

## BOUNDEDNESS OF THE MARCINKIEWICZ INTEGRAL ON GRAND VARIABLE HERZ SPACES

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*Abstract.* We prove the boundedness of the Marcinkiewicz integral on grand variable Herz spaces.

### 1. Introduction

The notion of Herz spaces was introduced in 1968 by C. Herz in [10]. There are two versions of such spaces, viz. the homogeneous and non-homogeneous Herz spaces denoted by  $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$  and  $K_q^{\alpha,p}(\mathbb{R}^n)$  and defined by the norms

$$\|f\|_{\dot{K}_q^{\alpha,p}} := \left\{ \sum_{k \in \mathbb{Z}} 2^{k\alpha p} \left( \int_{R_{2^{k-1}, 2^k}} |f(x)|^q dx \right)^{p/q} \right\}^{1/p},$$

$$\|f\|_{K_q^{\alpha,p}} := \|f\|_{L^q(B(0,1))} + \left\{ \sum_{k \in \mathbb{N}} 2^{k\alpha p} \left( \int_{R_{2^{k-1}, 2^k}} |f(x)|^q dx \right)^{p/q} \right\}^{1/p},$$

respectively, where  $R_{t,\tau}$  denotes the annulus  $R_{t,\tau} := B(0,\tau) \setminus B(0,t)$ . These spaces were studied in many papers, see for instance [5, 7, 9, 12, 13, 22, 14, 15] and references therein.

Last two decades, under the influence of some applications revealed in [32], there was a vast boom of research in the so-called variable exponent spaces, see e.g. [29]. For the time being, the theory of such variable exponent Lebesgue, Orlicz, Lorentz, and Sobolev function spaces is widely developed, cf. [2, 4, 20, 21, 3]. Herz spaces with variables exponent have been introduced in [1, 12, 13]. Samko in [34] introduced continual variable exponent Herz spaces with all parameters variable, see also [30]. Another approach for variable smoothness and integrability to study Herz type Hardy spaces was used in [26].

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Grand Lebesgue spaces on bounded sets were introduced in [8, 11]. Grand spaces proved to be useful in application to partial differential equations. Various operators of harmonic analysis were intensively studied in the last years, cf. [6, 16, 17, 18, 19, 20, 24, 25, 35] and the references therein. Grand Lebesgue sequence spaces were introduced recently in [31], where various operators of harmonic analysis were studied in these spaces, e.g. maximal, convolutions, Hardy, Hilbert, and fractional operators, among others.

The aim of this paper is to obtain the boundedness of Marcinkiewicz integral operator on the space  $K_{q(\cdot)}^{\alpha(\cdot), p(\cdot), \theta}(\mathbb{R}^n)$ , which was recently introduced in [28]. Throughout the paper, constants (often different constant in the same series of inequalities) will mainly be denoted by  $c$  or  $C$ .  $f \lesssim g$  means that  $f \leq cg$  and  $f \asymp g$  means that  $f \lesssim g \lesssim f$ .

In what follows, we denote  $\mathbf{1}_k = \mathbf{1}_{R_k}$ ,  $R_k = B_k \setminus B_{k-1}$  and  $B_k = \{x \in \mathbb{R}^n : |x| < 2^k\}$  for all  $k \in \mathbb{Z}$ .

## 2. Preliminaries

### 2.1. Lebesgue space with variable exponent

For the current section we refer to [2, 4, 33, 23] unless and until stated otherwise. Let  $X \subseteq \mathbb{R}^n$  be an open set and  $p(\cdot)$  be a real-valued measurable function on  $X$  with values in  $[1, \infty)$ . We suppose that

$$1 \leq p^-(X) \leq p(x) \leq p^+(X) < \infty, \tag{1}$$

where  $p^-(X) := \text{ess inf}_{x \in X} p(x)$  and  $p^+(X) := \text{ess sup}_{x \in X} p(x)$ . By  $L^{p(\cdot)}(X)$  we denote the space of measurable function  $f$  on  $X$  such that

$$I_{p(\cdot)}(f) = \int_X |f(x)|^{p(x)} dx < \infty.$$

It is a Banach space equipped with norm [see, e.g. [4, 33]]:

$$\|f\|_{L^{p(\cdot)}(X)} = \inf \left\{ \eta > 0 : I_{p(\cdot)} \left( \frac{f}{\eta} \right) \leq 1 \right\}.$$

By  $p'(x) = p(x)/(p(x) - 1)$ , we denote the conjugate exponent of  $p(\cdot)$ . For the following lemma we refer to e.g. [2].

LEMMA 1. (Generalized Hölder’s inequality) *Let  $X$  be a measurable subset of  $\mathbb{R}^n$ . Suppose that  $1 \leq p^-(X) \leq p^+(X) < \infty$  and  $q_-, r_- > 1$ . Then*

$$\|fg\|_{L^{r(\cdot)}(X)} \lesssim \|f\|_{L^{p(\cdot)}(X)} \|g\|_{L^{q(\cdot)}(X)}$$

holds, where  $f \in L^{p(\cdot)}(X)$  and  $g \in L^{q(\cdot)}(X)$  and  $\frac{1}{r(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}$  for every  $x \in X$ .

In the sequel we use the well known log-condition

$$|q(x) - q(y)| \leq \frac{A}{-\ln|x - y|}, \quad |x - y| \leq \frac{1}{2}, \quad x, y \in X, \tag{2}$$

where  $A = A(q) > 0$  does not depend on  $x, y$ , and the decay condition: there exists a number  $q_\infty \in (1, \infty)$ , such that

$$|q(x) - q_\infty| \leq \frac{A}{\ln(e + |x|)}, \quad (3)$$

and also the decay condition

$$|q(x) - q_0| \leq \frac{A}{\ln|x|}, \quad |x| \leq \frac{1}{2}, \quad (4)$$

holds for some  $q_0 \in (1, \infty)$  in case of homogeneous Herz spaces.

With respect to classes of variable exponents used in this paper, we adopt the following notations:

- (i)  $L_{\text{loc}}^{q(\cdot)}(X) := \left\{ f : f \in L^{q(\cdot)}(K) \text{ for all compact subsets } K \subset X \right\}$ .
- (ii) The set  $\mathcal{P}(X)$  consists of all  $q(\cdot)$  satisfying  $q^- > 1$  and  $q^+ < \infty$ .
- (iii)  $\mathcal{P}^{\text{log}} = \mathcal{P}^{\text{log}}(X)$  is the class of functions  $q \in \mathcal{P}(X)$  satisfying the conditions (1) and (2).
- (iv)  $\mathcal{P}_\infty(X)$  and  $\mathcal{P}_{0,\infty}(X)$  are subsets of exponents in  $\mathcal{P}(X)$  with values in  $[1, \infty)$  which satisfy condition (3) and both the conditions (3) and (4) respectively.

LEMMA 2. ([34]) *Let  $D > 1$  and  $q \in \mathcal{P}_{0,\infty}(\mathbb{R}^n)$ . Then*

$$\frac{1}{c_0} r^{\frac{n}{q(0)}} \leq \|\mathbf{1}_{R_r D r}\|_{q(\cdot)} \leq c_0 r^{\frac{n}{q(0)}} \text{ for } 0 < r \leq 1 \quad (5)$$

and

$$\frac{1}{c_\infty} r^{\frac{n}{q_\infty}} \leq \|\mathbf{1}_{R_r D r}\|_{q(\cdot)} \leq c_\infty r^{\frac{n}{q_\infty}} \text{ for } r \geq 1, \quad (6)$$

respectively, where  $c_0 \geq 1$  and  $c_\infty \geq 1$  depend on  $D$ , but do not depend on  $r$ .

## 2.2. Grand Lebesgue sequence space

In this section we recall grand Lebesgue sequence space. For the following definition and statements, see [31].  $\mathbb{X}$  will stand for one of the sets  $\mathbb{Z}^n$ ,  $\mathbb{Z}$ ,  $\mathbb{N}$  and  $\mathbb{N}_0$ .

DEFINITION 1. Let  $1 \leq p < \infty$  and  $\theta > 0$ . The grand Lebesgue sequence space  $l^{p,\theta}$  is defined by the norm

$$\|\mathbf{x}\|_{l^{p,\theta}(\mathbb{X})} := \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k \in \mathbb{X}} |x_k|^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} = \sup_{\varepsilon > 0} \varepsilon^{\frac{\theta}{p(1+\varepsilon)}} \|\mathbf{x}\|_{l^{p(1+\varepsilon)}(\mathbb{X})},$$

where  $\mathbf{x} = \{x_k\}_{k \in \mathbb{X}}$ .

Note that the following nesting properties hold:

$$l^{p(1-\varepsilon)} \hookrightarrow l^p \hookrightarrow l^{p,\theta_1} \hookrightarrow l^{p,\theta_2} \hookrightarrow l^{p(1+\delta)} \quad (7)$$

for  $0 < \varepsilon < \frac{1}{p'}$ ,  $\delta > 0$  and  $0 < \theta_1 \leq \theta_2$ .

### 2.3. Grand variable Herz space

We recall the definition of grand variable Herz space as given in [28].

DEFINITION 2. Let  $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ ,  $1 \leq p < \infty$ ,  $q(\cdot) : \mathbb{R}^n \rightarrow [1, \infty)$ , and  $\theta > 0$ . We define the grand variable Herz space by

$$\dot{K}_{q(\cdot)}^{\alpha(\cdot), p, \theta}(\mathbb{R}^n) = \{f \in L_{loc}^{q(\cdot)}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot), p, \theta}(\mathbb{R}^n)} < \infty\}$$

where

$$\|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot), p, \theta}(\mathbb{R}^n)} = \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k \in \mathbb{Z}} \|2^{k\alpha(\cdot)} f \mathbf{1}_k\|_{L^{q(\cdot)}}^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} = \sup_{\varepsilon > 0} \varepsilon^{\frac{\theta}{p(1+\varepsilon)}} \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot), p(1+\varepsilon)}(\mathbb{R}^n)}.$$

### 3. Marcinkiewicz integral on grand variable Herz space

Let  $\mathbb{S}^{n-1}$  denote the unit sphere in  $\mathbb{R}^n$  equipped with the normalized Lebesgue measure. Let  $\Omega \in L^r(\mathbb{S}^{n-1})$  be a homogeneous function of degree zero such that

$$\int_{\mathbb{S}^{n-1}} \Omega(y') d\sigma(y') = 0, \tag{8}$$

where  $y' = y/|y|$  for any  $y \neq 0$ . The Marcinkiewicz integral was introduced by Stein in [36] in connection with Littewood-Paley  $g$ -function on  $\mathbb{R}^n$  as:

$$\mu_\Omega(f)(x) = \left( \int_0^\infty |F_{\Omega, s}(f)(x)|^2 \frac{ds}{s^3} \right)^{1/2},$$

where

$$F_{\Omega, s}(f)(x) = \int_{|x-y| \leq s} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy.$$

By extrapolation arguments, cf. [37], we have

$$\|\mu_\Omega f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \tag{9}$$

when  $p \in \mathcal{P}(\mathbb{R}^n)$ ,  $r > (p^-)'$  and  $\Omega \in L^r(\mathbb{S}^{n-1})$ .

To obtain Theorem 1, we need the following lemma.

LEMMA 3. [27] If  $a > 0$ ,  $1 \leq s \leq \infty$ ,  $0 < d \leq s$  and  $-n + (n-1)d/s < v < \infty$ , then

$$\left( \int_{|y| \leq a|x|} |y|^v |\Omega(x-y)|^d dy \right)^{1/d} \lesssim |x|^{(v+n)/d} \|\Omega\|_{L^s(\mathbb{S}^{n-1})}.$$

Now, we show the boundedness of Marcinkiewicz integral on grand variable Herz space.

THEOREM 1. Let  $0 < v \leq 1$ ,  $\alpha(\cdot), q(\cdot) \in \mathcal{P}_{0,\infty}(\mathbb{R}^n)$  with  $1 < q^- \leq q^+ < \infty$ ,  $1 \leq p < \infty$ . Let  $\Omega$  be homogeneous of degree zero and  $\Omega \in L^s(\mathbb{S}^{n-1})$ ,  $s > q^+$ . Let  $\alpha$  be such that:

- (a)  $\frac{-n}{q(0)} - v - \frac{n}{s} < \alpha(0) < \frac{n}{q'(0)} - v - \frac{n}{s}$ , and  
 (b)  $\frac{-n}{q_\infty} - v - \frac{n}{s} < \alpha_\infty < \frac{n}{q'_\infty} - v - \frac{n}{s}$ .

Then the operator  $\mu_\Omega$  is bounded on  $\dot{K}_{q(\cdot)}^{\alpha(\cdot), p, \theta}(\mathbb{R}^n)$ .

*Proof.* We decompose  $f$  as  $f(x) = \sum_{i=-\infty}^{\infty} f(x) \mathbf{1}_i(x) = \sum_{i=-\infty}^{\infty} f_i(x)$ . We have

$$\begin{aligned} & \|\mu_\Omega(f)\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot), p, \theta}(\mathbb{R}^n)} \\ &= \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k \in \mathbb{Z}} \|2^{k\alpha(\cdot)} \mu_\Omega(f) \mathbf{1}_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=-\infty}^{\infty} \left( \sum_{l=-\infty}^{k-2} \|2^{k\alpha(\cdot)} \mathbf{1}_k \mu_\Omega(f \mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\quad + \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=-\infty}^{\infty} \left( \sum_{l=k-1}^{k+1} \|2^{k\alpha(\cdot)} \mathbf{1}_k \mu_\Omega(f \mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\quad + \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=-\infty}^{\infty} \left( \sum_{l=k+2}^{\infty} \|2^{k\alpha(\cdot)} \mathbf{1}_k \mu_\Omega(f \mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{10}$$

For  $I_2$ , we have

$$\begin{aligned} I_2 &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k \in \mathbb{Z}} \left( \sum_{l=k-1}^{k+1} \|2^{k\alpha(\cdot)} \mathbf{1}_k \mu_\Omega(f \mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=-\infty}^{-1} \left( \sum_{l=k-1}^{k+1} \|2^{k\alpha(\cdot)} \mathbf{1}_k \mu_\Omega(f \mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\quad + \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=0}^{\infty} \left( \sum_{l=k-1}^{k+1} \|2^{k\alpha(\cdot)} \mathbf{1}_k \mu_\Omega(f \mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &=: I_{21} + I_{22}. \end{aligned}$$

Here we use the fact  $2^{k\alpha(x)} \approx 2^{k\alpha(0)}$ ,  $k < 0$ ,  $x \in R_k$  equivalent to say that

$$\|2^{k\alpha(\cdot)} f \mathbf{1}_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \asymp 2^{k\alpha(0)} \|f \mathbf{1}_k\|_{L^{q(\cdot)}(\mathbb{R}^n)},$$

and using the fact that  $\mu_\Omega$  is a bounded operator on  $L^{q(\cdot)}(\mathbb{R}^n)$ , we get

$$\begin{aligned} I_{21} &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=-\infty}^{-1} \left( \sum_{l=k-1}^{k+1} \|2^{k\alpha(\cdot)} \mathbf{1}_k \mu_\Omega(f \mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\asymp \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p(1+\varepsilon)} \left( \sum_{l=k-1}^{k+1} \|\mathbf{1}_k \mu_\Omega(f \mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p(1+\varepsilon)} \left( \sum_{l=k-1}^{k+1} \|\mu_\Omega(f \mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p(1+\varepsilon)} \left( \sum_{l=k-1}^{k+1} \|f \mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p(1+\varepsilon)} \|f \mathbf{1}_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k \in \mathbb{Z}} \|2^{k\alpha(\cdot)} f \mathbf{1}_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot), p, \theta}(\mathbb{R}^n)}. \end{aligned}$$

For estimate  $I_{22}$ , we use the fact  $2^{k\alpha(x)} \asymp 2^{k\alpha_\infty}$ ,  $k \geq 0$ ,  $x \in R_k$  and  $\mu_\Omega$  is a bounded operator on  $L^{q(\cdot)}(\mathbb{R}^n)$ , we have

$$\begin{aligned} I_{22} &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=0}^{\infty} \left( \sum_{l=k-1}^{k+1} \|2^{k\alpha(\cdot)} \mathbf{1}_k \mu_\Omega(f \mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\asymp \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=0}^{\infty} 2^{k\alpha_\infty p(1+\varepsilon)} \left( \sum_{l=k-1}^{k+1} \|\mathbf{1}_k \mu_\Omega(f \mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=0}^{\infty} 2^{k\alpha_\infty p(1+\varepsilon)} \left( \sum_{l=k-1}^{k+1} \|\mu_\Omega(f \mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=0}^{\infty} 2^{k\alpha_\infty p(1+\varepsilon)} \left( \sum_{l=k-1}^{k+1} \|f \mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \end{aligned}$$

$$\begin{aligned}
&\lesssim \sup_{\varepsilon>0} \left( \varepsilon^\theta \sum_{k=0}^{\infty} 2^{k\alpha_\infty p(1+\varepsilon)} \|f \mathbf{1}_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
&\lesssim \sup_{\varepsilon>0} \left( \varepsilon^\theta \sum_{k \in \mathbb{Z}} \|2^{k\alpha(\cdot)} f \mathbf{1}_k\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
&\lesssim \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot), p, \theta}(\mathbb{R}^n)}.
\end{aligned}$$

Combining these estimates we get  $I_2 \lesssim \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot), p, \theta}(\mathbb{R}^n)}$ .

Now we turn to estimate  $I_1$ . We consider

$$\begin{aligned}
|\mu_\Omega(f \mathbf{1}_l)(x)| &\leq \left( \int_0^{|x|} \left| \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f_l(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\quad + \left( \int_{|x|}^{\infty} \left| \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f_l(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&=: I_{11} + I_{12}.
\end{aligned}$$

Let  $x \in R_k$ ,  $y \in R_l$  and  $l \leq k-2$ , we know that  $|x-y| \asymp |x| \asymp 2^k$  and by the mean value theorem we have

$$\left| \frac{1}{|x-y|^2} - \frac{1}{|x|^2} \right| \lesssim \frac{|y|}{|x-y|^3}. \quad (11)$$

By the integral Minkowski inequality, inequality (11) and the generalized Hölder's inequality we have

$$\begin{aligned}
I_{11} &\lesssim \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f_l(y)| \left( \int_{|x-y|}^{|x|} \frac{dt}{t^3} \right)^{1/2} dy \\
&\lesssim \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f_l(y)| \left| \frac{1}{|x-y|^2} - \frac{1}{|x|^2} \right|^{1/2} dy \\
&\lesssim \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f_l(y)| \frac{|y|^{1/2}}{|x-y|^{3/2}} dy \\
&\lesssim \frac{2^{l/2}}{|x|^{n+1/2}} \int_{R_l} |\Omega(x-y)| |f(y)| dy \\
&\lesssim 2^{(l-k)/2} 2^{-kn} \|f \mathbf{1}_l\|_{L^{q(\cdot)}} \|\Omega(x-\cdot) \mathbf{1}_l(\cdot)\|_{L^{q'(\cdot)}}.
\end{aligned}$$

Similarly, we consider  $I_{12}$ . Noting that  $|x-y| \asymp |x| \asymp 2^k$ , by Minkowski and the generalized Hölder's inequality we have

$$\begin{aligned}
I_{12} &\lesssim \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} |f_l(y)| \left( \int_{|x|}^{\infty} \frac{dt}{t^3} \right)^{1/2} dy \\
&\lesssim \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n} |f_l(y)| dy
\end{aligned}$$

$$\begin{aligned} &\lesssim \frac{1}{|x|^n} \int_{R_l} |\Omega(x-y)| |f(y)| dy \\ &\lesssim 2^{-kn} \|f \mathbf{1}_l\|_{L^{q(\cdot)}} \|\Omega(x-\cdot) \mathbf{1}_l(\cdot)\|_{L^{q'(\cdot)}}. \end{aligned}$$

So, we have

$$\begin{aligned} |\mu_\Omega(f \mathbf{1}_l)(x)| &\lesssim 2^{(l-k)/2} 2^{-kn} \|f \mathbf{1}_l\|_{L^{q(\cdot)}} \|\Omega(x-\cdot) \mathbf{1}_l(\cdot)\|_{L^{q'(\cdot)}} \\ &\quad + 2^{-kn} \|f \mathbf{1}_l\|_{L^{q(\cdot)}} \|\Omega(x-\cdot) \mathbf{1}_l(\cdot)\|_{L^{q'(\cdot)}} \tag{12} \\ &\lesssim 2^{-kn} \|f \mathbf{1}_l\|_{L^{q(\cdot)}} \|\Omega(x-\cdot) \mathbf{1}_l(\cdot)\|_{L^{q'(\cdot)}}. \end{aligned}$$

Noting that  $s > q^+$ , we define  $q(\cdot)$  by the relation  $\frac{1}{q^l(x)} = \frac{1}{q(x)} + \frac{1}{s}$ . By Lemma 3 and generalized Hölder’s inequality we have

$$\begin{aligned} \|\Omega(x-\cdot) \mathbf{1}_l(\cdot)\|_{L^{q^l(\cdot)}} &\lesssim \|\Omega(x-\cdot) \mathbf{1}_l(\cdot)\|_{L^s} \|\mathbf{1}_l(\cdot)\|_{L^{q(\cdot)}} \\ &\lesssim 2^{-lv} \left( \int_{2^{l-1} < |y| < 2^l} |\Omega(x-y)|^s |y|^{sv} dy \right)^{1/s} \|\mathbf{1}_l\|_{L^{q(\cdot)}} \\ &\lesssim 2^{-lv} 2^{k(v+\frac{n}{s})} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \|\mathbf{1}_l\|_{L^{q(\cdot)}}. \tag{13} \end{aligned}$$

Moreover, splitting  $I_1$  by means of Minkowski’s inequality we have

$$\begin{aligned} I_1 &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=-\infty}^{-1} \left( \sum_{l=-\infty}^{k-2} \|2^{k\alpha(\cdot)} \mathbf{1}_k \mu_\Omega(f \mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\quad + \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=0}^{\infty} \left( \sum_{l=-\infty}^{k-2} \|2^{k\alpha(\cdot)} \mathbf{1}_k \mu_\Omega(f \mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &:= I'_{11} + I'_{12}. \end{aligned}$$

For  $I'_{11}$ , via (13) and (5), we have

$$\begin{aligned} I'_{11} &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p(1+\varepsilon)} \left( \sum_{l=-\infty}^{k-2} 2^{(l-k)(n/q^l(0)-v-\frac{n}{s})} \|f \mathbf{1}_l\|_{L^{q(\cdot)}} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=-\infty}^{-1} \left( \sum_{l=-\infty}^{k-2} 2^{\alpha(0)l} \|f \mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{b(l-k)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \end{aligned}$$

where  $b := \frac{n}{q^l(0)} - v - \frac{n}{s} - \alpha(0) > 0$ . Then we use Hölder’s inequality, Fubini’s theorem



for series and  $2^{-p(1+\varepsilon)} < 2^{-p}$  to obtain

$$\begin{aligned}
 I'_{11} &\lesssim \sup_{\varepsilon>0} \left( \varepsilon^\theta \sum_{k=-\infty}^{-1} \left( \sum_{l=-\infty}^{k-2} 2^{\alpha(0)p(1+\varepsilon)l} \|f\mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} 2^{bp(1+\varepsilon)(l-k)/2} \right) \right. \\
 &\quad \left. \times \left( \sum_{l=-\infty}^{k-2} 2^{b(p(1+\varepsilon))'(l-k)/2} \right)^{p(1+\varepsilon)/(p(1+\varepsilon))'} \frac{1}{p(1+\varepsilon)} \right) \\
 &\lesssim \sup_{\varepsilon>0} \left( \varepsilon^\theta \sum_{k=-\infty}^{-1} \sum_{l=-\infty}^{k-2} 2^{\alpha(0)p(1+\varepsilon)l} \|f\mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} 2^{bp(1+\varepsilon)(l-k)/2} \right)^{\frac{1}{p(1+\varepsilon)}} \\
 &\lesssim c \sup_{\varepsilon>0} \left( \varepsilon^\theta \sum_{l=-\infty}^{-1} 2^{\alpha(0)p(1+\varepsilon)l} \|f\mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} \sum_{k=l+2}^{-1} 2^{bp(1+\varepsilon)(l-k)/2} \right)^{\frac{1}{p(1+\varepsilon)}} \\
 &\lesssim \sup_{\varepsilon>0} \left( \varepsilon^\theta \sum_{l=-\infty}^{-1} 2^{\alpha(0)p(1+\varepsilon)l} \|f\mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} \sum_{k=l+2}^{-1} 2^{bp(l-k)/2} \right)^{\frac{1}{p(1+\varepsilon)}} \\
 &\lesssim \sup_{\varepsilon>0} \left( \varepsilon^\theta \sum_{l \in \mathbb{Z}} \|2^{l\alpha(\cdot)} f\mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
 &\lesssim \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot), p, \theta}(\mathbb{R}^n)}.
 \end{aligned}$$

Now for  $I'_{12}$  using Minkowski's inequality we have:

$$\begin{aligned}
 I'_{12} &\lesssim \sup_{\varepsilon>0} \left( \varepsilon^\theta \sum_{k=0}^{\infty} \left( \sum_{l=-\infty}^{-1} \|2^{k\alpha(\cdot)} \mathbf{1}_k \mu_\Omega(f\mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
 &\quad + \sup_{\varepsilon>0} \left( \varepsilon^\theta \sum_{k=0}^{\infty} \left( \sum_{l=0}^{k-2} \|2^{k\alpha(\cdot)} \mathbf{1}_k \mu_\Omega(f\mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
 &:= A_1 + A_2.
 \end{aligned}$$

The estimate for  $A_2$  follows in a similar manner as in  $I'_{11}$  with  $q'(0)$  replaced by  $q'_{\infty}$  and using the fact that  $b := \frac{n}{q'_{\infty}} - \nu - \frac{n}{s} - \alpha_{\infty} > 0$ . For  $A_1$  using (5), (6), (12) and (13) we have

$$\begin{aligned}
 \|\mu_\Omega(f\mathbf{1}_l)\mathbf{1}_k\|_{L^{q(\cdot)}} &\lesssim 2^{-kn} \|f\mathbf{1}_l\|_{L^{q(\cdot)}} \|\Omega(x - \cdot)\mathbf{1}_l(\cdot)\|_{L^{q(\cdot)}} \|\mathbf{1}_k\|_{L^{q(\cdot)}} \\
 &\lesssim 2^{-kn} 2^{-l\nu} 2^{k(\nu + \frac{n}{s})} \|f\mathbf{1}_l\|_{L^{q(\cdot)}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \|\mathbf{1}_l\|_{L^{q(\cdot)}} \|\mathbf{1}_k\|_{L^{q(\cdot)}} \\
 &\lesssim 2^{-kn} 2^{-l\nu} 2^{k(\nu + \frac{n}{s})} 2^{ln/q(0)} 2^{kn/q_{\infty}} \|f\mathbf{1}_l\|_{L^{q(\cdot)}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \\
 &\lesssim 2^{l(\frac{n}{q(0)} - \nu)} 2^{k(\nu + \frac{n}{s} - \frac{n}{q'_{\infty}})} \|f\mathbf{1}_l\|_{L^{q(\cdot)}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})}. \tag{14}
 \end{aligned}$$

Now using (14) and the fact that  $\alpha_\infty + \nu + \frac{\eta}{s} - \frac{\eta}{q_\infty} < 0$  we have,

$$\begin{aligned} A_1 &\leq \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=0}^\infty 2^{k\alpha_\infty p(1+\varepsilon)} \left( \sum_{l=-\infty}^{-1} \|\mathbf{1}_k \mu_\Omega(f \mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=0}^\infty 2^{k\alpha_\infty p(1+\varepsilon)} \left( \sum_{l=-\infty}^{-1} 2^{l(\frac{\eta}{q(0)} - \nu)} 2^{k(\nu + \frac{\eta}{s} - \frac{\eta}{q_\infty})} \|f \mathbf{1}_l\|_{L^{q(\cdot)}} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=0}^\infty 2^{k(\alpha_\infty + \nu + \frac{\eta}{s} - \frac{\eta}{q_\infty}) p(1+\varepsilon)} \left( \sum_{l=-\infty}^{-1} 2^{l(\frac{\eta}{q(0)} - \nu)} \|f \mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \left( \sum_{l=-\infty}^{-1} 2^{l(\frac{\eta}{q(0)} - \nu)} \|f \mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \left( \sum_{l=-\infty}^{-1} 2^{l\alpha(0)} \|f \mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{l(\frac{\eta}{q(0)} - \frac{\eta}{s} - \nu - \alpha(0))} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}}. \end{aligned}$$

Now applying Hölder’s inequality and using the fact that  $\frac{\eta}{q(0)} - \frac{\eta}{s} - \nu - \alpha(0) > 0$  we have

$$\begin{aligned} A_1 &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \left( \sum_{l=-\infty}^{-1} 2^{l\alpha(0) p(1+\varepsilon)} \|f \mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} \right) \right. \\ &\quad \times \left. \left( \sum_{l=-\infty}^{-1} 2^{l(\frac{\eta}{q(0)} - \frac{\eta}{s} - \nu - \alpha(0))(p(1+\varepsilon))'} \right)^{\frac{p(1+\varepsilon)}{(p(1+\varepsilon))'}} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{l \in \mathbb{Z}} \|2^{l\alpha(\cdot)} f \mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot), p, \theta}(\mathbb{R}^n)}. \end{aligned}$$

Let us now estimate  $I_3$ . Note that  $x \in R_k$ ,  $y \in R_l$  and  $l \geq k + 2$ , so we know that  $|x - y| \asymp |y| \asymp 2^l$ . We consider

$$\begin{aligned} |\mu_\Omega(f_l)(x)| &\leq \left( \int_0^{|y|} \left| \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f_l(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &\quad + \left( \int_{|y|}^\infty \left| \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f_l(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ &=: I_{31} + I_{32}. \end{aligned}$$

Similarly to the estimate for  $I_{11}$ , we get

$$I_{31} \lesssim 2^{(k-l)/2} 2^{-ln} \|f_l\|_{L^{q(\cdot)}} \|\Omega(x - \cdot) \mathbf{1}_l(\cdot)\|_{L^{q'(\cdot)}}.$$

Similarly to the estimate for  $I_{12}$ , we get

$$I_{32} \lesssim 2^{-ln} \|f_l\|_{L^{q(\cdot)}} \|\Omega(x - \cdot) \mathbf{1}_l(\cdot)\|_{L^{q'(\cdot)}}.$$

So, we have

$$|\mu_\Omega(f_l)(x)| \lesssim 2^{-ln} \|f_l\|_{L^{q(\cdot)}} \|\Omega(x - \cdot) \mathbf{1}_l(\cdot)\|_{L^{q'(\cdot)}}.$$

Similar to  $I_1$ , splitting  $I_3$  we have

$$\begin{aligned} I_3 &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=-\infty}^{-1} \left( \sum_{l=k+2}^{\infty} \|2^{k\alpha(\cdot)} \mu_\Omega(f \mathbf{1}_l) \mathbf{1}_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\quad + \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=0}^{\infty} \left( \sum_{l=k+2}^{\infty} \|2^{k\alpha(\cdot)} \mu_\Omega(f \mathbf{1}_l) \mathbf{1}_k\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &:= I'_{31} + I'_{32}. \end{aligned}$$

For  $I'_{32}$  by condition (6), we obtain

$$\begin{aligned} \|\mu_\Omega(f \mathbf{1}_l) \mathbf{1}_k\|_{L^{q(\cdot)}} &\lesssim 2^{-ln} \|f \mathbf{1}_l\|_{L^{q(\cdot)}} \|\Omega(x - \cdot) \mathbf{1}_l(\cdot)\|_{L^{q'(\cdot)}} \|\mathbf{1}_k\|_{L^{q(\cdot)}} \\ &\lesssim 2^{-ln} 2^{-lv} 2^{k(v+\frac{n}{s})} \|f \mathbf{1}_l\|_{L^{q(\cdot)}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \|\mathbf{1}_l\|_{L^{q(\cdot)}} \|\mathbf{1}_k\|_{L^{q(\cdot)}} \\ &\lesssim 2^{-ln} 2^{-lv} 2^{k(v+\frac{n}{s})} 2^{ln/q_\infty} 2^{kn/q_\infty} \|f \mathbf{1}_l\|_{L^{q(\cdot)}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \\ &\lesssim 2^{(k-l)} 2^{\left(\frac{n}{q_\infty} + v + \frac{n}{s}\right)} \|f \mathbf{1}_l\|_{L^{q(\cdot)}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})}. \end{aligned} \quad (15)$$

Hence,

$$\begin{aligned} I'_{32} &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=0}^{\infty} 2^{k\alpha_\infty p(1+\varepsilon)} \left( \sum_{l=k+2}^{\infty} 2^{(k-l)} 2^{\left(\frac{n}{q_\infty} + v + \frac{n}{s}\right)} \|f \mathbf{1}_l\|_{L^{q(\cdot)}} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=0}^{\infty} \left( \sum_{l=k+2}^{\infty} 2^{\alpha_\infty l} \|f \mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{d(k-l)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \end{aligned}$$

where  $d := \frac{n}{q_\infty} + v + \frac{n}{s} + \alpha_\infty > 0$ . Then we use Hölder's inequality, Fubini's theorem for series and  $2^{-p(1+\varepsilon)} < 2^{-p}$  to obtain

$$\begin{aligned} I'_{32} &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=0}^{\infty} \left( \sum_{l=k+2}^{\infty} 2^{\alpha_\infty p(1+\varepsilon)l} \|f \mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} 2^{dp(1+\varepsilon)(k-l)/2} \right) \right. \\ &\quad \left. \times \left( \sum_{l=k+2}^{\infty} 2^{d(p(1+\varepsilon))'(k-l)/2} \right)^{p(1+\varepsilon)/(p(1+\varepsilon))'} \right)^{\frac{1}{p(1+\varepsilon)}} \end{aligned}$$

$$\begin{aligned}
 &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=0}^{\infty} \sum_{l=k+2}^{\infty} 2^{\alpha_\infty p(1+\varepsilon)l} \|f\mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} 2^{d p(1+\varepsilon)(k-l)/2} \right)^{\frac{1}{p(1+\varepsilon)}} \\
 &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{l=0}^{\infty} 2^{\alpha_\infty p(1+\varepsilon)l} \|f\mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} \sum_{k=0}^{l-2} 2^{d p(1+\varepsilon)(k-l)/2} \right)^{\frac{1}{p(1+\varepsilon)}} \\
 &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{l \in \mathbb{Z}} \|2^{\alpha(\cdot)} f\mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
 &\lesssim \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot), p, \theta}(\mathbb{R}^n)}.
 \end{aligned}$$

Now for  $I'_{31}$  we have:

$$\begin{aligned}
 I'_{31} &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=-\infty}^{-1} \left( \sum_{l=k+2}^{-1} \|2^{k\alpha(\cdot)} \mathbf{1}_k \mu_\Omega(f\mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
 &\quad + \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=-\infty}^{-1} \left( \sum_{l=0}^{\infty} \|2^{k\alpha(\cdot)} \mathbf{1}_k \mu_\Omega(f\mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
 &:= I_{31}^* + I_{32}^*.
 \end{aligned}$$

For  $I_{32}^*$  by condition (5) and (6) we have

$$\begin{aligned}
 \|\mu_\Omega(f\mathbf{1}_l)\mathbf{1}_k\|_{L^{q(\cdot)}} &\lesssim 2^{-ln} \|f\mathbf{1}_l\|_{L^{q(\cdot)}} \|\Omega(x-\cdot)\mathbf{1}_l(\cdot)\|_{L^{q'(\cdot)}} \|\mathbf{1}_k\|_{L^{q(\cdot)}} \\
 &\lesssim 2^{-ln} 2^{-lv} 2^{k(v+\frac{n}{s})} \|f\mathbf{1}_l\|_{L^{q(\cdot)}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \|\mathbf{1}_l\|_{L^{q(\cdot)}} \|\mathbf{1}_k\|_{L^{q(\cdot)}} \\
 &\lesssim 2^{-ln} 2^{-lv} 2^{k(v+\frac{n}{s})} 2^{ln/q_\infty} 2^{kn/q(0)} \|f\mathbf{1}_l\|_{L^{q(\cdot)}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})} \\
 &\lesssim 2^{-l(\frac{n}{q_\infty} + v + \frac{n}{s})} 2^{k(\frac{n}{q(0)} + v + \frac{n}{s})} \|f\mathbf{1}_l\|_{L^{q(\cdot)}} \|\Omega\|_{L^s(\mathbb{S}^{n-1})}.
 \end{aligned} \tag{16}$$

Now using (16) and the fact that  $\frac{n}{q(0)} + v + \frac{n}{s} + \alpha(0) > 0$  we have,

$$\begin{aligned}
 I_{32}^* &\leq \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p(1+\varepsilon)} \left( \sum_{l=0}^{\infty} \|\mathbf{1}_k \mu_\Omega(f\mathbf{1}_l)\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
 &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k\alpha(0)p(1+\varepsilon)} \left( \sum_{l=0}^{\infty} 2^{-l(\frac{n}{q_\infty} + v + \frac{n}{s})} 2^{k(\frac{n}{q(0)} + v + \frac{n}{s})} \|f\mathbf{1}_l\|_{L^{q(\cdot)}} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\
 &\lesssim \sup_{\varepsilon > 0} \left( \varepsilon^\theta \sum_{k=-\infty}^{-1} 2^{k(\frac{n}{q(0)} + v + \frac{n}{s} + \alpha(0))p(1+\varepsilon)} \right. \\
 &\quad \left. \times \left( \sum_{l=0}^{\infty} 2^{-l(\frac{n}{q_\infty} + v + \frac{n}{s})} \|f\mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}}
 \end{aligned}$$

$$\begin{aligned} &\lesssim \sup_{\varepsilon>0} \left( \varepsilon^\theta \left( \sum_{l=0}^{\infty} 2^{-l(\frac{n}{q_\infty} + \nu + \frac{n}{s})} \|f \mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0} \left( \varepsilon^\theta \left( \sum_{l=0}^{\infty} 2^{l\alpha_\infty} \|f \mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)} 2^{-l(\frac{n}{q_\infty} + \nu + \frac{n}{s} + \alpha_\infty)} \right)^{p(1+\varepsilon)} \right)^{\frac{1}{p(1+\varepsilon)}}. \end{aligned}$$

Now applying Hölder's inequality and the fact that  $\frac{n}{q_\infty} + \nu + \frac{n}{s} + \alpha_\infty > 0$  we have

$$\begin{aligned} I_{32}^* &\lesssim \sup_{\varepsilon>0} \left( \varepsilon^\theta \left( \sum_{l=0}^{\infty} 2^{l\alpha_\infty p(1+\varepsilon)} \|f \mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} \right) \right. \\ &\quad \left. \times \left( \sum_{l=0}^{\infty} 2^{-l(\frac{n}{q_\infty} + \nu + \frac{n}{s} + \alpha_\infty)(p(1+\varepsilon))'} \right)^{\frac{p(1+\varepsilon)}{p((1+\varepsilon))'}} \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \sup_{\varepsilon>0} \left( \varepsilon^\theta \left( \sum_{l \in \mathbb{Z}} \|2^{l\alpha(\cdot)} f \mathbf{1}_l\|_{L^{q(\cdot)}(\mathbb{R}^n)}^{p(1+\varepsilon)} \right) \right)^{\frac{1}{p(1+\varepsilon)}} \\ &\lesssim \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot), p, \theta}(\mathbb{R}^n)}. \end{aligned}$$

The estimate for  $I_{31}^*$  follows in similar manner as in  $I_{32}^*$  with simply  $q_\infty$  replaced by  $q(0)$  and using the fact that  $\frac{n}{q(0)} + \nu + \frac{n}{s} + \alpha(0) > 0$ . Combining the estimates for  $I_1, I_2$  and  $I_3$ , yields

$$\|\mu_\Omega(f)\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot), p, \theta}(\mathbb{R}^n)} \lesssim \|f\|_{\dot{K}_{q(\cdot)}^{\alpha(\cdot), p, \theta}(\mathbb{R}^n)},$$

which ends the proof.  $\square$

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#### REFERENCES

- [1] A. ALMEIDA AND D. DRIHEM, *Maximal, potential and singular type operators on Herz spaces with variable exponents*, J. Math. Anal. Appl., **394** (2): 781–795, 2012.
- [2] D. CRUZ-URIBE, A. FIORENZA, *Variable Lebesgue Space*, Foundations and Harmonic Analysis, Birkhäuser, 2013.
- [3] D. CRUZ-URIBE, A. FIORENZA, J. MARTELL AND C. PÉREZ, *The boundedness of classical operators on variable  $L^p$  spaces*, Ann. Acad. Sci. Fenn. Math., 31 (1): 239–264, 2006.
- [4] L. DIENING, P. HARJULEHTO, HÄSTÖ, AND M. RUZICKA, *Lebesgue and Sobolev Spaces with variable exponents*, Springer-Verlag, Lecture Notes in Mathematics, vol. 2017, Berlin, 2011.
- [5] H. G. FEICHTINGER AND F. WEISZ, *Herz spaces and summability of Fourier transforms*, Math. Nachr., **281** (3): 309–324, 2008.

- [6] A. FIORENZA, B. GUPTA AND P. JAIN, *The maximal theorem for weighted grand Lebesgue spaces*, *Studia Math.*, **188**, no. 2, 123–133, 2008.
- [7] L. GRAFAKOS, X. LI, AND D. YANG, *Bilinear operators on Herz-type Hardy spaces*, *Trans. Amer. Math. Soc.*, **350** (3): 1249–1275, 1998.
- [8] L. GRECO, T. IWANIEC AND C. SBORDONE, *Inverting the  $p$ -harmonic operator*, *Manuscripta Math.*, **92**, no. 2, 249–258, 1997.
- [9] E. HERNANDEZ AND D. YANG, *Interpolation of Herz spaces and application*, *Math. Nachr.*, **205** (1): 69283–87, 1999.
- [10] C. HERZ, *Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transform*, *J. Math. Mech.*, **18**: 283–323, 1968/69.
- [11] T. IWANIEC AND C. SBORDONE, *On the integrability of the Jacobian under minimal hypotheses*, *Arch. Ration. Mech. Anal.*, **119**, no. 2, 129–143, 1992.
- [12] M. IZUKI, *Boundedness of vector-valued sublinear operators on Herz-Morrey spaces with variable exponent*, *Math. Sci. Res. J.*, **13** (10): 243–253, 2009.
- [13] M. IZUKI, *Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization*, *Analysis Mathematica*, **13** (36): 33–50, 2010.
- [14] R. JOHNSON, *Temperature, Riesz potential and the Lipschitz spaces of Herz*, *Proc. London Math. Soc.*, **27** (2): 290–316, 1973.
- [15] R. JOHNSON, *Lipschitz spaces, Littlewood-Paley spaces and convoluteurs*, *Proc. London Math. Soc.*, **29** (1): 127–141, 1974.
- [16] V. KOKILASHVILI, *Boundedness criteria for singular integrals in weighted grand Lebesgue spaces*, *J. Math. Sci. (N. Y.)*, **170**, no. 1, 20–33, 2010.
- [17] V. KOKILASHVILI AND A. MESKHI, *A note on the boundedness of the Hilbert transform in weighted grand Lebesgue spaces*, *Georgian Math. J.*, **16**, no. 3, 547–551, 2009.
- [18] V. KOKILASHVILI AND A. MESKHI, *Trace inequalities for fractional integrals in grand Lebesgue spaces*, *Studia Math.*, **210**, no. 2, 159–176, 2012.
- [19] V. KOKILASHVILI AND A. MESKHI, *Potentials with product kernels in grand Lebesgue spaces: One-weight criteria*, *Lith. Math. J.*, **23**, no. 1, 27–39, 2013.
- [20] V. KOKILASHVILI AND A. MESKHI, H. RAFEIRO, S. SAMKO, *Integral Operators in Non-Standard Function Spaces, Vol. 1*, *Oper. Theory Adv. Appl.* 248, Birkhäuser, Cham, 2016.
- [21] V. KOKILASHVILI AND A. MESKHI, H. RAFEIRO, S. SAMKO, *Integral Operators in Non-Standard Function Spaces, Vol. 2*, *Oper. Theory Adv. Appl.* 249, Birkhäuser, Cham, 2016.
- [22] Y. KOMORI, *Notes on singular integrals on some inhomogeneous Herz spaces*, *Taiwanese J. Math.*, **8** (3): 547–556, 2004.
- [23] O. KAVACIK AND J. RAKOSNIK, *On spaces  $L^{p(x)}$  and  $W^{k,p(x)}$* , *Czechoslovak Math. J.*, **41** (116): 592–618, 1991.
- [24] A. MESKHI, *Weighted criteria for the Hardy transform under the  $B_p$  condition in grand Lebesgue spaces and some application*, *J. Math. Sci. (N. Y.)*, **178**, no. 6, 622–636, 2011.
- [25] A. MESKHI, *Criteria for the boundedness of potential operators in grand Lebesgue spaces*, *Proc. A. Razmadeze Math. inst.*, **169**, 119–132, 2015.
- [26] A. MESKHI, H. RAFEIRO, AND M. A. ZAIGHUM, *Central Calderon-Zygmund Operators On Herz-type Hardy Spaces Of Variable Smoothness and Integrability*, *Ann. Funct. Anal.*, 2017.
- [27] B. MUCKENHOUT, R. L. WHEEDEN, *Weighted norm inequalities for singular and fractional integrals*, *Trans. Am. Maths. Soc.* 249–258, 161 (1971).
- [28] H. NAFIS, H. RAFEIRO, M. A. ZAIGHUM, *A note on the Boundedness of Sublinear Operators on Grand Variable Herz Spaces*, *J. Inequal. Appl.*, 2020, 1 (2020).
- [29] V. D. RADULESCU, D. REPOVES, *Partial differential equations with variable exponents: variational methods and qualitative analysis, 1st edition*, Chapman and Hall/CRC, pub. June 24, 2015.
- [30] H. RAFEIRO, AND S. SAMKO, *Riesz potential operator in continual variable exponents Herz spaces*, *Math. Nachr.*, 288 (4), 465–475, 2015.
- [31] H. RAFEIRO, S. SAMKO, AND S. UMARKHADZHIEV, *Grand Lebesgue sequence spaces*, *Georgian Math. J.*, 2018.
- [32] M. RUZICKA, *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer, Lecture Notes in Math., vol. **1748**, 176 pages, 2000.
- [33] S. G. SAMKO, *On a progress in the theory of Lebesgue spaces with Variable exponents: maximal and singular operators*, *Integr. Transf. and Spec. Funct.*, **16** (5–6): 461–482, 2005.

- [34] S. G. SAMKO, *Variable exponents Herz space*, *Mediterr. J. Math.*, **10** (4): 2007–2025, 2013.
- [35] F. SORIA AND G. WEISS, *A remark on singular integrals and power weights*, *Indiana University Mathematical Journal*, vol. **43**, no. 1, pp. 187–204, 1994.
- [36] E. STEIN, *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz*, *Trans. Amer. Math. Soc.*, **88**: 430–466, 1958.
- [37] H. WANG, *Commutators of Marcinkiewicz integrals on Herz spaces with variable exponent*, *Czechoslovak Mathematical Journal* **66**: 251–269, 2016.

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