\frac{\pi^2}{3}-4\zeta(3)=-2.1875\cdots,$$

$$R_1^2(x)=1-\left(\frac{\pi^2}{6}+1\right)x+\left(2\,\zeta\,(3)+\frac{\pi^2}{6}\right)x^2.$$

Let $n=2$. Then inequality (3.12) leads to Corollary 3.12.

COROLLARY 3.12. *The two-sided inequality*

$$R_2^3(x)+B_2(2)(1-2x)^4 \leqslant R(x)-[1-x(1-x)]B(x) \leqslant R_2^3(x)+B_2(2)(1-2x)^4+\left[K_2\left(0^+\right)-B_2(2)\right](1-2x)^5 \tag{3.14}$$

takes place for all $x\in(0,1/2]$, *and each inequality of (3.14) becomes equality if and only if* $x=1/2$, *where*

$$B_2(2)=62\zeta(5)-\frac{5\pi^5}{32}-\frac{\pi^3}{2}=0.969\cdots,$$

$$K_2\left(0^+\right) = \frac{3\pi^3}{32} + \pi - 4\ln(2) - \frac{7\zeta(3)}{2} + 1 = 0.0684\cdots,$$

$$R_2^3(x) = 4\ln(2) - \frac{3\pi}{4} + \frac{1}{4}\left(14\zeta(3) - \pi - \frac{3\pi^3}{8}\right)(1-2x)^2.$$

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(Received January 16, 2020)

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