# CONCAVITY AND BOUNDS INVOLVING GENERALIZED
# ELLIPTIC INTEGRAL OF THE FIRST KIND

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*Abstract.* In the article, we provide a sufficient condition for value range of the constant $c$ such that the function $x \to \mathcal{K}_a(\sqrt{x})/\log(c/\sqrt{1-x})$ is strictly concave on $(0,1)$ for $a \in (0,1/2]$, which generalize a very recently obtained result that the function $x \to \mathcal{K}(\sqrt{x})/\log(c/\sqrt{1-x})$ is strictly concave on $(0,1)$ if and only if $c = e^{4/3}$. As applications, we present new bounds for $\mathcal{K}_a(x)$, $\mathcal{K}_a(\sqrt{1-x^2})/\mathcal{K}_a(\sqrt{x})$ and $\mathcal{K}_a(\sqrt{1-x^2})\mathcal{K}_a(\sqrt{x})$, where $\mathcal{K}_a(x)$ is the generalized elliptic integral of the first kind and $\mathcal{K}(x) = \mathcal{K}_{1/2}(x)$.

## 1. Introduction

Let $a, b, c \in \mathbb{R}$ with $c \neq 0, -1, -2, \cdots$ and $x \in (-1, 1)$. Then the Gaussian hypergeometric function $F(a,b;c;x)$ [28, 32, 39, 40, 42, 47, 50, 53, 54, 55, 71] is defined by

$$F(a,b;c;x) =_2 F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} x^n, \tag{1.1}$$

where $(a,0) = 1$ for $a \neq 0$, $(a,n) = a(a+1)(a+2)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ for $n \in \mathbb{N}$ is the shifted factorial function and $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt$ is the classical Euler gamma function [20, 62, 63, 67, 68]. $F(a,b;c;x)$ is said to be *zero-balanced* if $c = a+b$. The behavior of $F(a,b;c;x)$ near $x = 1$ is given by

$$\begin{cases} F(a,b;c;1) = \dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, & a+b < c, \\ B(a,b)F(a,b;c;x) + \log(1-x) = R(a,b) + O((1-x)\log(1-x)), & a+b = c, \\ F(a,b;c;x) = (1-x)^{c-a-b}F(c-a,c-b;c;x), & a+b > c, \end{cases} \tag{1.2}$$

which can be found in the literature [8, Theorems 1.19 and 1.48], where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function, $B(p,q) = [\Gamma(p)\Gamma(q)]/\Gamma(p+q)$ is the beta function, and

$$R(a,b) = -\psi(a) - \psi(b) - 2\gamma, \quad R(1/2,1/2) = 4\log 2 \tag{1.3}$$

and $\gamma = \lim_{n\to\infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) = 0.57721\cdots$ is the Euler-Mascheroni constant.

One of the most important special cases of $F(a,b;c;x)$ is the complete elliptic integral $\mathscr{K}(r)$ [14, 15, 16, 17, 18, 19, 29, 43, 46, 49, 56, 57, 60, 61, 64, 65, 70] of the first kind with the modulus $r \in (0,1)$, which is given by

$$\mathscr{K}(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - r^2 \sin^2 t}} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right). \tag{1.4}$$

The complete elliptic integrals and Gaussian hypergeometric function play very important roles in many branches of modern mathematics such as classical analysis, number theory, geometric function theory, and conformal and quasi-conformal mappings [30, 31, 38, 41, 45, 48, 51, 52, 59, 69]. Recently, many inequalities [2, 3, 4, 9, 10, 11, 22, 23, 24, 25, 26, 33, 34, 35, 36, 37] in the convexity theory have been established via the complete elliptic integrals and Gaussian hypergeometric function.

Let $r \in (0,1)$ and $a \in (0,1)$. Then the generalized elliptic integral $\mathscr{K}_a(r)$ [5, 21] of the first kind is defined by

$$\mathscr{K}_a = \mathscr{K}_a(r) = \frac{\pi}{2} F(a, 1-a; 1, r^2). \tag{1.5}$$

Note that $\mathscr{K}_a(0) = \pi/2$, $\mathscr{K}_a(1^-) = \infty$ and the generalized elliptic integrals $\mathscr{K}_a(r)$ reduces to the complete elliptic integrals $\mathscr{K}(r)$ if $a = 1/2$. From (1.5) we clearly see that we only need to discuss the case of $a \in (0, 1/2]$ due to its symmetry.

It is well known that the generalized Grötzsch ring function $\mu_a(r)$ and its related special function $m_a(r)$ [5, (1.3) and (1.12)] in the theory of Ramanujan generalized modular equation can be expressed by the generalized elliptic integral as follows

$$\mu_a(r) = \frac{\pi}{2\sin(\pi a)} \frac{\mathscr{K}_a(r')}{\mathscr{K}_a(r)}, \quad m_a(r) = \frac{2}{\sin(\pi a)} r'^2 \mathscr{K}_a(r)\mathscr{K}_a(r'), \tag{1.6}$$

where and in what follows we denote $r' = \sqrt{1 - r^2}$.

From (1.4) and the second formula of (1.2), we clearly see that $\mathscr{K}(r)$ has the following asymptotic expansion

$$\mathscr{K}(r) \sim \log \frac{4}{r'}, \qquad (r \to 1^-). \tag{1.7}$$

Inspired by (1.7), many remarkable inequalities for $\mathscr{K}(r)$ in terms of $\log(4/r')$ and monotonicity properties for the ratio $\mathscr{K}(r)/\left[\log(4/r')\right]$ were established in [6, 7, 12, 13].

Very recently, Yang and Tian [66] established the following Theorem A.

THEOREM A. *The function*

$$x \to \frac{\mathscr{K}(\sqrt{x})}{\log(c/\sqrt{1-x})}$$

*is strictly concave on* $(0,1)$ *if and only if* $c = e^{4/3}$.

The main purpose of the article is to generalize Theorem A to the generalized elliptic integral $\mathscr{K}_a(r)$, and provide several bounds for $\mathscr{K}_a(r)$, $\mu_a(r)$ and $m_a(r)$. Our main result is the following Theorem 1.1.

THEOREM 1.1. *Let* $a \in (0, 1/2]$ *and* $\varphi(x)$ *be defined by*

$$\varphi(x) = \frac{\mathscr{K}_a(\sqrt{x})}{\log(c/\sqrt{1-x})}.$$

*Then* $\varphi(x)$ *is strictly concave on* $(0,1)$ *if* $e^{1/(1-a^2)} \leqslant c \leqslant e^{R(a)/2+4/3-2\log 2}$, *where* $R(a) = R(a, 1-a)$ *given in* (1.3).

## 2. Preliminaries

In order to prove our main result, we need to introduce some notations and establish several technical lemmas which we present in this section.

For $k \in \mathbb{N}$, we denote by $F_k = F_k(x) = F(a, 1-a; k, x)$ for short. Recall that the derivative formulae for $F(a, b; c; x)$ [27, 15.5.5,15.5.21] are given by

$$\begin{cases} \dfrac{d}{dx}F(a,b;c;x) = \dfrac{ab}{c}F(a+1,b+1;c+1;x), \\ (1-x)\dfrac{d}{dx}F(a,b;c;x) = \dfrac{(c-a)(c-b)}{c}F(a,b;c+1;x) + (a+b-c)F(a,b;c;x). \end{cases} \tag{2.8}$$

From the second formula of (2.8) we clearly see that the derivative of $F_k$ satisfies the following recurrence relation

$$(1-x)\frac{dF_k}{dx} = \frac{(a+k-1)(k-a)}{k}F_{k+1} - (k-1)F_k. \tag{2.9}$$

Making use of the power series expansion (1.1), we obtain

$$F_1(x) = \sum_{n=0}^{\infty} W_n^2 x^n, \quad F_k(x) = \sum_{n=0}^{\infty} \frac{(k-1)!W_n^2}{(n+1)\cdots(n+k-1)}x^n \quad (k \geqslant 2), \tag{2.10}$$

where

$$W_n = W_n(a) = \frac{\sqrt{\sin(a\pi)\Gamma(a+n)\Gamma(1-a+n)}}{\Gamma(1/2)\Gamma(n+1)} \tag{2.11}$$

is the generalized Wallis' ratio. In particular, $W_n(1/2)$ reduces to the classical Wallis' ratio.

It is easy to verify that $W_n$ satisfies the recurrence relation

$$W_{n+1} = \frac{\sqrt{(a+n)(1-a+n)}}{n+1}W_n. \tag{2.12}$$

A special value of hypergeometric function $F(a, 1-a,; 1; x)$ is given in [1, 15.1.26],

$$\mathcal{K}_a(1/\sqrt{2}) = \frac{\pi}{2} F\left(a; 1-a; 1; \frac{1}{2}\right) = \frac{\Gamma(\frac{1-a}{2})\Gamma(\frac{a}{2})\sin(a\pi)}{4\sqrt{\pi}}, \tag{2.13}$$

which will be used later.

For $a \in (0, 1/2]$, the Ramanujan constant function $R(a)$ is defined as in (1.3) if $b = 1-a$, more precisely,

$$R(a) = R(a, 1-a) = -\psi(a) - \psi(1-a) - 2\gamma, \quad R(1/2) = 4\log 2. \tag{2.14}$$

Throughout the paper, we give some notations related to $a$ for $a \in (0, 1/2]$, which help us to simplify our expressions later. We divide them into $\delta$-symbols, $\sigma$-symbols and $\tau$-symbols as follows.

### $\delta$-symbols

$$\delta_0 = a(1-a), \quad \delta_1 = (1+a)(2-a), \quad \delta_2 = a(1-a^2)(2-a),$$
$$\delta_3 = (a+1)(3-a)(4-a^2), \quad \delta_4 = a^4 - 11a^2 + 22,$$
$$\delta_5 = (1-a^2)(4-a^2)(9-a^2), \quad \delta_6 = \delta_3 + 4\delta_1 + 2,$$
$$\delta_7 = \delta_0\delta_3 + 9\delta_3 - 18\delta_1 - 36, \quad \delta_8 = \delta_0\delta_3(\delta_0 + 12).$$

Clearly, $\delta_j > 0$ for $j = 0, 1, 2, \cdots, 8$.

### $\sigma$-symbols

$$\sigma_0 = \frac{\delta_0 \log c}{3(2\log c - 1)}, \quad \sigma_1 = 2(4\log^2 c - 1),$$
$$\sigma_2 = 8(1 - \delta_0)\log^2 c + 4(9 + \delta_0)\log c - 2(1 + \delta_0),$$
$$\sigma_3 = 4(4\delta_0 - 2\delta_3 + 5)\log^2 c + 4(13\delta_0 + 54)\log c - 2(3\delta_0 + \delta_3 + 23),$$
$$\sigma_4 = 4(4\delta_0\delta_3 - 20\delta_3 + 78\delta_0 + 142)\log^2 c$$
$$\quad\quad + 2(4\delta_0\delta_3 - 7\delta_3 + 134\delta_0 + 282)\log c + (50\delta_0 - 17\delta_3 + 104),$$
$$\sigma_5 = 4(31\delta_7 - 2\delta_6^2 - 219\delta_3 + 642\delta_0 + 2420)\log^2 c$$
$$\quad\quad + 2(20\delta_7 - 225\delta_3 + 726\delta_0 + 2088)\log c + 3(74\delta_0 - 15\delta_3 + 164),$$
$$\sigma_6 = 16(16\delta_7 - \delta_6^2 - 105\delta_3 + 294\delta_0 + 1168)\log^2 c$$
$$\quad\quad + 48(\delta_7 + 12\delta_6 - 24\delta_3 - 12\delta_0 - 12)\log c + 36(10\delta_0 - \delta_6 + 22),$$
$$\sigma_7 = \frac{\delta_0^2\delta_1^2(a^2 - 7a + 12)(1 - a + 6\sigma_0)}{2\log c - 1}.$$

### $\tau$-symbols

$$\tau_0 = 1 + \sigma_0, \quad \tau_1 = 11 + 30\sigma_0 - 6\delta_1\sigma_0 + 2(12\delta_1\sigma_0 - \delta_2)\log c,$$
$$\tau_2 = 6\left[1 + 6\sigma_0 - 3\delta_1\sigma_0 + (12\delta_1\sigma_0 - \delta_2 - 2\delta_3\sigma_0)\log c\right],$$
$$\tau_3 = 12\delta_1\sigma_0 - \delta_2, \quad \tau_4 = 3(12\delta_1\sigma_0 - \delta_2 - 2\delta_3\sigma_0).$$

The following lemma provides a simple criterion to determine the sign of a class of special power series.

LEMMA 2.1. ([66, Lemma 2.1]) *Let* $\{a_k\}_{k=0}^{\infty}$ *be a nonnegative real sequence with* $a_m > 0$ *and* $\sum_{k=m+1}^{\infty} a_k > 0$, *and*

$$S(x) = -\sum_{k=0}^{m} a_k x^k + \sum_{k=m+1}^{\infty} a_k x^k$$

*be a convergent power series on the interval* $(0,r)$. *Then one has*

1. *If* $S(r^-) \leqslant 0$, *then* $S(x) < 0$ *for all* $x \in (0,r)$.

2. *If* $S(r^-) > 0$. *then there exists* $x_0 \in (0,r)$ *such that* $S(x) < 0$ *for* $x \in (0,x_0)$ *and* $S(x) > 0$ *for* $x \in (x_0,r)$.

LEMMA 2.2. *Let* $a \in (0,1/2]$. *Then we have*

1. $R(a)/2 < 1/(2a) + 5a^2/4 + 27a^4/20$;

2. $1/[3a(1-a)] \leqslant R(a)/2 + 4/3 - 2\log 2$;

3. $R(a)/2 + 4/3 - 2\log 2 \leqslant 1/[a(2-a)]$.

*Proof.* (i) As shown in [44, Theorem 2.2], $R(a)$ can be written as

$$R(a) = \frac{1}{a} + \sum_{k=1}^{\infty} 2\zeta(2k+1)a^{2k}, \tag{2.15}$$

where $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ is the well-known Riemann zeta function.
Let

$$\phi_1(a) = \frac{R(a)}{2} - \left(\frac{1}{2a} + \frac{5a^2}{4} + \frac{27a^4}{20}\right).$$

Then it follows from (2.15) that

$$\phi_1(a) = \left[\zeta(3) - \frac{5}{4}\right]a^2 + \left[\zeta(5) - \frac{27}{20}\right]a^4 + \sum_{k=3}^{\infty}\zeta(2k+1)a^{2k}. \tag{2.16}$$

Numerical calculations lead to

$$\zeta(3) - \frac{5}{4} = -0.04794\cdots, \quad \zeta(5) - \frac{27}{20} = -0.31307\cdots,$$

$$\phi_1(1/2) = 2\log 2 - \frac{447}{320} = -0.01058\cdots.$$

This in conjunction with (2.16) and Lemma 2.1 yields $\phi_1(a) < 0$ for $a \in (0,1/2]$.

(ii) Let
$$\phi_2(a) = \frac{1}{3a(1-a)} - \left[ \frac{R(a)}{2} + \frac{4}{3} - 2\log 2 \right].$$

Then differentiating $\phi_2(a)$ yields

$$
\begin{aligned}
2\phi_2'(a) &= \frac{2}{3} \left[ \frac{1}{(1-a)^2} - \frac{1}{a^2} \right] + \Psi'(a) - \Psi'(1-a) \\
&= \frac{2}{3} \left[ \frac{1}{(1-a)^2} - \frac{1}{a^2} \right] + \sum_{k=0}^{\infty} \left[ \frac{1}{(a+k)^2} - \frac{1}{(1-a+k)^2} \right] \\
&= \frac{1}{3} \left[ \frac{1}{a^2} - \frac{1}{(1-a)^2} \right] + \sum_{k=1}^{\infty} \left[ \frac{1}{(a+k)^2} - \frac{1}{(1-a+k)^2} \right] > 0
\end{aligned}
$$

for $a \in (0, 1/2)$. This in conjunction with $\phi_2(1/2) = 0$ yields $\phi_2(a) \leqslant 0$ for $a \in (0, 1/2]$.

(iii) Let
$$\phi_3(a) = \frac{R(a)}{2} + \frac{4}{3} - 2\log 2 - \frac{1}{a(2-a)}.$$

Then taking the differentiation of $\phi_3(a)$ yields

$$
\begin{aligned}
2\phi_3'(a) &= -\Psi'(a) + \Psi'(1-a) + \frac{1}{a^2} - \frac{1}{(2-a)^2} \\
&= \sum_{k=0}^{\infty} \left[ \frac{1}{(1-a+k)^2} - \frac{1}{(a+k)^2} \right] + \frac{1}{a^2} - \frac{1}{(2-a)^2} \\
&= \sum_{k=1}^{\infty} \left[ \frac{1}{(1-a+k)^2} - \frac{1}{(a+k)^2} \right] + \frac{1}{(1-a)^2} - \frac{1}{(2-a)^2} \qquad (2.17)
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_3''(a) &= -\frac{1}{2} \left[ \Psi''(a) + \Psi''(1-a) \right] - \frac{1}{a^3} - \frac{1}{(2-a)^3} \\
&= \sum_{k=2}^{\infty} \left[ \frac{1}{(a+k)^3} + \frac{1}{(1-a+k)^3} \right] + \frac{1}{(1+a)^3} + \frac{1}{(1-a)^3} > 0. \qquad (2.18)
\end{aligned}
$$

It follows from (2.17) that $\phi_3'(0) = -1/4$ and $\phi_3'(1/2) = 32/9$. This in conjunction with (2.18) implies that there exists $a_1 \in (0, 1/2)$ such that $\phi_3(a)$ is strictly decreasing on $(0, a_1)$ and strictly increasing on $(a_1, 1/2]$.

Moreover,

$$
\begin{aligned}
\phi_3(0^+) &= -\frac{1}{2} \lim_{a \to 0^+} \left[ \Psi(a) + \Psi(1-a) + 2\gamma + \frac{1}{a} + \frac{1}{2-a} \right] + \frac{4}{3} - 2\log 2 \\
&= -\frac{1}{2} \lim_{a \to 0^+} \left[ \sum_{k=1}^{\infty} \frac{a}{k(k+a)} + \sum_{k=1}^{\infty} \frac{1-a}{k(k+1-a)} - \frac{1}{(1-a)(2-a)} \right] + \frac{4}{3} - 2\log 2 \\
&= -\sum_{k=1}^{\infty} \frac{1}{2k(k+1)} + \frac{19}{12} - 2\log 2 = \frac{13}{12} - 2\log 2 = -0.302961\cdots
\end{aligned}
$$

and $\phi_3(1/2) = 0$, which together with the piecewise monotonicity of $\phi_3(a)$ yields $\phi_3(a) \leqslant 0$ for $a \in (0, 1/2]$. $\square$

LEMMA 2.3. *Let the sequence $\{\beta_n\}$ be defined by*

$$\beta_n = \sum_{k=0}^{n-1} \left[ \frac{(\tau_3 k + \tau_4) W_k^2}{(k+1)(k+2)(k+3)} \cdot \frac{1}{n-k} \right].$$

*Then $\beta_n$ satisfies the recurrence formula*

$$\beta_{n+1} - \lambda_n \beta_n = -\frac{\delta_0 \delta_1 \left[ \sigma_1 n^5 + \sigma_2 n^4 + \sigma_3 n^3 + \sigma_4 n^2 + \sigma_5 n + \sigma_6 \right]}{(2\log c - 1)^2 (\tau_3 n + \tau_4)(n+1)^2(n+2)(n+3)(n+4)} W_n^2, \quad (2.19)$$

*where*

$$\lambda_n = \frac{(\tau_3 n + \tau_3 + \tau_4)(a + n)(1 - a + n)}{(\tau_3 n + \tau_4)(n+1)(n+4)}. \quad (2.20)$$

*Proof.* We first denote by

$$\beta_{k,n-k} = \frac{(\tau_3 k + \tau_4) W_k^2}{(k+1)(k+2)(k+3)} \cdot \frac{1}{n-k}.$$

Elaborated computations lead to

$$\beta_{n+1} - \lambda_n \beta_n = \sum_{k=0}^{n} \beta_{k,n+1-k} - \lambda_n \sum_{k=0}^{n-1} \beta_{k,n-k} = \beta_{0,n+1} + \sum_{k=0}^{n-1} \left( \beta_{k+1,n-k} - \beta_{k,n-k} \right)$$

$$= \frac{\tau_4}{6(n+1)} + \sum_{k=0}^{n-1} \left[ \frac{\tau_3(k+1) + \tau_4}{(k+2)(k+3)(k+4)} \cdot \frac{(a+k)(1-a+k)}{(k+1)^2} \right.$$

$$\left. - \frac{(\tau_3 n + \tau_3 + \tau_4)(a+n)(1-a+n)}{(\tau_3 n + \tau_4)(n+1)(n+4)} \frac{\tau_3 k + \tau_4}{(k+1)(k+2)(k+3)} \right] \frac{W_k^2}{n-k}$$

$$= \frac{\tau_4}{6(n+1)} - \frac{1}{(\tau_3 n + \tau_4)(n+1)(n+4)} \sum_{k=0}^{n-1} \frac{\zeta_k(n)}{(k+1)^2(k+2)(k+3)(k+4)} W_k^2,$$

where $\zeta_k(n) = \theta_2(n)(k^2 + k + \delta_0) + \theta_1(n)(k + a) + \theta_1(n)$ and

$$\theta_2(n) = \tau_3 \left[ 3\tau_3 n^2 + (3\tau_3 - \delta_0 \tau_3 + 4\tau_4)n + (4 - \delta_0)\tau_4 - \delta_0 \tau_3 \right],$$

$$\theta_1(n) = \tau_3(4\tau_4 - \delta_0 \tau_3)n^2 + \left[ 4\tau_4^2 + 2(4 - \delta_0)\tau_3 \tau_4 - 5\delta_0 \tau_3^2 \right]n$$

$$+ (4 - \delta_0)\tau_4^2 + (4 - 5\delta_0)\tau_3 \tau_4 - 4\delta_0 \tau_3^2,$$

$$\theta_0(n) = -\sigma_7 \left[ (1 + 2\log c)n^2 + \left( (1-a)(1+2\log c) + 4 - 2\delta_1 \log c \right)n \right.$$

$$\left. + (1-a)(4 - 2\delta_1 \log c) \right].$$

We denote by

$$\kappa_2(n) = \sum_{k=0}^{n-1} \frac{(a+k)(1-a+k)W_k^2}{(k+1)^2(k+2)(k+3)(k+4)},$$

$$\kappa_1(n) = \sum_{k=0}^{n-1} \frac{(a+k)W_k^2}{(k+1)^2(k+2)(k+3)(k+4)},$$

$$\kappa_0(n) = \sum_{k=0}^{n-1} \frac{W_k^2}{(k+1)^2(k+2)(k+3)(k+4)}.$$

Then

$$\beta_{n+1} - \lambda_n\beta_n = \frac{\tau_4}{6(n+1)} - \sum_{j=0}^{3} \frac{\theta_j(n)\kappa_j(n)}{(\tau_3 n + \tau_4)(n+1)(n+4)}. \tag{2.21}$$

From the third formula of (1.2) we clearly see that

$$\frac{1}{1-x}F(a,1-a;4;x) = (1-x)^2 F(a+3,4-a;4;x),$$

$$\frac{1}{1-x}F(-a,a;4;x) = (1-x)^3 F(a+4,4-a;4;x), \tag{2.22}$$

$$\frac{1}{1-x}F(-a,a-1;4;x) = (1-x)^4 F(a+4,5-a;4;x).$$

We can rewrite (2.22) in terms of power series and compare the coefficients of both sides of (2.22). For instance,

$$\frac{1}{1-x}F(a,1-a;4;x)$$
$$= \left(\sum_{n=0}^{\infty} x^n\right)\left[\sum_{n=0}^{\infty} \frac{6W_n^2}{(n+1)(n+2)(n+3)}x^n\right] = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} \frac{6W_k^2}{(k+1)(k+2)(k+3)}\right]x^n$$
$$= 1 + \sum_{n=1}^{\infty}\left[1 + \sum_{k=0}^{n-1} \frac{6(a+k)(1-a+k)W_k^2}{(k+1)^2(k+2)(k+3)(k+4)}\right]x^n$$

and

$$(1-x)^2 F(a+3,4-a;4;x) = (1-2x+x^2)\sum_{n=0}^{\infty} \frac{6(n+1)(n+2)(n+3)}{\delta_0\delta_3}W_{n+3}^2(a)x^n$$
$$= \sum_{n=0}^{\infty} \frac{6[2n^2+2(\delta_1+3)n+\delta_3](a+n)(1-a+n)}{\delta_0\delta_3(n+1)(n+2)(n+3)}W_n^2 x^n.$$

This gives

$$\kappa_2(n) = \frac{[2n^2+2(\delta_1+3)n+\delta_3](a+n)(1-a+n)W_n^2}{\delta_0\delta_3(n+1)(n+2)(n+3)} - \frac{1}{6}. \tag{2.23}$$

Note that

$$(-a,n) = \frac{a}{a-n}(1-a,n), \quad (a-1,n) = \frac{a-1}{a+n-1}(a,n).$$

Combining this, the similar computations as above yields

$$\kappa_1(n) = \frac{1}{6a} - \frac{(a+n)\left[6n^3 + 6(6-a^2)n^2 + 3\delta_4 n + \delta_5\right]W_n^2}{a(a+3)\delta_0\delta_3(n+1)(n+2)(n+3)}, \tag{2.24}$$

$$\kappa_0(n) = \frac{\left[24n^4 + 24(4+\delta_1)n^3 + 12\delta_6 n^2 + 4\delta_7 n + \delta_8\right]W_n^2}{\delta_0^2\delta_3(\delta_1+10)(n+1)(n+2)(n+3)} - \frac{1}{6\delta_0}. \tag{2.25}$$

By substituting (2.23)–(2.25) into (2.21) and simplifying, we obtain the recurrence relation (2.19). □

LEMMA 2.4. *For $a \in (0,1/2]$, we define*

$$\xi_1 = \frac{1}{1-a^2}, \quad \xi_2 = \frac{1}{3a(1-a)}, \quad \xi_3 = \frac{1}{2a} - \frac{1}{20} + \frac{5a^2}{4} + \frac{27a^4}{20} \tag{2.26}$$

*and*

$$J_1(x) = -\delta_0\delta_3(\delta_3 - \delta_0 + 8)x^2 + 5(12\delta_7 - \delta_8 - 108\delta_3 + 240\delta_0 + 888)x$$
$$- 10(12\delta_3 - \delta_7 - 30\delta_0 - 102),$$

$$J_2(x) = -2\delta_5(\delta_0 + 3a)(\delta_3 + 18)x^2 + 6[\delta_0(5\delta_7 - 3\delta_8 - 65\delta_3 + 210\delta_0 + 132) + 720]x$$
$$- 15(16\delta_7 - 3\delta_8 - 96\delta_3 + 144\delta_0 + 720),$$

$$J_3(x) = -2\delta_0\delta_3(\delta_3 + 24\delta_0 + 228)(\delta_3 + \delta_0 + 30)x^2$$
$$+ 14(\delta_0 + 6)[1440 + (12 - \delta_3)(2\delta_7 + 9\delta_3 - 51\delta_0 - 330)]x$$
$$- 21[2880 - \delta_0^2(4\delta_7 + 57\delta_3 + 452\delta_1 + 1584)].$$

*Then $J_1(x) > 0$ for $\xi_1 \leqslant x \leqslant \xi_2$ and $J_j(x) > 0$ $(j = 2,3)$ for $\xi_1 \leqslant x \leqslant \xi_3$.*

*Proof.* Clearly, $J_j(x)$ $(j = 1,2,3)$ can be regarded as a quadratic function in terms of $x$. It is easy to verify that $\delta_3 - \delta_0 = 11 + a^4 + (1-a)(1+8a+2a^2) > 0$ for $a \in (0,1/2]$. This in conjunction with $\delta_j > 0$ $(j = 0,1,2,\cdots,8)$ yields that the graph of $J_j(x)$ $(j = 1,2,3)$ is a parabola with opening downwards. In order to prove Lemma 2.4, it suffices to show that $J_j(\xi_1) > 0$ $(j = 1,2,3)$, $J_1(\xi_2) > 0$ and $J_j(\xi_3) > 0$ $(j = 2,3)$.

We now illustrate all the computations as follows. For $a \in (0,1/2]$,

◇ $J_1(\xi_1) = 60 - 120a - 14a^2 + 30a^3 + 12a^4 + 5a^5 - 4a^6 > 60 - 120a - 14a^2 + 28a^3 = 2(1-2a)(30 - 7a^2) > 0.$

◇ $J_2(\xi_1) = 2160 - 4320a - 396a^2 + 732a^3 + 463a^4 + 534a^5 - 236a^6 - 78a^7 + 25a^8 > 2160 - 4320a - 396a^2 + 732a^3 + 120a^4 = 12(1-2a)(180 - 33a^2 - 5a^3) > 0.$

$\diamond$   $J_3(\xi_1) = 60480 - 120960a - 7236a^2 + 13356a^3 + 10343a^4 + 23590a^5 - 7120a^6 - 4928a^7 + 1187a^8 + 238a^9 - 54a^{10} > 60480 - 120960a - 7236a^2 + 13356a^3 + 2232a^4 = 36(1 - 2a)(1680 - 201a^2 - 31a^3) > 0.$

$\diamond$   $9\delta_0 J_1(\xi_2) = 120 - 424a + 156a^2 + 851a^3 - 1139a^4 + 649a^5 + 129a^6 - 296a^7 + 74a^8 > 120 - 424a + 156a^2 + 851a^3 - 1139a^4 + 570a^5 = (1 - 2a)\big[169a^4 + (1 - 2a)(120 + 56a - 100a^2 + 227a^3)\big] > 0.$

$\diamond$   $200 J_2(\xi_3) = 36000 - 7680a - 386272a^2 + 378296a^3 + 409734a^4 - 366940a^5 + 845847a^6 - 1772566a^7 - 826469a^8 + 3525940a^9 + 486664a^{10} + 1094488a^{11} - 74958a^{12} - 1515440a^{13} - 58866a^{14} + 510108a^{15} - 7078a^{16} - 79380a^{17} + 8127a^{18} + 4374a^{19} - 729a^{20} > \mu_1(a) + a^{11}\mu_2(a)$, where

$$\mu_1(a) = 36000 - 7680a - 386272a^2 + 378296a^3 + 409734a^4 - 366940a^5$$
$$+ 845847a^6 - 1772566a^7 - 826469a^8 + 3525940a^9 + 486664a^{10},$$

$$\mu_2(a) = 1094488 - 74958a - 1515440a^2 - 58866a^3.$$

Firstly, we clearly see that $\mu_2(a) \geqslant \mu_2(1/2) = 2683163/4 > 0$ for $a \in (0, 1/2]$. Secondly, we have $\mu_1(a) = (1 - 3a)(36000 + 100320a - 85312a^2 + 122360a^3 + 776814a^4 + 1963502a^5 + 6736353a^6 + 18436493a^7 + 54483010a^8 + 166974970a^9) + 501411574a^{10} > 0$ for $a \in (0, 1/3)$. Further, we can get

$$\mu_1(a) = \frac{(21839039a - 10071525)^2 + 10716873512}{1397698496} + \frac{(2a - 1)^2}{256}\left[\frac{498483721}{81}\right.$$
$$+ \frac{1996055056}{27}\left(a - \frac{1}{3}\right) + \frac{23683928}{3}\left(a - \frac{1}{3}\right)^2$$
$$+ \frac{1521241024}{3}\left(a - \frac{1}{3}\right)^3 + 1032525376\left(a - \frac{1}{3}\right)^4\right]$$
$$+ \frac{(2a - 1)^6}{256}\big(14280147 + 70663440a + 49225348a^2 + 19943728a^3$$
$$+ 1946656a^4\big) > 0$$

for $a \in (1/3, 1/2]$. This gives $J_2(\xi_3) > 0$ for $a \in (0, 1/2]$.

$\diamond$   $200 J_3(\xi_3) = 1036800 - 691200a - 9013248a^2 + 10067704a^3 + 7357132a^4 - 7622362a^5 + 26565609a^6 - 52257599a^7 - 21751362a^8 + 98310717a^9 + 11233505a^{10} + 32847512a^{11} - 2336494a^{12} - 42823906a^{13} - 1215456a^{14} + 13860122a^{15} + 121986a^{16} - 2271646a^{17} + 115861a^{18} + 179361a^{19} - 19062a^{20} - 5103a^{21} + 729a^{22} > \mu_3(a) + a^{11}\mu_4(a)$, where

$$\mu_3(a) = 1036800 - 691200a - 9013248a^2 + 10067704a^3 + 7357132a^4$$
$$- 7622362a^5 + 26565609a^6 - 52257599a^7 - 21751362a^8$$
$$+ 98310717a^9 + 11233505a^{10},$$

$$\mu_4(a) = 32847512 - 2336494a - 42823906a^2 - 1215456a^3.$$

We clearly see that $\mu_4(a) \geqslant \mu_4(1/2) = 41642713/2$ for $a \in (0, 1/2]$. Moreover,

$$\mu_3(a) = (2 - 5a)\left[518400 + \frac{1282928a}{25} + \frac{4a}{25}(2 - 5a)(2809634 + 365885a)\right.$$
$$+ 2946796a^4 + 3555809a^5 + 22172327a^6 + 29302018a^7$$
$$\left. + 62379364a^8\right] + 410207537a^9 + 11233505a^{10} > 0$$

for $a \in (0, 2/5]$ and

$$\mu_3(a) = \frac{32\left[19088336886739633228 + (95843786365a - 48734653064)^2\right]}{37438979048828125}$$
$$+ \frac{318518457432}{15625}t^3 + \frac{113566994684}{3125}t^4 + \frac{92932230806}{625}t^5$$
$$+ \frac{46463495939}{125}t^6 + \frac{13267027273}{25}t^7 + \frac{2065242276}{5}t^8$$
$$+ 143244737t^9 + 11233505t^{10} > 0$$

for $a \in (2/5, 1/2]$, where $t = a - 2/5$. This gives $J_3(\xi_3) > 0$ for $a \in (0, 1/2]$. $\quad\square$

LEMMA 2.5. *For $a \in (0, 1/2]$, define*

$$f(x) = \frac{\delta_0}{6}\left[2\left(\delta_3 F_4 - 6\delta_1 F_3\right)\log(c/\sqrt{1-x}) + 3\left(\delta_1 F_3 - 2F_2\right)\right].$$

*Then $f(x) > 0$ for $x \in (0, 1)$.*

*Proof.* For $k \in \mathbb{N}$, we first prove $F_{k+1}(x)/F_k(x)$ is strictly decreasing on $(0, 1)$. Making use of the power series expansion, one has

$$\frac{F_{k+1}(x)}{F_k(x)} = \frac{\sum_{n=0}^{\infty} \frac{(a,n)(1-a,n)}{(k+1,n)n!}x^n}{\sum_{n=0}^{\infty} \frac{(a,n)(1-a,n)}{(k,n)n!}x^n}.$$

This in conjunction with the monotonicity criterion for the ratio of power series [58, Theorem 2.1] yields that the monotonicity of $F_{k+1}(x)/F_k(x)$ follows easily from the sequence

$$\{\beta_n\}_{n=0}^{\infty} = \left\{\frac{(k,n)}{(k+1,n)}\right\}_{n=0}^{\infty} = \left\{\frac{k}{k+n}\right\}_{n=0}^{\infty}.$$

Note that

$$\delta_3 F_4(1) = 6\delta_1 F_3(1) = 12F_2(1) = \frac{12\sin(a\pi)}{\delta_0 \pi}.$$

Combining this with the monotonicity of $F_{k+1}/F_k$, we conclude that

$$\frac{\delta_3 F_4(x)}{6\delta_1 F_3(x)} > \frac{\delta_3 F_4(1)}{6\delta_1 F_3(1)} = 1 \quad \text{and} \quad \frac{\delta_1 F_3(x)}{2F_2(x)} > \frac{\delta_1 F_3(1)}{2F_2(1)} = 1.$$

In other words, $\delta_3 F_4 - 6\delta_1 F_3 > 0$ and $\delta_1 F_3 - 2F_2 > 0$. This gives the desired result. $\quad\square$

LEMMA 2.6. *For $a \in (0,1/2]$ and $1/(1-a^2) \leqslant \log c \leqslant R(a)/2 + 4/3 - 2\log 2$, define*

$$\Phi(x) = f(x)g(x) - h(x),$$

*where $f(x)$ is defined as in Lemma* 2.5 *and*

$$g(x) = \frac{2\log(c/\sqrt{1-x})}{1 - 2\log(c/\sqrt{1-x})}, \quad h(x) = \delta_2 F_3 \log(c/\sqrt{1-x}) - F_1.$$

*Then $\Phi(x) > 0$ for $x \in (0,1)$.*

*Proof.* It is clear that $\log(c/\sqrt{1-x})$ is strictly increasing on $(0,1)$, which gives

$$1 - 2\log(c/\sqrt{1-x}) < 1 - 2\log c \leqslant 1 - \frac{2}{1-a^2} = -\frac{1+a^2}{1-a^2} < 0 \qquad (2.27)$$

and we keep in mind of $2\log c - 1 > 0$ in what follows.

Combining with (2.27), we clearly see that $g(x) = 2/[1/\log(c/\sqrt{1-x}) - 2]$ is strictly increasing on $(0,1)$, which yields

$$g(x) = \frac{2\log(c/\sqrt{1-x})}{1 - 2\log(c/\sqrt{1-x})} > -\frac{2\log c}{2\log c - 1} \qquad (2.28)$$

for $x \in (0,1)$.

From Lemma 2.5 and (2.28), we clearly see that

$$\begin{aligned}
\Phi(x) &> -\frac{2\log c}{2\log c - 1} f(x) - h(x) \\
&= F_1 - 3\sigma_0(\delta_1 F_3 - 2F_2) - [2\delta_3\sigma_0 F_4 + (\delta_2 - 12\delta_1\sigma_0)F_3]\log(c/\sqrt{1-x}) \\
&:= \widehat{\Phi}(x).
\end{aligned} \qquad (2.29)$$

Making use of (2.10), $\widehat{\Phi}(x)$ can be rewritten, in terms of power series, as follows

$$\begin{aligned}
\widehat{\Phi}(x) &= \sum_{n=0}^{\infty} \frac{(n+1)(n+2) + 6\sigma_0(n+2) - 6\delta_1\sigma_0}{(n+1)(n+2)} W_n^2 x^n \\
&\quad - \left[\sum_{n=0}^{\infty} \frac{12\delta_3\sigma_0 + 2(\delta_2 - 12\delta_1\sigma_0)(n+3)}{(n+1)(n+2)(n+3)} W_n^2 x^n\right] \left(\log c + \frac{x}{2}\sum_{n=0}^{\infty} \frac{x^n}{n+1}\right) \\
&= \sum_{n=0}^{\infty} \frac{n^3 + 6\tau_0 n^2 + \tau_1 n + \tau_2}{(n+1)(n+2)(n+3)} W_n^2 x^n + \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \left[\frac{(\tau_3 k + \tau_4)W_k^2}{(k+1)(k+2)(k+3)} \cdot \frac{1}{n-k}\right] x^n \\
&= \frac{\tau_2}{6} + \sum_{n=1}^{\infty} A_n x^n,
\end{aligned} \qquad (2.30)$$

where

$$A_n = \frac{n^3 + 6\tau_0 n^2 + \tau_1 n + \tau_2}{(n+1)(n+2)(n+3)} W_n^2 + \beta_n$$

and $\beta_n$ is defined as in Lemma 2.3.

Let $\lambda_n$ be defined as in (2.20). Lemma 2.3 and (2.12) enable us to know that

$$
\begin{aligned}
&A_{n+1} - \lambda_n A_n \\
&= \frac{(n+1)^3 + 6\tau_0(n+1)^2 + \tau_1(n+1) + \tau_2}{(n+2)(n+3)(n+4)} \cdot \frac{(a+n)(1-a+n)}{(n+1)^2} W_n^2 \\
&\quad - \frac{(\tau_3 n + \tau_3 + \tau_4)(a+n)(1-a+n)}{(\tau_3 n + \tau_4)(n+1)(n+4)} \cdot \frac{n^3 + 6\tau_0 n^2 + \tau_1 n + \tau_2}{(n+1)(n+2)(n+3)} W_n^2 + \beta_{n+1} - \lambda_n \beta_n \\
&= \frac{\delta_0 \delta_1 (\varepsilon_0 + \varepsilon_1 n + \varepsilon_2 n^2 + \varepsilon_3 n^3 + \varepsilon_4 n^4)}{(2\log c - 1)^2 (\tau_3 n + \tau_4)(n+1)^2(n+2)(n+3)(n+4)} W_n^2,
\end{aligned}
\tag{2.31}
$$

where

$$
\varepsilon_0 = 4(5\delta_7 - 51\delta_6 + 336\delta_0 + 954)\log^2 c - 12(4\delta_3 - 17\delta_0 - 42)\log c + 12(3\delta_0 + 2),
$$

$$
\varepsilon_1 = 8(\delta_7 - 9\delta_6 + 67\delta_0 + 188)\log^2 c - 2(20\delta_6 - 147\delta_0 - 314)\log c + (45\delta_0 + 86),
$$

$$
\varepsilon_2 = 40\delta_1 \log^2 c - 2(4\delta_3 - 11\delta_0 + 8)\log c + 17(\delta_0 + 4),
$$

$$
\varepsilon_3 = 2(4\delta_1 \log^2 c + \delta_1 + 8),
$$

$$
\varepsilon_4 = 2(2\log c + 1).
$$

We now claim that $\varepsilon_j > 0$ $(j = 0, 1, 2, 3, 4)$ for $a \in (0, 1/2]$ and $1/(1-a^2) \leqslant \log c \leqslant R(a)/2 + 4/3 - 2\log 2$. It is clear that $\varepsilon_3 > 0$ and $\varepsilon_4 > 0$ and it remains to show $\varepsilon_j > 0$ for $j = 0, 1, 2$.

First we clearly see that $\varepsilon_0, \varepsilon_1$ and $\varepsilon_2$ can be regarded as a quadratic function in terms of $\log c$. All the signs of $\varepsilon_j$ for $j = 0, 1, 2$ are determined as follows.

$\diamond$ $5\delta_7 - 51\delta_6 + 336\delta_0 + 954 = (1+a)(2-a)\left[6 + 5a - 10a^3 + 19(1-a^2) + 5a^4\right] > 0$ and

$$
\begin{aligned}
&\frac{12(4\delta_3 - 17\delta_0 - 42)}{8(5\delta_7 - 51\delta_6 + 336\delta_0 + 954)} - \frac{1}{1-a^2} \\
&= -\frac{a(6 + 57a - 46a^2 - 11a^3 + 4a^4 + 2a^5)}{2\delta_0 \delta_1 (6 + 24a - 19a^2 - 10a^3 + 5a^4)} < 0
\end{aligned}
$$

for $a \in (0, 1/2]$. This is to say, the graph of $\varepsilon_0$ is a parabola with opening upwards and the symmetric axis lying left side of the range. Thus

$$
\begin{aligned}
\varepsilon_0 &> \frac{4(5\delta_7 - 51\delta_6 + 336\delta_0 + 954)}{(1-a^2)^2} - \frac{12(4\delta_3 - 17\delta_0 - 42)}{1-a^2} + 12(3\delta_0 + 2) \\
&= \frac{8a(9 - 4a - 2a^2 + a^3 - a^4)}{(1-a)^2(1+a)} > 0.
\end{aligned}
$$

$\diamond$ $\delta_7 - 9\delta_6 + 67\delta_0 + 188 = (1+a)(2-a)(13 + 6a - 5a^2 - 2a^3 + a^4) > 0$ and
$\Delta_1 = 4(20\delta_6 - 147\delta_0 - 314)^2 - 32(\delta_7 - 9\delta_6 + 67\delta_0 + 188)(45\delta_0 + 86)$
$= -4\left(2012 - 1782a^3 + 3124a - 2309a^2 + 1231a^4 - 296a^5 - 88a^6 + 160a^7 - 40a^8\right) < 0$.

◇  $\delta_1 > 0$ and $\Delta_2 = 4(4\delta_3 - 11\delta_0 + 8)^2 - 2720\delta_1(\delta_0 + 4) = -4(2304 + 1728a - 1937a^2 + 250a^3 + 279a^4 - 440a^5 + 72a^6 + 64a^7 - 16a^8) < 0$.

Furhermore, it follows from Lemma 2.2(iii) and $\log c \leqslant R(a)/2 + 4/3 - 2\log 2$ that $\log c \leqslant 1/[a(2-a)]$, which yields

$$3\tau_3 + \tau_4 = \frac{2\delta_0\delta_1}{2\log c - 1}[3 - a(1-a)\log c] \geqslant \frac{2\delta_0\delta_1}{2\log c - 1}\left[3 - \frac{a(1-a)}{a(2-a)}\right]$$

$$= \frac{2\delta_0\delta_1(5 - 2a)}{(2-a)(2\log c - 1)} > 0. \tag{2.32}$$

Combining the sign of $\varepsilon_j$ $(j = 0,1,2,3,4)$ with (2.31) and (2.32), we conclude that

$$A_{n+1} - \lambda_n A_n > 0 \tag{2.33}$$

for $n \geqslant 3$.

From Lemma 2.3 and (2.11), we clearly see that

$$A_1 = \frac{\delta_0^2(3 + \delta_1\log c)(1 - \delta_1\log c)}{6(2\log c - 1)},$$

$$A_2 = \frac{\delta_0\delta_2\left[10 - 5\delta_1\log c - \delta_1(1 + \delta_0)\log^2 c\right]}{60(2\log c - 1)},$$

$$A_3 = \frac{\delta_0\delta_2\left[210 + 45\delta_0 - 2(39\delta_1 + 9\delta_2)\log c - 2\delta_0\delta_3\log^2 c\right]}{2160(2\log c - 1)},$$

$$A_4 = \frac{\delta_0\delta_2}{60480(2\log c - 1)}\left[21(4\delta_0^2 + 57\delta_0 + 178) - 14\delta_1(2\delta_0^2 + 27\delta_0 + 75)\log c\right.$$
$$\left. + 2\delta_1(1 - \delta_0)(\delta_3 + 10\delta_1 + 40)\log^2 c\right],$$

$$A_5 = \frac{\delta_0\delta_2}{1209600(2\log c - 1)}\left[14(5\delta_0^3 + 154\delta_0^2 + 1392\delta_0 + 3648)\right.$$
$$- 4\delta_1(5\delta_0^3 + 148\delta_0^2 + 1236\delta_0 + 2748)\log c$$
$$\left. + \delta_1(2 - \delta_0)(\delta_0\delta_3 + 30\delta_3 + 180\delta_0 + 1080)\log^2 c\right].$$

We now verify $A_j < 0$ $(j = 1,2)$ and $A_5 > 0$.

◇  It follows from $\log c \geqslant 1/(1 - a^2)$ that

$$1 - \delta_1\log c \leqslant 1 - \frac{\delta_1}{1 - a^2} = -\frac{1}{1 - a} < 0$$

and

$$10 - 5\delta_1\log c - \delta_1(1 + \delta_0)\log^2 c$$

$$\leqslant 10 - \frac{5\delta_1}{1 - a^2} - \frac{\delta_1(1 + \delta_0)}{(1 - a^2)^2} = -\frac{[2(1 - 2a^3) + 3a(2 - a)]}{(a+1)(a-1)^2} < 0,$$

which yield $A_1 < 0$ and $A_2 < 0$.

◇ Since $1209600(2\log c - 1)A_5/(\delta_0\delta_2)$ can be regarded as a quadratic function of $\log c$ with the positive quadratic term coefficient,

$$
\begin{aligned}
\Delta &= 16\delta_1^2(5\delta_0^3 + 148\delta_0^2 + 1236\delta_0 + 2748)^2 \\
&\quad - 56\delta_1(2-\delta_0)(\delta_0\delta_3 + 30\delta_3 + 180\delta_0 + 1080)(5\delta_0^3 + 154\delta_0^2 + 1392\delta_0 + 3648) \\
&= -8(2-a)(1+a)\Big[8788416 + 28921248(1-a) + 5628000(1-a^2) \\
&\quad + a^3\big(28361876 + 2761704(1-a) + 26811076(1-a^2)\big) \\
&\quad + 1611886a^6 + a^7\big(4492731 + 429751(1-a) + 571125(1-a^2)\big) \\
&\quad + 57465a^{10} + a^{11}\big(25205 + 3613(1-a) + 595(1-a^2)\big) + 85a^{14}\Big] < 0,
\end{aligned}
$$

which gives $A_5 > 0$.

To this end, we divide the proof into three cases.

*Case* 1. $A_3 \geqslant 0$. From Lemma 2.2(ii) we clearly see that $1/(1-a^2) \leqslant 1/[3a(1-a)] \leqslant R(a)/2 + 4/3 - 2\log 2$. For $\log c \geqslant 1/[3a(1-a)]$, a simple calculation yields

$$
\begin{aligned}
\frac{2160(2\log c - 1)A_3}{\delta_0\delta_2} &= 210 + 45\delta_0 - 2(39\delta_1 + 9\delta_2)\log c - 2\delta_0\delta_3\log^2 c \\
&\leqslant 210 + 45\delta_0 - \frac{2(39\delta_1 + 9\delta_2)}{3a(1-a)} - \frac{2\delta_0\delta_3}{9a^2(1-a)^2} \\
&= -\frac{(1-2a)\big[218 + 274(1-2a) + 87a^2 + 872a^3\big] + 1395a^4}{9a(1-a)} < 0.
\end{aligned}
$$

In other words, $A_3 \geqslant 0$ gives $\xi_1 \leqslant \log c < \xi_2$, where $\xi_1$ and $\xi_2$ are defined as in (2.26). In this case, it follows from (2.33) that $A_n > 0$ for $n \geqslant 3$.

Therefore, from (2.30) and Lemma 2.4 we clearly see that

$$
\widehat{\Phi}(x) > \frac{\tau_2}{6} + A_1 x + A_2 x > \frac{\tau_2}{6} + A_1 + A_2 = \frac{J_1(\log c)}{60(2\log c - 1)} > 0
$$

for $x \in (0,1)$.

*Case* 2. $A_3 < 0$ and $A_4 \geqslant 0$. It follows from Lemma 2.2(i) and $4/3 - 2\log 2 + 1/20 = -0.002961\cdots < 0$ that $R(a)/2 + 4/3 - 2\log 2 < \xi_3$, where $\xi_3$ is defined as in (2.26).

In this case, we clearly see from (2.33) that $A_n > 0$ for $n \geqslant 4$. This in conjunction with (2.30), Lemma 2.4 and $\xi_1 \leqslant \log c < \xi_3$ yields

$$
\widehat{\Phi}(x) > \frac{\tau_2}{6} + A_1 x + A_2 x + A_3 x^2 > \frac{\tau_2}{6} + A_1 + A_2 + A_3 = \frac{J_2(\log c)}{2160(2\log c - 1)} > 0
$$

for $x \in (0,1)$.

*Case* 3. $A_3 < 0$ and $A_4 < 0$. It follows from $A_5 > 0$ and (2.33) that $A_n > 0$ for $n \geqslant 5$. Combining this with (2.30), Lemma 2.4 and $\xi_1 \leqslant \log c < \xi_3$, we conclude that

$$\widehat{\Phi}(x) > \frac{\tau_2}{6} + A_1 x + A_2 x + A_3 x^2 + A_4 x^3 > \frac{\tau_2}{6} + A_1 + A_2 + A_3 + A_4$$
$$= \frac{J_3(\log c)}{60480(2\log c - 1)} > 0$$

for $x \in (0,1)$.

The proof is completed from (2.29) and the sign of $\widehat{\Phi}(x)$.  □

## 3. Proof of Theorem 1.1

*Proof.* Differentiation of $\varphi(x)$ yields

$$\frac{2}{\pi}\varphi'(x) = \frac{2\delta_0 F_2 \log(c/\sqrt{1-x}) - F_1}{2(1-x)\log^2(c/\sqrt{1-x})},$$

$$\frac{2}{\pi}\varphi''(x) = \frac{\delta_2 F_3}{2(1-x)^2 \log(c/\sqrt{1-x})} - \frac{F_1 + 2\delta_0 F_2}{(1-x)^2 \log^2(c/\sqrt{1-x})}$$
$$+ \frac{F_1}{(1-x)^2 \log^3(c/\sqrt{1-x})}$$

$$= \frac{\widehat{\varphi}(x)}{2(1-x)\log^3(c/\sqrt{1-x})}, \tag{3.34}$$

where

$$\widehat{\varphi}(x) = \frac{\delta_2 F_3 \log^2(c/\sqrt{1-x}) - [F_1 + 2\delta_0 F_2]\log(c/\sqrt{1-x}) + F_1}{1-x}. \tag{3.35}$$

If $e^{1/(1-a^2)} \leqslant c \leqslant e^{R(a)/2 + 4/3 - 2\log 2}$, then it follows from Lemma 2.2(iii) that $1/(1-a^2) \leqslant \log c \leqslant 1/[a(2-a)]$, which gives

$$\widehat{\varphi}(0^+) = \delta_2 \left[\log c - \frac{1}{1-a^2}\right]\left[\log c - \frac{1}{a(2-a)}\right] \leqslant 0. \tag{3.36}$$

On the other hand, differentiating $\widehat{\varphi}(x)$ yields

$$\widehat{\varphi}'(x) = \frac{1 - 2\log(c/\sqrt{1-x})}{2(1-x)^2}\Phi(x), \tag{3.37}$$

where $\Phi(x)$ is defined as in Lemma 2.6.

Lemma 2.6, (2.27) and (3.37) enable us to know that $\widehat{\varphi}(x)$ is strictly decreasing on $(0,1)$. This in conjunction with (3.36) yields $\widehat{\varphi}(x) < 0$ for $x \in (0,1)$.

Therefore, Theorem 1.1 follows from (3.34) and $\widehat{\varphi}(x) < 0$.  □

As an application of Theorem 1.1, the properties of a concave function together with (2.13) enable us to obtain

$$\sqrt{\varphi(x) \cdot \varphi(1-x)} \leqslant \frac{\varphi(x) + \varphi(1-x)}{2} \leqslant \varphi(1/2) = \frac{\Gamma(\frac{1-a}{2})\Gamma(\frac{a}{2})\sin(a\pi)}{4\sqrt{\pi}\log(\sqrt{2}c)} :\triangleq \upsilon_a(c)$$

(3.38)

for $e^{1/(1-a^2)} \leqslant c \leqslant e^{R(a)/2+4/3-2\log2}$, which gives the following corollaries by substituting $x = r^2$ in (3.38) together with (1.6).

COROLLARY 3.1. *The inequality*

$$\frac{\mathscr{K}_a(r)}{\log(c/r')} + \frac{\mathscr{K}_a(r')}{\log(c/r)} \leqslant 2\upsilon_a(c)$$

*or equivalently,*

$$\mu_a(r) \leqslant \frac{\pi}{2\sin(\pi a)}\left[\frac{2\upsilon_a(c)\log(c/r)}{\mathscr{K}_a(r)} - \frac{\log(c/r)}{\log(c/r')}\right]$$

(3.39)

*holds for $r \in (0,1)$ with $e^{1/(1-a^2)} \leqslant c \leqslant e^{R(a)/2+4/3-2\log2}$, where $\upsilon_a(c)$ given in (3.38) is the best constant.*

COROLLARY 3.2. *The inequality*

$$\mathscr{K}_a(r)\mathscr{K}_a(r') \leqslant 2\upsilon_a^2(c)\log(c/r)\log(c/r')$$

*or equivalently,*

$$m_a(r) \leqslant \frac{4\upsilon_a^2(c)}{\pi\sin(\pi a)}r'^2\log(c/r)\log(c/r') := \mathscr{V}_a(r;c)$$

(3.40)

*holds for $r \in (0,1)$ with $e^{1/(1-a^2)} \leqslant c \leqslant e^{R(a)/2+4/3-2\log2}$, where $\upsilon_a(c)$ given in (3.38) is the best constant. In particular, the best upper bound of (3.40) with respect to $c$ in our setting is given by $\mathscr{V}_a(r;e^{R(a)/2+4/3-2\log2})$.*

*Proof.* In order to investigate the best upper bound of (3.40), it suffices to consider the monotonicity of $\mathscr{V}_a(r;c)$ with respect to $c \in [e^{1/(1-a^2)}, e^{R(a)/2+4/3-2\log2}]$.

Let

$$\mathscr{L}(c,r) = \frac{\log(c/r)\log(c/r')}{\log^2(\sqrt{2}c)},$$

$$\mathscr{Z}_1(c,r) = -\log(c/r)\log(c/\sqrt{1-r^2}) + \log(\sqrt{2}c)\left[\log(c/r) + \log(c/\sqrt{1-r^2})\right],$$

$$\mathscr{Z}_2(c,r) = (2r^2 - 1)\log(\sqrt{2}c) - 2r^2\log(c/r) + 2(1-r^2)\log(c/\sqrt{1-r^2}),$$

$$\mathscr{Z}_3(c,r) = 1 + \log(\sqrt{2}c) - \log(c/r) - \log(c/\sqrt{1-r^2}).$$

Then a simple calculation yields

$$\mathscr{Z}_1(c, 1/\sqrt{2}) = 0, \quad \mathscr{Z}_2(c, 1/\sqrt{2}) = 0, \tag{3.41}$$

$$\mathscr{Z}_3(c, 1/\sqrt{2}) = 1 - \log\sqrt{2} - \log c \leqslant -\frac{a^2}{1-a^2} - \log\sqrt{2} < 0 \tag{3.42}$$

for $a \in (0, 1/2]$ with $c \geqslant e^{1/(1-a^2)}$ and

$$\frac{\partial \mathscr{L}(c,r)}{\partial c} = \frac{\mathscr{Z}_1(c,r)}{c\log^3(\sqrt{2}c)}, \tag{3.43}$$

$$\frac{\partial \mathscr{Z}_1(c,r)}{\partial r} = \frac{\mathscr{Z}_2(c,r)}{r(1-r^2)}, \quad \frac{\partial \mathscr{Z}_2(c,r)}{\partial r} = 4r\mathscr{Z}_3(c,r), \tag{3.44}$$

$$\frac{\partial \mathscr{Z}_3(c,r)}{\partial r} = \frac{1-2r^2}{r(1-r^2)}. \tag{3.45}$$

It follows from (3.45) that $\partial\mathscr{Z}_3(c,r)/\partial r > 0$ for $r \in (0, 1/\sqrt{2})$ and $\partial\mathscr{Z}_3(c,r)/\partial r < 0$ for $r \in (1/\sqrt{2}, 1)$, which in conjunction with (3.42) yields $\mathscr{Z}_3(c,r) < 0$ for $r \in (0,1)$. Combining this with (3.41) and (3.44), we clearly see that $\mathscr{Z}_1(c,r) < 0$ for $r \in (0,1)$ and $e^{1/(1-a^2)} \leqslant c \leqslant e^{R(a)/2+4/3-2\log 2}$. It follows from (3.43) that $\mathscr{L}(c,r)$ is strictly decreasing with respect to $c$. So is $\mathscr{V}_a(r;c)$.

This completes the proof. $\quad\square$

COROLLARY 3.3. *For $a \in (0, 1/2]$, the function $(1-x)^{\iota(a)}\mathscr{K}_a(\sqrt{x})$ is strictly log-concave on $(0,1)$, where*

$$\iota(a) = \frac{1}{2}\left(2 - a^2 - \sqrt{2 - 2a^2 + a^4}\right).$$

*As a consequence, the inequality*

$$(rr')^{2\iota(a)}\mathscr{K}_a(r)\mathscr{K}_a(r') \leqslant \frac{1}{4^{\iota(a)}}\left[\mathscr{K}_a\left(1/\sqrt{2}\right)\right]^2 = \frac{[\Gamma(\frac{1-a}{2})\Gamma(\frac{a}{2})\sin(\pi a)]^2}{4^{\iota(a)+2}\pi} \tag{3.46}$$

*holds for $r \in (0,1)$.*

*Proof.* According to Theorem 1.1, the function $\mathscr{K}_a(\sqrt{x})/\log\left(e^{\frac{1}{1-a^2}}/\sqrt{1-x}\right)$ is strictly concave on $(0,1)$. Moreover,

$$\frac{d^2}{dx^2}\left[(1-x)^{\iota(a)}\log\left(e^{\frac{1}{1-a^2}}/\sqrt{1-x}\right)\right] = \frac{1}{2}\iota(a)[1-\iota(a)](1-x)^{\iota(a)-2}\log(1-x) < 0$$

since $0 < \iota(a) < 1$. This implies that $(1-x)^{\iota(a)}\log\left(e^{\frac{1}{1-a^2}}/\sqrt{1-x}\right)$ is positive and concave on $(0,1)$, and is also log-concave on $(0,1)$.

Therefore, the function

$$(1-x)^{t(a)}\mathcal{K}_a(\sqrt{x}) = (1-x)^{t(a)}\log\left(e^{\frac{1}{1-a^2}}/\sqrt{1-x}\right)\cdot\frac{\mathcal{K}_a(r)}{\log\left(e^{\frac{1}{1-a^2}}/\sqrt{1-x}\right)}$$

is log-concave on $(0,1)$, which yields

$$(1-x)^{t(a)}\mathcal{K}_a(\sqrt{x})\cdot x^{t(a)}\mathcal{K}_a(\sqrt{1-x}) \leqslant \frac{1}{4^{t(a)}}\left[\mathcal{K}_a\left(1/\sqrt{2}\right)\right]^2. \qquad (3.47)$$

By substituting $x = r^2$ into (3.47), the desired inequality (3.46) is obtained. $\qquad\square$

From Theorem 1.1 and the monotonicity criterion in [8, Theorem 1.25], we clearly see that the functions

$$\frac{\varphi(x)-\varphi(0)}{x} \quad \text{and} \quad \frac{\varphi(1)-\varphi(x)}{1-x}$$

are strictly decreasing on $(0,1)$, which gives the following corollary.

COROLLARY 3.4. *For $e^{1/(1-a^2)} \leqslant c \leqslant e^{R(a)/2+4/3-2\log 2}$, the functions*

$$r^2 \to \frac{1}{r^2}\left[\frac{\mathcal{K}_a(r)}{\log(c/r')} - \frac{\pi}{2\log c}\right] \quad \text{and} \quad r^2 \to \frac{1}{r'^2}\left[\sin(\pi a) - \frac{\mathcal{K}_a(r)}{\log(c/r')}\right]$$

*are strictly decreasing on $(0,1)$. As a consequence, the double inequality*

$$\sin(\pi a)+\left[\frac{\pi}{2\log c} - \sin(\pi a)\right]r^2 < \frac{\mathcal{K}_a(r)}{\log(c/r')} < \frac{\pi}{2\log c} + \frac{\pi[2a(1-a)\log c - 1]}{4\log^2 c}(1-r'^2)$$

$$(3.48)$$

*holds for $r \in (0,1)$, where the lower and upper bounds are sharp.*

REMARK 3.5. Inequalities (3.48) give a new sharp bounds for $\mathcal{K}_a(r)$ and other inequalities given in [44].

As shown in the proof of Lemma 2.2(iii), we clearly see that there exists $a_1 \in (0,1/2)$ such that $R(a)/2 - 1/[a(2-a)]$ is strictly decreasing on $(0,a_1)$ and strictly increasing on $(a_1,1/2)$. Moreover,

$$\lim_{a\to 0^+}\left[\frac{R(a)}{2} - \frac{1}{a(2-a)}\right] = \frac{1}{4} - \sum_{k=1}^{\infty}\frac{1}{2k(k+1)} = -\frac{1}{4}$$

and

$$\lim_{a\to\frac{1}{2}}\left[\frac{R(a)}{2} - \frac{1}{a(2-a)}\right] = \log 4 - \frac{4}{3} = 0.05296\cdots,$$

which gives the following lemma.

LEMMA 3.6. *For $a \in (0, 1/2]$, there exists $a_0 \in (0, 1/2]$ such that $R(a)/2 < 1/[a(2-a)]$ for $(0, a_0)$ and $R(a)/2 > 1/[a(2-a)]$ for $(a_0, 2]$. In particular, $a_0 = 0.46787\cdots$ as an approximation.*

Note that

$$\delta_2 F_3(1^-) = 2\delta_0 F_2(1^-) = \frac{2\sin(\pi a)}{\pi}$$

and

$$\lim_{x \to 1^-} \frac{\delta_2 F_3(x) - 2\sin(\pi a)/\pi}{(1-x)\log(1-x)} = -\lim_{x \to 1^-} \frac{\delta_0 \delta_2 F(1+a, 2-a; 4; x)}{3[1 + \log(1-x)]} = 0.$$

Combining this with (1.2), we have the following asymptotic expansions

$$\delta_2 F_3(x)\log(c/\sqrt{1-x}) - F_1(x) = \frac{\sin(a\pi)}{\pi}\left[2\log c - R(a)\right] + o_1[(1-x)\log^2(1-x)] \tag{3.49}$$

and

$$2\delta_0 F_2(x)\log(c/\sqrt{1-x}) - F_1(x) = \frac{\sin(a\pi)}{\pi}\left[2\log c - R(a)\right] + o_2[(1-x)\log^2(1-x)] \tag{3.50}$$

as $x \to 1^-$.

Let $\widehat{\varphi}(x)$ be defined as in (3.35). Then it follows from (3.49) and (3.50) that

$$
\begin{aligned}
&(1-x)\widehat{\varphi}(x) \\
&= \left[\delta_2 F_3(x)\log(c/\sqrt{1-x}) - F_1(x)\right]\log(c/\sqrt{1-x}) - \left[2\delta_0 F_2(x)\log(c/\sqrt{1-x}) - F_1(x)\right] \\
&\sim \frac{\sin(a\pi)}{\pi}\left[2\log c - R(a)\right]\left[\log(c/\sqrt{1-x}) - 1\right]. \qquad (x \to 1^-)
\end{aligned}
\tag{3.51}
$$

From the proof of Theorem 1.1, we clearly see that $\varphi(x)$ is concave on $(0, 1)$ if and only if $\widehat{\varphi}(x) \leqslant 0$. The necessary condition requires us to satisfy $\widehat{\varphi}(0^+) \leqslant 0$ and $\widehat{\varphi}(1^-) \leqslant 0$. This in conjunction with (3.36) and (3.51) yields

$$\frac{1}{1-a^2} \leqslant \log c \leqslant \frac{1}{a(2-a)} \quad \text{and} \quad \log c \leqslant \frac{R(a)}{2},$$

which is equivalent to

$$\frac{1}{1-a^2} \leqslant \log c \leqslant \min_{a \in (0, 1/2]}\left\{\frac{1}{a(2-a)}, \frac{R(a)}{2}\right\} \tag{3.52}$$

follows from Lemma 3.6.

To this end, numerical computations and (3.52) allow us to pose the following conjecture

CONJECTURE. *For $a \in (0, 1/2$ and $c \geqslant 1$, the function*

$$\frac{\mathscr{K}_a(\sqrt{x})}{\log(c/\sqrt{1-x})}$$

is concave on $(0,1)$ if and only if $\frac{1}{1-a^2} \leqslant \log c \leqslant \min_{a \in (0,1/2]} \left\{ \frac{1}{a(2-a)}, \frac{R(a)}{2} \right\}$.

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