Journal of Mathematical Inequalities
Volume 15, Number 2 (2021), 725–737
doi:10.7153/jmi-2021-15-51

A NOTE ON THE COMPLETE CONSISTENCY FOR THE WEIGHTED LINEAR ESTIMATOR OF NONPARAMETRIC REGRESSION MODELS

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(*Communicated by T. Burić*)

Abstract. In this paper, we study the complete consistency for the estimator of nonparametric regression models based on extended negatively dependent random errors by using the exponential inequalities and the truncation method. In particular, if $E|X|^{1+p} < \infty$ for some $p > 1$, then the result also holds, which improves the corresponding one in the literature. As an application, the complete consistency for the nearest neighbor estimator is obtained. Finally, the simulation study is provided to verify the validity of the theoretical result.

1. Introduction

Consider the following nonparametric regression model:

$$Y_{ni} = g(x_{ni}) + \varepsilon_{ni}, \quad i = 1,2,\ldots,n, \ n \geqslant 1, \tag{1.1}$$

where $g(\cdot)$ is an unknown regression function defined on A, $A \subset \mathbb{R}^d$ is a given compact set for some $d \geqslant 1$, x_{ni} are known fixed design points and ε_{ni} are random errors. As an estimator of $g(\cdot)$, the following general weighted linear regression estimator was proposed:

$$g_n(x) = \sum_{i=1}^{n} W_{ni}(x) Y_{ni}, \quad x \in A \subset \mathbb{R}^d, \tag{1.2}$$

where $W_{ni}(x) = W_{ni}(x; x_{n1}, x_{n2}, \ldots, x_{nn})$, $i = 1,2,\ldots,n$ are the weight functions.

The above estimator was first proposed by Stone (1977) and adapted by Georgiev (1985) to the fixed design case, and then constantly studied by many authors. For instance, when ε_{ni} are assumed to be independent, the consistency and asymptotic normality have been studied by Georgiev and Greblicki (1986), Georgiev (1988) and Müller (1987) among others. When ε_{ni} are assumed to be dependent, the consistency and asymptotic normality have also been studied by many authors in recent years. For

Mathematics subject classification (2020): 62G20.

Keywords and phrases: Nonparametric regression model, complete consistency, END random variables, nearest neighbor estimator.

Supported by the National Natural Science Foundation of China (11671012), the Provincial Natural Science Research Project of Anhui Colleges (KJ2019A0003), the Students Innovative Training Project of Anhui University (S201910357345) and the Project on Reserve Candidates for Academic and Technical Leaders of Anhui Province (2017H123).

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example, Roussas (1989) discussed strong consistency and quadratic mean consistency for $g_n(x)$ under mixing conditions; Fan (1990) established some asymptotic properties for the estimator based on L_q-mixingale sequence for some $1 \leqslant q \leqslant 2$; Roussas et al. (1992) established the asymptotic normality of $g_n(x)$ based on the strictly stationary and strong mixing errors; Tran et al. (1996) discussed again the asymptotic normality of $g_n(x)$ under weakly stationary linear processes based on a martingale difference sequence; Liang and Jing (2005) proved some consistencies and asymptotic normality for the estimator $g_n(x)$ based on negatively associated (NA, for short) errors; Shen et al. (2015) established the Rosenthal-type inequality for negatively superadditive dependent (NSD, for short) random variables and applied it to nonparametric regression models; Wang et al. (2015) and Wu et al. (2017) obtained the complete consistency for the weighted estimator based on extended negatively dependent (END, for short) and ρ^*-mixing errors, respectively; Shen (2016) presented some results on complete convergence for weighted sums of END random variables, and gave its application to nonparametric regression models; Yang et al. (2018) provided the complete consistency and the convergence rate for the estimator based on END errors; Chen et al. (2019) established the complete consistency for the weighted estimator based on asymptotically negatively associated (ANA, for short) random errors; Yan (2019) provided some sufficient conditions for the complete convergence for the maximum of weighted sums of END random variables and gave some applications to the nonparametric regression models, and so forth.

Now let us recall the concepts of stochastic domination and END random variables.

DEFINITION 1.1. An array $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_{ni}| > x) \leqslant CP(|X| > x)$$

for all $x \geqslant 0$, $i \geqslant 1$ and $n \geqslant 1$.

DEFINITION 1.2. A finite collection of random variables $\{X_i, 1 \leqslant i \leqslant n\}$ is said to be END if there exists a constant $M > 0$ such that both

$$P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n) \leqslant M \prod_{i=1}^{n} P(X_i > x_i)$$

and

$$P(X_1 \leqslant x_1, X_2 \leqslant x_2, \ldots, X_n \leqslant x_n) \leqslant M \prod_{i=1}^{n} P(X_i \leqslant x_i)$$

hold for all real numbers x_1, x_2, ..., x_n. An infinite sequence $\{X_n, n \geqslant 1\}$ is said to be END if every finite subcollection is END.

An array $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ of random variables is called END if for every $n \geqslant 1$, $\{X_{ni}, 1 \leqslant i \leqslant n\}$ is a sequence of END random variables.

The above concept of END sequence was introduced by Liu (2009). If $M = 1$, then the notion of END random variables reduces to the well-known notion of so-called negatively dependent (ND, for short) random variables, which was introduced by Lehmann (1966). It is not that the concept of the END seems to be a simple generalization of the concept of negative correlation, but the extended negative correlation structure is actually more comprehensive. As Liu (2009) mentioned, the terminal structure can reflect not only the negative correlation structure, but also the positive correlation structure (the inequality defined by the ND random variable also has the opposite direction), and some extensions. Since the concept of END sequence was proposed by Liu (2009), many authors were devoted to studying the probability limit theorems and statistical large sample properties for END random variables, including the probability inequalities, moment inequalities and applications. For example, Liu (2009) obtained the precise large deviations for dependent random variables with heavy tails; Chen et al. (2010) proved the strong law of large numbers for END random variables and gave its applications to risk theory and renewal theory; Wu and Guan (2012), Qiu et al. (2013) and Hu et al. (2015) studied complete convergence for weighted sums and arrays of rowwise END random variables; Wu et al. (2014) established the complete convergence and complete moment convergence for arrays of rowwise END random variables; Wang et al. (2015) studied the complete consistency for the estimator of nonparametric regression models based on END errors; Yang et al. (2018) established the complete consistency for the estimator based on END errors, and so on.

In this paper, the complete consistency for the weighted estimator (1.2) in the model (1.1) based on extended negatively dependent errors is investigated under some mild conditions, especially the moment condition $E|X|^{1+p} < \infty$ for some $p \geqslant 1$. This moment condition is weaker than the corresponding one of Wang et al. (2015), which needs the moment condition $E|X|^{2p} < \infty$ for some $p \geqslant 1$.

The work is organized as follows. In next section, we list our preliminaries. The main results and the simulation are presented in Section 3. In Section 4, we provide the proofs of main results.

Throughout the paper, C denotes a positive constant not depending on n, which may be different in various places. Let $\lfloor x \rfloor$ denote the integer of x and $I(A)$ be the indicator function of set A. Denote $\log x = \log \max(x, e)$, $x^+ = xI(x \geqslant 0)$ and $x^- = -xI(x \leqslant 0)$. Unless other specified, we assume throughout the paper that $g_n(x)$ is defined by (1.2). For any function $g(x)$, we assume $c(g)$ to denote all continuity points of the function g on A. The norm $||x||$ means the Euclidean norm.

2. Preliminaries

In this section, we will present some important lemmas which will be used to prove the main results. The first one is a basic property for END random variables, which was presented by Liu (2009).

LEMMA 2.1. *Let random variables X_1, X_2, $\cdots$, X_n be END.*

(i) If f_1, f_2, $\cdots$, f_n are all nondecreasing (or nonincreasing) functions, then random variables $f_1(X_1)$, $f_2(X_2)$, $\ldots$, $f_n(X_n)$ are END.

(ii) For each $n \geqslant 1$, there exists a constant $\lambda > 0$ such that

$$E\left(\prod_{j=1}^{n} X_j^+\right) \leqslant \lambda \prod_{j=1}^{n} EX_j^+. \tag{2.1}$$

The following lemma can be referred to Wang et al. (2015)

LEMMA 2.2. *Let $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of rowwise END random variables with $EX_{ni} = 0$ and $\{b_n, n \geqslant 1\}$ be a sequence of positive constants. Suppose that*

(i) $\max_{1\leqslant i\leqslant n} |X_{ni}| = O(b_n)$ *a.s.,*
(ii) $\sum_{i=1}^{n} EX_{ni}^2 = o(b_n)$,
(iii) $\sum_{n=1}^{\infty} e^{-\alpha/b_n} < \infty$ *for some* $\alpha > 0$.
Then $\sum_{i=1}^{n} X_{ni}$ converges completely to zero.

The last one is a basic property for stochastic domination. For the proof, one can refer to Wu (2006, 2010), or Shen (2013).

LEMMA 2.3. *Let $\{X_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following two statements hold:*

$$E|X_{ni}|^{\alpha} I(X_{ni} \leqslant b) \leqslant C_1[E|X|^{\alpha} I(|X| \leqslant b) + b^{\alpha} P(|X| > b)], \tag{2.2}$$

and

$$E|X_{ni}|^{\alpha} I(X_{ni} > b) \leqslant C_2 E|X|^{\alpha} I(|X| > b), \tag{2.3}$$

where C_1 and C_2 are positive constants. Consequently, $E|X_{ni}|^{\alpha} \leqslant CE|X|^{\alpha}$.

3. Main results and simulation

3.1. Main results

First, we give the following assumptions on weight functions $W_{ni}(x)$ for any fixed design point $x \in A$:
(A_1) $\sum_{i=1}^{n} W_{ni}(x) \to 1$ as $n \to \infty$;
(A_2) $\sum_{i=1}^{n} |W_{ni}(x)| \leqslant C < \infty$ for all n;
(A_3) $\sum_{i=1}^{n} |W_{ni}(x)| \cdot |g(x_{ni}) - g(x)| I(||x_{ni} - x|| \geqslant a) \to 0$ as $n \to \infty$ for all $a > 0$.

Let $\{\varepsilon_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of rowwise END random variables with the same constant $M > 0$ in each row. Based on the assumptions above, we can get the following complete consistency for the nonparametric regression estimator $g_n(x)$.

THEOREM 3.1. *Let $\{\varepsilon_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of rowwise END random variables, which is stochastically by a random variable X with $E|X|^{1+p} < \infty$ for some $p \geqslant 1$ and $E\varepsilon_{ni} = 0$. Assume that $(A_1) - (A_3)$ hold. If*

$$\max_{1\leqslant i\leqslant n} |W_{ni}(x)| = O(n^{-1/p}), \tag{3.1}$$

then for any $x \in c(g)$,

$$g_n(x) \to g(x) \ \textit{completely, as} \ n \to \infty. \tag{3.2}$$

REMARK 3.2. In the case of $E|X|^{2p} < \infty$ for some $p \geqslant 1$, Wang et al. (2015) got the same result as Theorem 3.1. Hence our result improves the corresponding one of Wang et al. (2015).

As an application of Theorem 3.1, we give the complete consistency for the nearest neighbour estimator of $g(x)$. Put $A = [0,1]$ and take $x_{ni} = i/n, i = 1,2,\ldots, n$. For any $x \in A$, we rewrite

$$|x_{n1} - x|, |x_{n2} - x|, \ldots, |x_{nn} - x|$$

as follows:

$$|x^{(n)}_{R_1(x)} - x| \leqslant |x^{(n)}_{R_2(x)} - x| \leqslant \cdots \leqslant |x^{(n)}_{R_n(x)} - x|, \tag{3.3}$$

if $|x_{ni} - x| = |x_{nj} - x|$, then $|x_{ni} - x|$ is permuted before $|x_{nj} - x|$ when $x_{ni} < x_{nj}$. Let $1 \leqslant k_n \leqslant n$ and define the nearest neighbor weight function as follows:

$$\widetilde{W}_{ni}(x) = \begin{cases} \frac{1}{k_n}, & \text{if } |x_{ni} - x| \leqslant |x^{(n)}_{R_{k_n}(x)} - x|, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that conditions $(A_1) - (A_3)$ and (3.1) are satisfied, where $W_{ni}(x)$ is replaced by $\widetilde{W}_{ni}(x)$.

For any $x \in [0,1]$, if follows from the definitions of $R_i(x)$ and $\widetilde{W}_{ni}(x)$ that

$$\sum_{i=1}^{n} \widetilde{W}_{ni}(x) = \sum_{i=1}^{n} \widetilde{W}_{nR_i}(x) = \sum_{i=1}^{k_n} \frac{1}{k_n} = 1, \tag{3.4}$$

and

$$\max_{1 \leqslant i \leqslant n} \widetilde{W}_{ni}(x) = \frac{1}{k_n}, \ \widetilde{W}_{ni}(x) \geqslant 0. \tag{3.5}$$

Assume that g is continuous on the compact set A, and thus $\{|g(x_{ni}) - g(x)| : 1 \leqslant i \leqslant n, n \geqslant 1\}$ is bounded on set A. Hence,

$$\begin{aligned} &\sum_{i=1}^{n} |\widetilde{W}_{ni}(x)| \cdot |g(x_{ni}) - g(x)| I(|x_{ni} - x| > a) \\ &\leqslant C \sum_{i=1}^{n} \frac{(x_{ni} - x)^2 |\widetilde{W}_{ni}(x)|}{a^2} = C \sum_{i=1}^{k_n} \frac{(x^{(n)}_{R_i(x)} - x)^2}{k_n a^2} \\ &\leqslant C \sum_{i=1}^{k_n} \frac{(i/n)^2}{k_n a^2} \leqslant C\Big(\frac{k_n}{na}\Big)^2, \ \forall a > 0. \end{aligned} \tag{3.6}$$

If we take $k_n = \lfloor n^{1/p} \rfloor$ for some $p > 1$, then the conditions $(A_1) - (A_3)$ and (3.1) are satisfied.

Based on the notations above, we can get the following result by using Theorem 3.1.

COROLLARY 3.3. *Let* $\{\varepsilon_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *be an array of rowwise END random variables with mean zero, which is stochastically dominated by a random variable* X*. Assume that* g *is continuous on the compact set* A*. If there exists some* $p > 1$ *such that* $k_n = \lfloor n^{1/p} \rfloor$ *and* $E|X|^{1+p} < \infty$*, then*

$$g_n(x) \to g(x) \ \ completely, \ as \ \ n \to \infty, \tag{3.7}$$

where $g_n(x) = \sum_{i=1}^{n} \widetilde{W}_{ni}(x) Y_{ni}$.

3.2. Simulation

The data are generated from the following model:

$$Y_{ni} = g(x_{ni}) + \varepsilon_{ni}, \ \ i = 1, 2, \ldots, n, \ \ n \geqslant 1. \tag{3.8}$$

For any fixed $n \geqslant 3$, let $(\varepsilon_{n1}, \varepsilon_{n2}, \ldots, \varepsilon_{nn}) \sim N(0, \Sigma)$, where 0 is a zero vector, and

$$\Sigma = \begin{pmatrix} 1+\theta^2 & -\theta & 0 & \ldots & 0 & 0 & 0 \\ -\theta & 1+\theta^2 & -\theta & \ldots & 0 & 0 & 0 \\ 0 & -\theta & 1+\theta & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \ldots & 1+\theta^2 & -\theta & 0 \\ 0 & 0 & 0 & \ldots & -\theta & 1+\theta^2 & -\theta \\ 0 & 0 & 0 & \ldots & 0 & -\theta & 1+\theta^2 \end{pmatrix},$$

where $0 < \theta < 1$. It is obvious that $\varepsilon_{n1}, \varepsilon_{n2}, \ldots, \varepsilon_{nn}$ generated as the above method are NA by Joag-Dev and Proschan (1983). We choose $\theta = 0.5$ and $k_n = \lfloor n^{0.48} \rfloor$ in Corollary 3.3. It's easy to check that conditions $(A_1) - (A_3)$ are satisfied. Taking the points $x = 0.2$, 0.5, 0.8 and the sample sizes n as $n = 100$, 200, 400, and then we use Matlab to compute $g_n(x)$ and $g(x)$ with $g(x) = sin(2\pi x)$ for 500 times and obtain the fitting plots of $g_n(x)$ and $g(x)$ in Figures 1-3 and the mean square errors (MSE) of $g_n(x)$ in Table 1.

Table 1: *MSE of the estimator* $g_n(x)$.

	x		
n	0.2	0.5	0.8
100	0.0454	0.0492	0.0408
200	0.0310	0.0306	0.0316
400	0.0198	0.0209	0.0230

It can be seen that the predicted and actual values fit well. Table 1 reflects that MSE of $g_n(x)$ decreases as n increases. These show a good fit of our theoretical result.

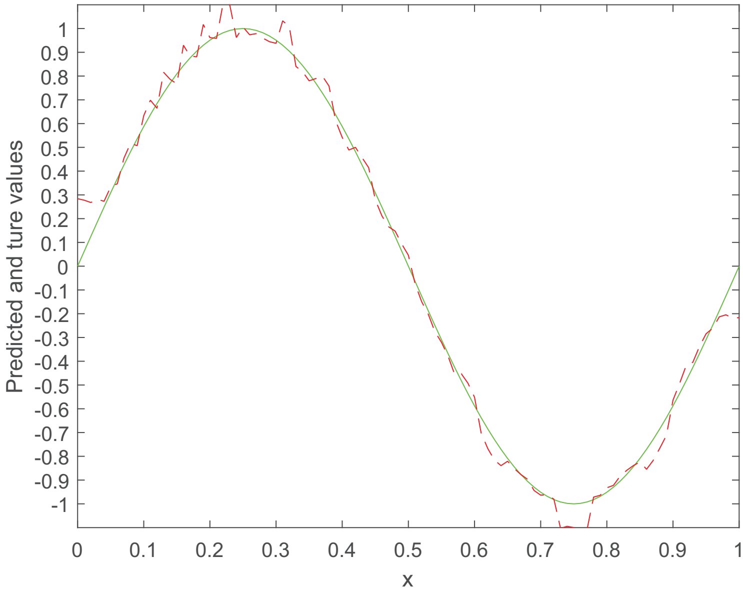

Figure 1: *fitting plots of* $g(x)$ *and* $g_n(x)$ *with* n=*100*

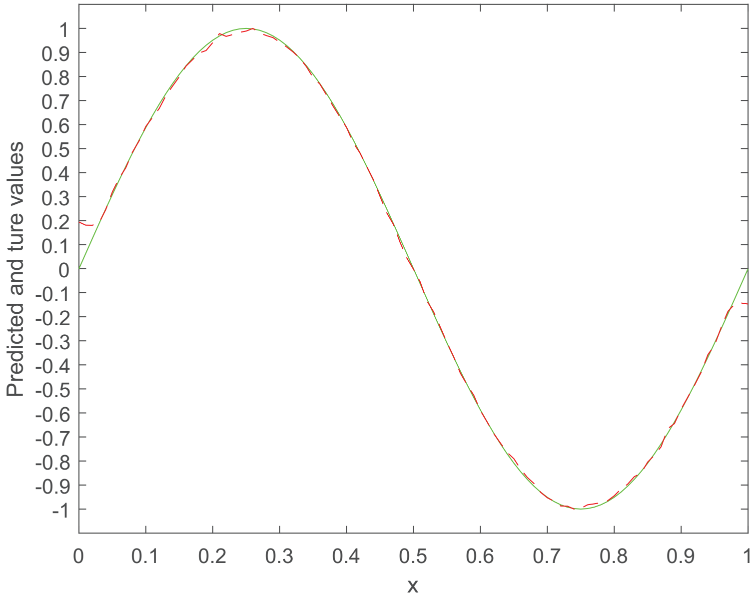

Figure 2: *fitting plots of* $g(x)$ *and* $g_n(x)$ *with* n=*200*

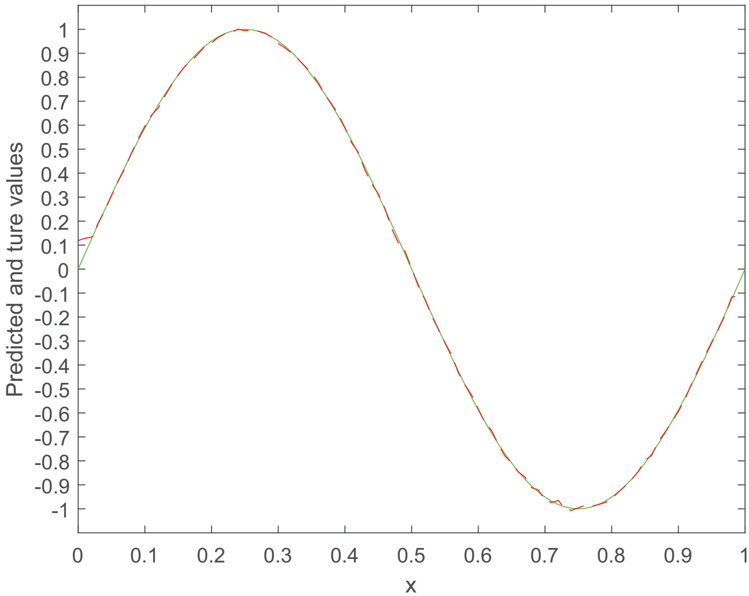

Figure 3: *fitting plots of* $g(x)$ *and* $g_n(x)$ *with* n=*400*

4. Proofs of main results

Proof of Theorem 3.1. For $x \in c(g)$ and $a > 0$, we have by equations (1.1) and (1.2) that

$$\begin{aligned} |Eg_n(x) - g(x)| \leqslant & \sum_{i=1}^{n} |W_{ni}(x)| \cdot |g(x_{ni}) - g(x)| I(\|x_{ni} - x\| \leqslant a) \\ & + \sum_{i=1}^{n} |W_{ni}(x)| \cdot |g(x_{ni}) - g(x)| I(\|x_{ni} - x\| > a) \\ & + |g(x)| \cdot \left| \sum_{i=1}^{n} W_{ni}(x) - 1 \right|. \end{aligned} \tag{4.1}$$

Since $x \in c(g)$, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|g(x^*) - g(x)| < \varepsilon$ when $\|x^* - x\| < \delta$. Setting $a \in (0, \delta)$ in (4.1), we can get that

$$\begin{aligned} |Eg_n(x) - g(x)| \leqslant & \ \varepsilon \sum_{i=1}^{n} |W_{ni}(x)| + |g(x)| \cdot \left| \sum_{i=1}^{n} W_{ni}(x) - 1 \right| \\ & + \sum_{i=1}^{n} |W_{ni}(x)| \cdot |g(x_{ni}) - g(x)| I(\|x_{ni} - x\| > a), \end{aligned} \tag{4.2}$$

which together with conditions $(A_1)-(A_3)$ yields that

$$\lim_{n\to\infty} Eg_n(x) = g(x), \ \ x \in c(g). \tag{4.3}$$

Without loss of generality, we assume that $W_{ni}(x) > 0$ in what follows. Otherwise, we will use $W_{ni}^{+}(x)$ and $W_{ni}^{-}(x)$ instead of $W_{ni}^{+}(x)$, respectively, since $W_{ni}(x) = W_{ni}^{+}(x) - W_{ni}^{-}(x)$.

In view of (4.3), in order to prove (3.2), it suffices to show that for any $x \in c(g)$,

$$g_n(x) - Eg_n(x) = \sum_{i=1}^{n} W_{ni}(x)\varepsilon_{ni}(x) \doteq \sum_{i=1}^{n} R_{ni} \to 0 \text{ completely, } as \ \ n \to \infty, \tag{4.4}$$

where $R_{ni}(x) = W_{ni}(x)\varepsilon_{ni}$. That is to say, we only need to show that for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} R_{ni}\right| > 4\varepsilon\right) < \infty. \tag{4.5}$$

By (3.1), we assume that $\max\limits_{1\leqslant i\leqslant n} W_{ni}(x) \leqslant n^{-1/p}$. For any $\varepsilon > 0$, choose $p < q < 1+p$ and some positive integer N (to be specified later), and denote that

$$X_{ni}(1) = -n^{1/q} I(\varepsilon_{ni} < -n^{1/q}) + \varepsilon_{ni} I(|\varepsilon_{ni}| \leqslant n^{1/q}) + n^{1/q} I(\varepsilon_{ni} > n^{1/q}),$$

$$X_{ni}(2) = (\varepsilon_{ni} - n^{1/q}) I(n^{1/q} < \varepsilon_{ni} \leqslant \frac{\varepsilon n^{\frac{2}{1+p}}}{N}),$$

$$X_{ni}(3) = (\varepsilon_{ni} + n^{1/q}) I\Big(\frac{-\varepsilon n^{\frac{2}{1+p}}}{N} \leqslant \varepsilon_{ni} < -n^{1/q}\Big),$$

$$X_{ni}(4) = (\varepsilon_{ni} - n^{1/q}) I\Big(\varepsilon_{ni} > \frac{\varepsilon n^{\frac{2}{1+p}}}{N}\Big) + (\varepsilon_{ni} + n^{1/q}) I\Big(\varepsilon_{ni} < -\frac{\varepsilon n^{\frac{2}{1+p}}}{N}\Big),$$

where $1 \leqslant i \leqslant n$. Noting that $X_{ni}(1) + X_{ni}(2) + X_{ni}(3) + X_{ni}(4) = \varepsilon_{ni}$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} R_{ni}\right| > 4\varepsilon\right) \\
&\leqslant \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} W_{ni}(x) X_{ni}(1)\right| > \varepsilon\right) + \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} W_{ni}(x) X_{ni}(2)\right| > \varepsilon\right) \\
&\quad + \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} W_{ni}(x) X_{ni}(3)\right| > \varepsilon\right) + \sum_{n=1}^{\infty} P\left(\left|\sum_{i=1}^{n} W_{ni}(x) X_{ni}(4)\right| > \varepsilon\right) \\
&\doteq I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{4.6}$$

In order to prove (4.5), we only need to show that $I_1 < \infty$, $I_2 < \infty$, $I_3 < \infty$ and $I_4 < \infty$.

Note that

$$\begin{aligned}
E|\varepsilon_{ni}|^{1+p} I(|\varepsilon_{ni}| > n^{1/q}) &= E|\varepsilon_{ni}|^{p} \cdot |\varepsilon_{ni}| I(|\varepsilon_{ni}| > n^{1/q}) \\
&\geqslant n^{p/q} E|\varepsilon_{ni}| I(|\varepsilon_{ni}| > n^{1/q}),
\end{aligned} \tag{4.7}$$

which together with $E\varepsilon_{ni}=0$, Markov's inequality, equation (4.7) and $E|X|^{1+p}<\infty$, yields that

$$\begin{aligned}\left|\sum_{i=1}^{n}W_{ni}(x)EX_{ni}(1)\right| &\leqslant n^{1/q}\sum_{i=1}^{n}W_{ni}(x)P(|\varepsilon_{ni}|>n^{1/q})+\sum_{i=1}^{n}W_{ni}(x)\left|E\varepsilon_{ni}I(|\varepsilon_{ni}|\leqslant n^{1/q})\right|\\ &= n^{1/q}\sum_{i=1}^{n}W_{ni}(x)P(|\varepsilon_{ni}|>n^{1/q})+\sum_{i=1}^{n}W_{ni}(x)\left|E\varepsilon_{ni}I(|\varepsilon_{ni}|> n^{1/q})\right|\\ &\leqslant n^{1/q}\sum_{i=1}^{n}W_{ni}(x)P(|\varepsilon_{ni}|>n^{1/q})+\sum_{i=1}^{n}W_{ni}(x)E|\varepsilon_{ni}|I(|\varepsilon_{ni}|> n^{1/q})\\ &\leqslant n^{-p/q}\sum_{i=1}^{n}W_{ni}(x)E|\varepsilon_{ni}|^{1+p}\\ &\quad +n^{-p/q}\sum_{i=1}^{n}W_{ni}(x)E|\varepsilon_{ni}|^{1+p}I(|\varepsilon_{ni}|>n^{1/q})\\ &\leqslant CE|X|^{1+p}n^{-p/q}\longrightarrow 0,\ as\ n\to\infty. \end{aligned}\tag{4.8}$$

Hence, to prove $I_1<\infty$, we only need to show that

$$\sum_{n=1}^{\infty}P\left(\left|\sum_{i=1}^{n}W_{ni}(x)(X_{ni}(1)-EX_{ni}(1))\right|>\frac{\varepsilon}{2}\right)<\infty.$$

By Lemma 2.1, we can see that $\{W_{ni}(x)(X_{ni}(1)-EX_{ni}(1)),1\leqslant i\leqslant n\}$ are still END random variables for fixed $n\geqslant 1$ and $x\in c(g)$. In the following, Lemma 2.2 will be applied to the array $\{W_{ni}(x)(X_{ni}(1)-EX_{ni}(1)),1\leqslant i\leqslant n\}$ and the sequence $\{(\log n)^{-1},n\geqslant 1\}$.

Noting that $p<q$, we obtain

$$\max_{1\leqslant i\leqslant n}|W_{ni}(x)(X_{ni}(1)-EX_{ni}(1))|\leqslant \frac{Cn^{1/q}}{n^{1/p}}=O((\log n)^{-1}),\tag{4.9}$$

which yields the condition (i) of Lemma 2.2.

By Lemma 2.3 and condition (A_2), we obtain

$$\begin{aligned}\sum_{i=1}^{n}E\left|W_{ni}(x)(X_{ni}(1)-EX_{ni}(1))\right|^2 &\leqslant \sum_{i=1}^{n}W_{ni}^2(x)E|X_{ni}(1)|^2\\ &\leqslant \max_{1\leqslant i\leqslant n}W_{ni}(x)\sum_{i=1}^{n}W_{ni}(x)E\varepsilon_{ni}^2\\ &\leqslant Cn^{-1/p}EX^2=o((\log n)^{-1}),\end{aligned}\tag{4.10}$$

which yields the condition (ii) of Lemma 2.2. It is easy to see that condition (iii) of Lemma 2.2 is satisfied by choosing $\alpha>1$. Hence, we can see that $I_1<\infty$ follows immediately from Lemma 2.2.

Next, we will show that $I_2 < \infty$. Noting that $0 < X_{ni}(2) < \frac{\varepsilon n^{2/(1+p)}}{N} - n^{1/q}$ and $0 < W_{ni}(x) \leqslant n^{-1/p}$, we can see that $|\sum_{i=1}^n X_{ni}(2)| = \sum_{i=1}^n X_{ni}(2) > \varepsilon$ implies that there exist at least N i's such that $X_{ni}(2) \neq 0$. Hence, we have by Lemma 2.3 that

$$\begin{aligned} P\left(\left|\sum_{i=1}^n X_{ni}(2)\right| > \varepsilon\right) &\leqslant \sum_{1\leqslant i_1<i_2<\cdots<i_N\leqslant n} P(X_{ni_1}(2)\neq 0,\cdots,X_{ni_N}(2)\neq 0) \\ &\leqslant \sum_{1\leqslant i_1<i_2<\cdots<i_N\leqslant n} P(\varepsilon_{ni_1} > n^{1/q},\cdots,\varepsilon_{ni_N} > n^{1/q}) \\ &\leqslant C \sum_{1\leqslant i_1<i_2<\cdots<i_N\leqslant n} P(\varepsilon_{ni_1} > n^{1/q})\cdots P(\varepsilon_{ni_N} > n^{1/q}) \\ &\leqslant C\Big(\sum_{i=1}^n P(\varepsilon_{ni} > n^{1/q})\Big)^N \\ &\leqslant C\Big(nP(|X| > n^{1/q})\Big)^N \\ &\leqslant C\Big((nE|X|^{1+p})/(n^{(1+p)/q})\Big)^N \\ &\leqslant Cn^{-(-1+(1+p)/q)N}, \end{aligned} \tag{4.11}$$

which is summable if we choose $N > \frac{q}{1+p-q}$ such that $(-1+(1+p)/q)N > 1$. Therefore, we can get that $I_2 < \infty$.

Noting that $n^{1/q} - \frac{\varepsilon n^{2/(1+p)}}{N} < X_{ni}(3) < 0$, we can see that

$$|\sum_{i=1}^n X_{ni}(3)| = -\sum_{i=1}^n X_{ni}(3) > \varepsilon$$

implies that there exist at least N i's such that $X_{ni}(3) \neq 0$. Similar to proof of $I_2 < \infty$, we can obtain that $I_3 < \infty$.

At last, we will show that $I_4 < \infty$. It follows by $E|X|^{1+p} < \infty$ that

$$\begin{aligned} I_4 &= \sum_{n=1}^\infty P\left(\left|\sum_{i=1}^n W_{ni}(x)X_{ni}(4)\right| > \varepsilon\right) \\ &\leqslant \sum_{n=1}^\infty \sum_{i=1}^n P\Big(|\varepsilon_{ni}| > \frac{\varepsilon n^{2/(1+p)}}{N}\Big) \\ &\leqslant C\sum_{n=1}^\infty n \sum_{k=n}^\infty P\Big(\frac{\varepsilon k^{2/(1+p)}}{N} < |X| < \frac{\varepsilon (k+1)^{2/(1+p)}}{N}\Big) \\ &= C\sum_{k=1}^\infty P\Big(\frac{\varepsilon k^{2/(1+p)}}{N} < |X| < \frac{\varepsilon (k+1)^{2/(1+p)}}{N}\Big)\sum_{n=1}^k n \\ &\leqslant C\sum_{k=1}^\infty k^2 P\Big(\frac{\varepsilon k^{2/(1+p)}}{N} < |X| < \frac{\varepsilon (k+1)^{2/(1+p)}}{N}\Big) \end{aligned}$$

$$\leqslant C\sum_{k=1}^{\infty} E|X|^{1+p} I\Big(\frac{\varepsilon k^{2/(1+p)}}{N} < |X| < \frac{\varepsilon (k+1)^{2/(1+p)}}{N}\Big)$$
$$\leqslant CE|X|^{1+p} < \infty. \tag{4.12}$$

This completes the proof of the theorem. □

Acknowledgements. The authors are most grateful to the Editor and anonymous referee for carefully reading the manuscript and valuable suggestions which helped in improving an earlier version of this paper.

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(Received March 9, 2020)

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