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VOLUMES OF SUB–LEVEL SETS AND THE DECAY OF OSCILLATORY INTEGRALS

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Abstract. In this paper, we present estimates for the volumes of sub-level sets and the decay of oscillatory integrals with phase functions in a family of functions definable in an o-minimal structure with compact domains.

1. Introduction

In this note, we are interested in the following problems:
– The estimate for the volumes of sub-level sets of a family of functions $(f_p)_{p\in P}$

$$\mathrm{Vol}\left(\{x : |f_p(x)| \leqslant t\}\right), \text{ when } t \to 0.$$

– The estimate for the decay of the oscillatory integrals with phase functions in a family $(f_p)_{p\in P}$

$$\left|\int_A e^{i\lambda f_p(x)} g(x)dx\right|, \text{ when } \lambda \to \infty,$$

where g is called the amplitude function.

These two estimates have a very close relationship with each other, and have been considered by many authors, especially in Harmonic Analysis and Asymptotic Analysis. Here we only refer to some of them: Stein in his book [18] (for smooth functions in one-dimensional and classical cases), [1] and the references therein (for results on asymptotic expansions, uniform estimates, ... for smooth functions), [3] (for the multidimensional van der Corput Lemma and sublevel sets of smooth functions on boxes), [5] and [4] (for the case when the phase functions being globally subanalytic, and the amplitude functions being constructible), [9] (for estimating the volumes of sublevels of subanalytic maps), [12] (for estimating the volumes of sublevels of definable maps). In this note we give the estimates for families of functions definable in o-minimal structures with compact domains. These classes of functions have many nice properties and they are widely applicable due to their generality. For the definition and properties of

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functions definable in o-minimal structures, we refer the readers to [6] (see also [2], [7] and [16]). We fix an o-minimal structure on $(\mathbb{R},+,\cdot)$ that contains all semialgebraic sets. “Definable” means definable in the structure. Let Φ denote the set of all odd, strictly increasing continuous definable bijections from $\mathbb{R}$ onto $\mathbb{R}$. $\mathrm{Vol}(X)$ denotes the Lebesgue measure of the subset X of $\mathbb{R}^n$. Our main results are the followings.

THEOREM 1.1. *Let $f: P\times A \to \mathbb{R}$ be a definable function. Set $f_p(x) = f(p,x)$, $(p,x)\in P\times A$. Suppose that A is compact, f_p is continuous and* $\mathrm{int}(\{x\in A : f_p(x) = 0\}) = \emptyset$*, for all $p\in P$. Then there exist $\varphi\in\Phi$ and $C: P\to\mathbb{R}$, being a positive definable function, such that*

$$\mathrm{Vol}(\{x\in A : |f_p(x)|\leqslant t\})\leqslant C(p)\varphi(t),\ \textit{for all } t\geqslant 0, p\in P.$$

In particular, when the structure is polynomially bounded, then

$$\mathrm{Vol}(\{x\in A : |f_p(x)|\leqslant t\})\leqslant C(p)t^{\alpha},\ \textit{for all } t\geqslant 0, p\in P,$$

for some $\alpha > 0$.

For $g:\mathbb{R}^n\to\mathbb{R}$ being C^1 function with compact support, set

$$\|g\|_\infty = \sup_{\mathbb{R}^n}|g|,\ \text{and}\ \|\nabla g\|_1 = \int_{\mathbb{R}^n}\|\nabla g\|.$$

THEOREM 1.2. *Let $f: P\times A\to\mathbb{R}$ be a definable function. Suppose that A is compact, f_p is continuous and* $\mathrm{int}(\{x\in A: f_p(x) = t\}) = \emptyset$*, for all $p\in P$ and $t\in\mathbb{R}$. Then there exist $\varphi\in\Phi$ and $C: P\to\mathbb{R}$ being a positive definable function such that for any C^1 function $g:\mathbb{R}^n\to\mathbb{R}$ with compact support contained in A, we have*

$$\left|\int_A e^{i\lambda f(p,x)}g(x)dx\right|\leqslant C(p)\varphi(\lambda^{-1})(\|g\|_\infty + \|\nabla g\|_1),\ \textit{for all } \lambda > 0.$$

In particular, when the structure is polynomially bounded, then

$$\left|\int_A e^{i\lambda f(p,x)}g(x)dx\right|\leqslant C(p)\lambda^{-\varepsilon}(\|g\|_\infty + \|\nabla g\|_1),\ \textit{for all } \lambda > 0,$$

for some $\varepsilon > 0$.

Note that the supposition on the emptiness of the interior of the level-sets in the theorems is necessary (see remark 3.9 below).

Now we give a succinct explanation of our proofs of the theorems. Theorem 1.1 is a version with parameters of [12, Theorem 3.3], where the volumes of the pre-images of segments under a definable map are estimated via the lengths of the segments. To estimate volumes of sub-level sets, first, we partition the sets into submanifolds, so that along each of which the level sets are transverse to one of the coordinate axes. Then, using Fubini's theorem, we need to estimate the lengths of the cuts of the sub-levels sets by the projections, and the problem is reduced to the case where the maps have

0-dimensional fibers. In this case, we can partition the domains into finite parts so that on each of them the maps are invertible. Relying on the Hölder-Łojasiewicz type inequalities for the family of the inverses of these maps, we get the estimates for the lengths of the cuts needed in Fubini's theorem. From that Theorem 1.1 is proved. To estimate the oscillatory integrals in Theorem 1.2, by the Stationary Phase Principle, we treat the behavior of the phase functions $f_p, p \in P$, nearby their critical points. Since each definable function on bounded sets has only finite asymptotic critical values (see 3.2) and for a definable family the numbers of these values are uniformly bounded, we can divide the domain of each of the phase functions into certain finite subsets. On the first subset, that contains the points 'far from' the critical set, the (generalized) derivative of f_p is bounded from below, by the Scaling Principle, using the van der Corput Lemma (see 3.7) we can estimate the integral. On each of the remaining subsets, f_p is related to one of the asymptotic critical values. For the points close to the critical points we apply the Bochnak-Lojasiewicz inequality (which says that the norm of the derivative of a function is greater than its value, see 3.1) to return to the case where the van der Corput Lemma can be applied. For the points 'very close to' the critical points (with the same critical value) we use the estimate for the volume of sub-level sets given in Theorem 1.1. In some sense, the spirit of the proof is the same as that of the van der Corput Lemma presented in [18, Ch. VIII, Proposition 2]. The details of the proofs are presented in the next two sections.

Notations and conventions: Throughout this note we fix an o-minimal structure on $(\mathbb{R}, +, \cdot)$ that contains all semialgebraic sets. "Definable" means definable in the structure. Let Φ denote the set of all odd, strictly increasing continuous definable bijections from $\mathbb{R}$ onto $\mathbb{R}$.

Let X be a subset of $P \times A$ and $f : X \to \mathbb{R}$ be a function. Then we set $X_p = \{x \in A : (p,x) \in X\}$ for the fiber of X over p, and $f_p : X_p \to \mathbb{R}$, is defined by $f_p(x) = f(p,x)$.

For a subset X in $\mathbb{R}^n$, we write χ_X for the characteristic function of X, $\mathscr{H}^k(X)$ for the k-dimensional Hausdorff measure of X (see [8]), and $\mathrm{Vol}(X)$ for $\mathscr{H}^n(X)$.

Let $\nabla g, \partial_k g$ denote the gradient, the partial derivative with respect to the k-th variable of the multivariable function g, respectively.

2. Proof of theorem 1.1

To prove the theorem, we prepare some lemmas. Most of them are related to definable families of functions.

LEMMA 2.1. (The Hölder-Łojasiewicz inequality) *Let P be a definable set and D be a definable subset of $P \times B$, where B is a bounded set. Let $g : D \to \mathbb{R}^m$ be a definable mapping. Suppose that $(D_p)_{p \in P}$ is a family of compact sets and g_p is continuous for all $p \in P$. Then there exist $\varphi \in \Phi$ and a positive definable function $C : P \to \mathbb{R}$, such that*

$$\|g(p,x) - g(p,y)\| \leqslant C(p)\varphi(\|x-y\|), \textit{ for all } x,y \in D_p, p \in P.$$

Proof. By [17, Th. 3], there is a cell partition of P, such that $g|_{D \cap (C \times B)}$ is continuous for each of the cell C. So the proof is reduced to case g being continuous.

Moreover, by Trivialization Theorem [6, Ch. 9 (1.2)], there exists a finite partition $P = \cup_i T_i$, where each P_i is a cell, such that

$$D_{P_i} = \{(p,x) \in D : p \in P_i\} \text{ is definably homeomorphic to } P_i \times D_{p_i}, \text{ for some } p_i \in P_i.$$

Since each cell is locally closed, so is D_{P_i}, and the proof is reduced to the case P being locally closed. In the case where g is continuous and P is locally closed, applying the Łojasiewicz inequality [11, Theorem 2] to $G(p,x,y) = g(p,x) - g(p,y)$, $F(p,x,y) = x - y$ on $D_P^2 = \{(p,x,y) : (p,x),(p,y) \in D_p\}$ (being locally closed and $F^{-1}(0) \subset G^{-1}(0)$), we get $\varphi, \psi \in \Phi$, such that

$$\|g(p,x) - g(p,y)\| \leqslant \varphi(C(p,x,y)\|x-y\|), \text{ for all } (p,x),(p,y) \in D_P,$$

where $C(p,x,y) = 1 + \psi(\|(p,x,y)\| + d((p,x,y), \overline{D_P^2} \setminus D_P^2)^{-1})$. Moreover, by the p-flatness of φ^{-1}, φ can be chosen to be concave on $\mathbb{R}_+$. Since B is bounded, $C(p) = \sup_{x,y\in B} C(p,x,y) \leqslant 1 + \psi(\max_{x,y\in B} \|x,y\| + \|p\| + d(p, \overline{P} \setminus P)^{-1}) < +\infty$. From the concavity of φ and $C(p) \geqslant 1$, we get $\varphi(C(p)t) \leqslant C(p)\varphi(t), \forall t \geqslant 0$. □

LEMMA 2.2. *Under the same notation in Lemma 2.1, let $(S_t)_{t\in T}$ be a definable family of subsets of B with $\dim S_t \leqslant 1$, for all $t \in T$. Then there exist $\varphi \in \Phi$ and a positive definable function $C_1 : P \to \mathbb{R}$, such that*

$$\mathscr{H}^1(g_p(S_t \cap D_p)) \leqslant C_1(p)\varphi((\mathrm{diam}(S_t)), \textit{ for all } t \in T, p \in P.$$

Proof. By Lemma 2.1, there exist $\varphi \in \Phi$ and $C : P \to \mathbb{R}$ being positive definable function such that

$$\mathrm{diam}(g_p(S_t \cap D_p)) \leqslant C(p)\varphi(\mathrm{diam}(S_t)),$$

i.e. $g_p(S_t \cap D_p)$ is contained in a ball of radius $C(p)\varphi(\mathrm{diam} S_t)$. By [15, Corollary 3.1], there exists $M = M(g,(S_t)_{t\in T}) > 0$, such that

$$\mathscr{H}^1(g_p(S_t \cap D_p)) \leqslant M\pi C(p)\varphi(\mathrm{diam} S_t) = C_1(p)\varphi(\mathrm{diam}(S_t)), \text{ for all } t \in T, p \in P,$$

where $C_1(p) = M\pi C(p)$. □

LEMMA 2.3. *Let $h : K \to \mathbb{R}^n$ be a definable map, where $K \subset \mathbb{R}^m$. Suppose that $\dim(h^{-1}(y)) \leqslant 0$ for all $y \in \mathbb{R}^n$. Let $\mathscr{A}$ be a finite collection of definable subsets of K. Then there exists a finite partition $\mathscr{C}$ of K into definable sets which is compatible with $\mathscr{A}$ such that for each $\Delta \in \mathscr{C}$, h is one-to-one on the closure of Δ in K.*

Proof. See [12, Lemma 3.6]. □

LEMMA 2.4. *Let E be a subset of $P \times A$, where A is a compact subset of $\mathbb{R}^n$. Suppose that $(E_p)_{p\in P}$ is a family of compact sets. Let $h : E \to \mathbb{R}^m$ be a continuous definable map. Let B be a compact subset of $F_{h_p}(0) = \{y \in \mathbb{R}^m : \dim h_p^{-1}(y) \leqslant 0\}$, for all $p \in P$. Then there exists $\varphi \in \Phi$ and a positive definable function $C : P \to \mathbb{R}$ such that for any definable family $(S_t)_{t\in T}$ of subsets of $\mathbb{R}^m$ with $\dim S_t \leqslant 1$, for all $t \in T$, we have*

$$\mathrm{Vol}\,(h_p^{-1}(S_t \cap B)) \leqslant C(p)\varphi(\mathrm{diam}(S_t)), \textit{ for all } t \in T, p \in P.$$

Proof. Let $\mathscr{C}$ Let be the partition of E obtained by applying Lemma 2.3 to the function $\bar{h}: E \to P \times \mathbb{R}^m$ defined by $\bar{h}(p,x) = (p, h(p,x))$. Note that, by compactness of E_p and continuity of h, for each $\Delta \in \mathscr{C}$, the fiber $\bar{\Delta}_p$ is compact and $h_p|_{\bar{\Delta}_p}: \bar{\Delta}_p \to h_p(\bar{\Delta}_p)$ is a definable homeomorphism, for all $p \in \pi(\bar{\Delta})$, where $\pi: P \times \mathbb{R}^n \to P$ is the natural projection. Set $D^\Delta = \{(p,y) : p \in \pi(\bar{\Delta}), y \in h_p(\bar{\Delta}_p)\}$ and $g^\Delta : D_\Delta \to A$ defined by $g^\Delta(p,y) = (h_p|_{\bar{\Delta}_p})^{-1}(y)$. Applying Lemma 2.2 to each g^Δ, for $\Delta \in \mathscr{C}$, we get $\varphi^\Delta \in \Phi$ and $C^\Delta : P \to \mathbb{R}$ being positive definable function, such that

$$\mathscr{H}^1((g^\Delta)_p(S_t \cap (D^\Delta)_p)) \leqslant C^\Delta(p)\varphi^\Delta((\operatorname{diam}(S_t))), \forall t \in T, p \in \pi(\Delta).$$

Therefore,

$$\mathscr{H}^1(h_p^{-1}(S_t \cap B)) \leqslant \sum_{\Delta \in \mathscr{C}} \mathscr{H}^1((g^\Delta)_p(S_t \cap (D^\Delta)_p)) \leqslant C(p)\varphi((\operatorname{diam}(S_t))), \forall t \in T, p \in P,$$

where $C = \operatorname{card}(\mathscr{C}) \max_{\Delta \in \mathscr{C}} C^\Delta$ and $\varphi = \max_{\Delta \in \mathscr{C}} \varphi^\Delta$. □

Note that, for any definable function $f : A \to \mathbb{R}$ with $A \subset \mathbb{R}^n$, we have

$$\operatorname{int}(\{x \in A : f(x) = t\}) \neq \emptyset \ \Leftrightarrow \ \dim f^{-1}(t) = n.$$

Moreover, for a definable family of functions, we have:

LEMMA 2.5. *Let $f : P \times A \to \mathbb{R}$ be a definable functions, where $A \subset \mathbb{R}^n$. Set $F_f(n) = \{(t,p) \in \mathbb{R} \times P : \dim f_p^{-1}(t) = n\}$. Then $F_f(n)$ is a definable family of sets whose the fibers $F_f(n)_p$, for $p \in P$, are finite, and there exists $N \in \mathbb{N}$ such that* $\operatorname{card}(F_f(n)_p) \leqslant N$, *for all $p \in P$.*

Proof. By [6, Ch. 4 (1.6)], $F_f(n)$ is definable and the dimension of the fiber over $p \in P$ satisfies

$$\dim F_f(n)_p = \dim\{t \in \mathbb{R} : \dim f_p^{-1}(t) = n\} = \dim f_p^{-1}(F_f(n)_p) - n \leqslant 0,$$

and hence the fibers are finite sets. The uniform finiteness of $\operatorname{card}(F_f(n)_p)$, $p \in P$, is followed from [6, Ch. 3 (2.13)]. □

LEMMA 2.6. *If the structure is polynomially bounded, then for every $\varphi \in \Phi$, there exists $\alpha \in \Lambda$ satisfying the following:*
For every $\tau_0 > 0$, there exist $C_1, C_2 > 0$, such that

$$C_1|t|^\alpha \leqslant \varphi(|t|) \leqslant C_2|t|^\alpha, \text{ whenever } |t| \leqslant \tau_0.$$

Proof. See [12, Proposition 1.2]. □

Before proving Theorem 1.1, we fix some more notations that will be used.

For $d \leqslant n$, let $G_d(\mathbb{R}^n)$ denote the Grassmannian of d-dimensional linear subspaces of $\mathbb{R}^n$. Let $(e_1,\cdots,e_n)$ denote the standard basis of $\mathbb{R}^n$ and $\pi_i : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the orthogonal projection along e_i. Put

$$\delta(L,e_i) = \inf\{\|x-e_i\| : x \in L, \|x\| = 1\}, \text{ for } L \in G_{n-1}(\mathbb{R}^n).$$

Proof of Theorem 1.1. The proof is an adaptation of that of [12, Theorem 3.3] (see also [9]).

By [17, Th. 3], there is a cell partition of P, such that $f|_{C\times A}$ is continuous for each cell C. So the proof is reduced to the case f being continuous. Choose $\delta > 0$, so that for each $L \in G_{n-1}(\mathbb{R}^n)$, there exists $i \in \{1,\cdots,n\}$ so that $\delta(L,e_i) > \delta$. By [13, Theorem 1.1], there exists a definable finite partition $P = \cup_j P_j$, such that for each j, there exists a definable Whitney stratification of $\mathscr{X}$ of $P_j \times A$ by cells over P_j satisfying the following conditions for each $p \in P_j$:

(1) $\mathscr{X}_p = \{\Gamma_p; \Gamma \in \mathscr{X}\}$ is also a Whitney stratification of A.

(2) For each $\Gamma \in \mathscr{X}$, $\operatorname{rank} f_p|_{\Gamma_p}$ is constant.

(3) When $\dim \Gamma_p = n, \operatorname{rank} f_p|_{\Gamma_p} = 1$, then

$$d(\Gamma_p,e_i)(x) = \delta(T_x(\Gamma_p \cap f_p^{-1}(f_p(x))),e_i) - \delta$$

has a constant sign for $x \in \Gamma_p$, for all $i \in \{1,\cdots,n\}$.

Therefore, the proof is reduced to the case where f is continuous and $P = P_j$. Using Lemma 2.5 and the notation therein, for $p \in P$, we put

$$\tau(p) = \begin{cases} \frac{1}{2}\min F_{|f|}(n)_p & \text{if } F_{|f|}(n)_p \neq \emptyset, \\ \max_A |f_p| & \text{if } F_{|f|}(n)_p = \emptyset. \end{cases}$$

Then, by the assumption, we have $\tau : P \to \mathbb{R}$ being definable and positive. For each $p \in P$, and $\Gamma \in \mathscr{X}$, we consider the volumes of $(f_p|_{\Gamma_p})^{-1}(I_t) \cap \Gamma_p$ of the definable family of the intervals $(I_t = [-t,t] : 0 \leqslant t \leqslant \tau(p))$. Since $\operatorname{int}(\{x \in A : f_p(x) = 0\}) = \emptyset$, for all $p \in P$, there are two cases to consider:

Case 1: $\dim \Gamma_p < n, \forall p \in P$. In this case $\mathscr{H}^n(f_p^{-1}(I_t) \cap \Gamma_p) = 0$ for all $0 \leqslant t < \tau(p)$.

Case 2: $\dim \Gamma_p = n$ and $\operatorname{rank} f_p|_{\Gamma_p} = 1, \forall p \in P$. In this case, there is $i \in \{1,\cdots,n\}$, such that $d(\Gamma_p,e_i)(x) > \delta, \forall x \in \Gamma_p$. Note that, from this fact and the Whitney condition (a), we have $\dim(f_p^{-1}(t) \cap \pi_i^{-1}(w) \cap \overline{\Gamma}_p) \leqslant 0$, for all $0 \leqslant |t| < \tau(p)$ and $w \in \mathbb{R}^{n-1}$. For each $i \in \{1,\cdots,n\}$, set

$$A^i = \{(p,x) : \exists \Gamma \in \mathscr{X}, x \in \overline{\Gamma}_p, \dim \Gamma_p = n, \operatorname{rank} f_p|_{\Gamma_p} = 1,\ d(\Gamma_p,e_i) > \delta\}.$$

Applying Fubini's theorem or the coarea formula, see [8, (3.2.22)], then Lemma 2.4 with $(A,h,(S_t)_{t\in T})$ replaced by $(A^i,(f,\pi_i),(I_t \times w)_{0\leqslant t<\tau(p),w\in\mathbb{R}^{n-1}})$, we get $\varphi_1 \in \Phi$

and $C_1 : P \to \mathbb{R}_+$ being definable, such that, for all $0 \leqslant t < \tau(p), p \in P$,

$$\begin{aligned}
\mathrm{Vol}(\{x : |f_p(x)| \leqslant t\}) &= \mathscr{H}^n(f_p^{-1}(I_t)) \\
&\leqslant \sum_{i=1}^{n} \mathscr{H}^n(A_p^i \cap f_p^{-1}(I_t)) \\
&= \sum_{i=1}^{n} \int_{\mathbb{R}^{n-1}} \mathscr{H}^1(A_p^i \cap f_p^{-1}(I_t) \cap \pi_i^{-1}(w))dw \\
&= \sum_{i=1}^{n} \int_{\pi_i(A^i)} \mathscr{H}^1(A_p^i \cap (f_p, \pi_i)^{-1}(I_t \times \{w\}))dw \\
&\leqslant \sum_{i=1}^{n} \int_{\pi_i(A^i)} C_1(p)\varphi_1(\mathscr{H}^1(\mathrm{diam}(I_t \times \{w\}))dw \\
&\leqslant \sum_{i=1}^{n} \mathscr{H}^{n-1}(\pi_i(A))C_1(p)\varphi_1(2t) \\
&= C_2(p)\varphi(t) ,
\end{aligned}$$

where $C_2(p) = n\max_{1 \leqslant i \leqslant n} \mathscr{H}^{n-1}(\pi_i(A))C_1(p)$ and $\varphi(t) = \varphi_1(2t)$. Since φ is increasing and $\mathrm{Vol}(\{x \in A : |f_p(x)| \leqslant t\}) \leqslant \mathrm{Vol}(A), \forall t$, taking $C(p) = \max(C_2(p), \frac{\mathrm{Vol}(A)}{\varphi(\tau(p))})$, we get

$$\mathrm{Vol}(\{x \in A : |f_p(x)| \leqslant t\}) \leqslant C(p)\varphi(t), \text{ for all } t \geqslant 0, p \in P.$$

When the structure is polynomially bounded, from this inequality and Lemma 2.6, we get the last inequality of the theorem. □

REMARK 2.7. For the case where P is compact, f is continuous, and there exists $\tau_0 > 0$, such that $\mathrm{int}(\{x \in A : f_p(x) = t\}) = \emptyset$, for all $p \in P$ and $t \in [0, \tau_0]$, the proof of Theorem 1.1 is much easier:

Let $\overline{f} : P \times A \to P \times \mathbb{R}$ defined by $\overline{f}(p,x) = (p, |f(p,x)|)$. Then $\overline{f}$ is a continuous definable map on compact set. Applying [12, Theorem 3.3] to $\overline{f}$ and the family of sets $S_{p,t} = \{p\} \times [0,t], (p,t) \in P \times [0, \tau_0]$, we get $\varphi \in \Phi$, such that

$$\mathrm{Vol}(\{x \in A : |f_p(x)| \leqslant t\}) = \mathrm{Vol}(\{(p,x) \in P \times A : \overline{f}^{-1}(S_{p,t})\}) \leqslant \varphi(t), \text{ for all } 0 \leqslant t \leqslant \tau_0.$$

In this case, we get a uniform estimate for the volumes of the sub-level sets of the family $(f_p)_{p \in P}$, i.e. $C(p)$, in the estimate of the theorem, can be chosen to be constant.

REMARK 2.8. In general, we do not have any uniform estimate in Theorem 1.1, i.e. C can be unbounded. For example, let $f_p(x) = p$, for $(p,x) \in (0,1) \times [0,1]$. Then

$$\mathrm{Vol}(\{x : |f_p(x)| \leqslant t\}) = \begin{cases} 1, & \text{if } p \leqslant t \leqslant 1, \\ 0, & \text{if } 0 < t < p. \end{cases}$$

Therefore, for any $\varphi \in \Phi$ and $C : (0,1) \to \mathbb{R}_+$, so that

$$\mathrm{Vol}(\{x : |f_p(x)| \leqslant t\}) \leqslant C(p)\varphi(t), \text{ for all } t \geqslant 0, p \in (0,1),$$

we have $C(p) \geqslant \dfrac{1}{\varphi(p)} \to \infty$, when $p \to 0^+$.

3. Proof of Theorem 1.2

As noted in the introduction, to estimate the oscillatory integrals, according to the Stationary Phase Principle and the Scaling Principle, we need to investigate the sets of critical points of the phase functions and where their gradients are bounded from below. Some useful information for a definable family of functions are given in the following lemmas.

LEMMA 3.1. (the Bochnak-Łojasiewicz inequality) *Let* $f : P \times A \to \mathbb{R}$ *be a definable function, where* A *is a bounded subset of* $\mathbb{R}^n$ *of dimension* n. *Let* $C^1(f_p) = \{x \in \operatorname{int} A : f_p \in C^1$ *on a neighborhood of* $x\}$, $p \in P$. *Then*

1. $(C^1(f_p))_{p \in P}$ *is a definable family.*
2. $\dim(A \setminus C^1(f_p)) < n$, *for all* $p \in P$.
3. *There exists a positive definable function* $\tau : P \to \mathbb{R}$, *such that*

$$\|\nabla f_p(x)\| \geqslant |f_p(x)|, \text{ whenever } |f_p(x)| \leqslant \tau(p), x \in C^1(f_p).$$

Proof. The arguments are similar to that of [11, Theorem 3].
Since $(C^1(f_p))_{p \in P}$ is the the family of fibers of the following definable set

$$\{(p,x) \in P \times A : p \in P, x \in \operatorname{int} A, f_p \in C^1 \text{ on a neighborhood of } x\},$$

it is a definable family. Statement (2) is followed from C^1-Cell Decomposition (see [6, Ch. 7 (3.2)]).

To prove (3), for $p \in P$ and $t > 0$, let

$$F(p,t) = \{x \in C^1(f_p) : |f_p(x)| = t\}.$$

By o-minimality, two cases are to be considered.

Case 1: $p \in P_1 = \{p' \in P : \text{ there is } \delta > 0, F(p,t) = \emptyset, \text{ for all } t \in (0,\delta)\}$. In this case, let $\tau_1(p) = \min(\sup\{\delta > 0 : F(p,t) = \emptyset, \forall t \in (0,\delta)\}, 1)$. Then $\tau_1 : P_1 \to \mathbb{R}$ is definable, and $F(p,t) = \emptyset$ for all $t \in (0, \tau_1(p))$. Therefore, the desired inequality holds for $\tau(p) = \tau_1(p)$.

Case 2: $p \in P_2 = \{p' \in P : \text{ there is } \delta > 0, F(p,t) \neq \emptyset, \text{ for all } t \in (0,\delta)\}$. Let $\tau_2(p) = \min(\sup\{\delta > 0 : F(p,t) \neq \emptyset, \forall t \in (0,\delta)\}, 1)$. Then $\tau_2 : P_2 \to \mathbb{R}$ is definable, and $F(p,t) \neq \emptyset$ for all $t \in (0, \tau_2(p))$. Let

$$v(p,t) = \inf\{\|\nabla f_p(x)\| : x \in F(p,t)\}, t \in (0, \tau_2(p)), p \in P_2.$$

By [11, Lemma 2.4], there exists $\delta > 0$ such that $v(p,t) > t$, for all $t \in (0,\delta)$. Let $\tau_3(p) = \min(\tau_2(p), \sup\{\delta > 0 : v(p,t) > t, \forall t \in (0,\delta)\})$.

Finally, let $\tau : P = P_1 \cup P_2 \to \mathbb{R}$, defined by $\tau(p) = \tau_1(p)$ if $p \in P_1$, and $\tau(p) = \tau_3(p)$ if $p \in P_2$. Then τ is definable and $\|\nabla f_p(x)\| \geqslant |f_p(x)|$, whenever $|f_p(x)| \leqslant \tau(p)$. □

DEFINITION 3.2. Let $g : A \to \mathbb{R}$ be a definable function on a subset A of $\mathbb{R}^n$. The set of the *asymptotic critical values* of g is defined by

$$K_a(g) = \{c : \text{there exists a sequence } (x_k)_{k\in\mathbb{N}} \text{ in } C^1(g), g(x_k) \to c, \nabla g(x_k) \to 0, \\ \text{when } k \to \infty\}.$$

LEMMA 3.3. *Let $f : P \times A \to \mathbb{R}$ be a definable function, where A is a bounded subset of $\mathbb{R}^n$. Then*

1. $(K_a(f_p))_{p\in P}$ *is a definable family.*
2. *There exists $M \in \mathbb{N}$ such that* $\operatorname{card}(K_a(f_p)) < M$, *for all $p \in P$.*
3. *There exists a positive definable function $\tau : P \to \mathbb{R}$, such that for all $p \in P, c \in K_a(f_p)$, we have*

$$\|\nabla f_p(x)\| \geqslant |f_p(x) - c|, \text{ whenever } |f_p(x) - c| \leqslant \tau(p), x \in C^1(f_p).$$

Proof. The family $(K_a(f_p))_{p\in P}$ is the fibers of the following set

$$K(f) = \{(p,c) : p \in P, \text{there exists a sequence } (x_k)_{k\in\mathbb{N}} \text{ in } C^1(f_p), g(x_k) \to c, \nabla g(x_k) \to 0\}.$$

It is easy to express $K(f)$ by (ε-δ) formulas on definable sets, and hence (1) follows.

By Lemma 3.1 (see also [10] or [11]), $K_a(f_p)$ is discrete, and hence, from the definability, it is finite. By Uniform Finiteness Property [6, Ch 3. (2.13)], we get (2). For each $c \in K_a(f_p)$, applying Lemma 3.1, we get the $\tau_c(p)$ corresponding to $f_p - c$. Taking $\tau(p) = \min_{c\in K_a(f_p)} \tau_c(p)$, we obtain the desired inequalities in (3). □

REMARK 3.4. If A is not bounded or f is not definable, then $K_a(f_p)$ can be infinite. See the following examples.

EXAMPLE 3.5. Let $f(x,y) = \dfrac{x}{y}$, for $(x,y) \in A = \mathbb{R} \times (0,\infty)$. Then $K_a(f) = \mathbb{R}$.

EXAMPLE 3.6. Let $f(x) = x\sin\frac{1}{x}$, for $x \neq 0$. Then $K_a(f)$ is infinite.

LEMMA 3.7. (van der Corput) *Let $f : (a,b) \to \mathbb{R}$ be a C^1 function. Fix $t > 0$. Suppose that $|f'(x)| \geqslant t, \forall x \in (a,b)$, and f' is monotonic. Then*

$$\left| \int_a^b e^{i\lambda f(x)} dx \right| \leqslant 3(\lambda t)^{-1}, \text{ for all } \lambda > 0.$$

Proof. See, for example, [18, Ch. VIII, Proposition 2]. □

LEMMA 3.8. *Let $f : P \times A \to \mathbb{R}$ be a definable function. Write $x = (x', x_n) \in A \subset \mathbb{R}^{n-1} \times \mathbb{R}$. Set V denote the set of all $(x', x_n, p, t) \in A \times P \times \mathbb{R}_+$ such that*

$$f_p(x', \cdot) \in C^1, \partial_n f_p(x', \cdot) \text{ is monotonic, and } |\partial_n f(x', \cdot)| \geqslant t, \text{ on a neighborhood of } x_n.$$

Then V is a definable set, and there exists $N \in \mathbb{N}$ such that the numbers of the connected components of the fibers $V_{(p,x',t)}$ are bounded by N.

Proof. The definability of V is obvious. By Cell Decomposition [6, Ch. 7 (3.2)] and Monotonicity Theorem [6, Ch. 3 (1.2)], $V_{(x',p,t)}$ has finite connected components. The uniform bound for the connected components is followed by Trivialization Theorem [6, Ch. 9 (1.2)]. □

Proof of Theorem 1.2. For each $p \in P$, taking τ as in Lemma 3.3, we set

$$A = \bigcup_{c \in K_a(f_p)} A_{p,c} \cup A'_p,$$

where $A_{p,c} = \{x \in A : |f_p(x) - c| \leqslant \tau(p)\}, A'_p = \{x \in A : |f_p(x) - c| > \tau(p), \forall c \in K_a(f_p)\}$. We will estimate the integrals on each set of the union.

For each $c \in K_a(f_p)$ and $0 < t \leqslant \tau(p)$, depending on λ that we will choose later, let

$$A_{p,c} = A_{p,c,t} \cup B_{p,c,t},$$

where $A_{p,c,t} = \{x \in A_{p,c} : |f_p(x) - c| \leqslant t\}, B_{p,c,t} = \{x \in A_{p,c} : |f_p(x) - c| > t\}$.

To estimate the integral on $A_{p,c,t}$, we apply Theorem 1.1 to get $\varphi_1 \in \Phi$ and $C_1 : P \to \mathbb{R}$ being positive definable function, such that

$$\begin{aligned} \left| \int_{A_{p,c,t}} e^{i\lambda f(p,x)} g(x) dx \right| &\leqslant \text{Vol}(A_{p,c,t}) \|g\|_\infty \leqslant \text{Vol}(\{x \in A : |f_p(x) - c| \leqslant t\}) \|g\|_\infty \\ &\leqslant C_1(p) \varphi_1(t) \|g\|_\infty, \text{ for all } t \geqslant 0. \end{aligned}$$

Note that, by Lemma 3.1, $B_{p,c,t} = C^1(f_p|_{B_{p,c,t}}) \cup (B_{p,c,t} \setminus C^1(f_p|_{B_{p,c,t}})$, in which the last set has measure 0. So, to estimate the integral on $B_{p,c,t}$, we simply consider it on $C^1(f_p|_{B_{p,c,t}})$. Since on this set $t < |f_p - c| \leqslant \tau(p)$, by the Bochnak-Łojasiewicz inequality in Lemma 3.3,

$$C^1(f_p|_{B_{p,c,t}}) \subset \{x \in C^1(f_p) : \|\nabla f_p(x)\| \geqslant t\}, \text{ when } 0 < t \leqslant \tau(p).$$

Therefore,

$$C^1(f_p|_{B_{p,c,t}}) \subset \bigcup_{k=1}^{n} \{x \in C^1(f_p) \cap A_{p,c} : |\partial_k f_p(x)| \geqslant \tfrac{t}{n}\},$$

For $k = n$, let $x = (x', x_n)$ denote a point in $A_{p,c} \subset \mathbb{R}^{n-1} \times \mathbb{R}, I$ the projection of $A_{p,c}$ to the last coordinate, and $\tilde{A}_{p,c}$ the projection of $A_{p,c}$ to the first $n-1$ coordinates. By Lemma 3.8, there exists $N \in \mathbb{N}$ such that for all $p \in P$ and $x' \in A_{p,c}$, the set of $x_n \in I_{p,x'} = \{s \in I : (p, x', s) \in A_{p,c}\}$ such that

$$f_p(x', \cdot) \in C^1, \partial_n f_p(x', \cdot) \text{ is monotonic, and } |\partial_n f_p(x', \cdot)| \geqslant \frac{t}{n}, \text{ on a neighborhood of } x_n,$$

is the union of N intervals atmost, say $I_1,\cdots,I_N$ (depending on p,x' and some of them may be empty). On each of the intervals, say $I_j=(a,b)$, let $F_p(x',x_n)=\int_a^{x_n} e^{i\lambda f_p(x',s)}ds$. Since $|\partial_n f_p(x',x_n)|\geqslant t/n$ on I_j, by the van der Corput Lemma 3.7, $|F_p(x',x_n)|\leqslant 3(\frac{\lambda t}{n})^{-1}$. Integrating by parts and using this inequality, we get

$$\left|\int_{I_j} e^{i\lambda f_p(x',x_n)}g(x',x_n)dx_n\right| = \left|\int_{I_j}\partial_n F_p(x',x_n)g(x',x_n)dx_n\right| \leqslant 3(\tfrac{\lambda t}{n})^{-1}2\|g\|_\infty + 3(\tfrac{\lambda t}{n})^{-1}\int_{I_j}|\partial_n g(x',x_n)|dx_n.$$

Applying the Fubini Theorem and the above estimation on each of the intervals, we get

$$\begin{aligned}&\left|\int_{|\partial_n f_p|\geqslant\frac{t}{n}} e^{i\lambda f_p(x)}g(x)\chi_{A_{p,c}}(x)dx\right|\\ &\leqslant \int_{\tilde{A}_{p,c}}\left(\sum_{j=1}^N\left|\int_{I_j} e^{i\lambda f_p(x',x_n)}g(x',x_n)dx_n\right|\right)dx'\\ &\leqslant \int_{\tilde{A}_{p,c}}\left(\sum_{j=1}^N\left(3(\tfrac{\lambda t}{n})^{-1}(2\|g\|_\infty+\textstyle\int_{I_j}|\partial_n g(x',x_n)|dx_n)\right)\right)dx'\\ &\leqslant C_2(\lambda t)^{-1}(\|g\|_\infty+\|\nabla g\|_1),\end{aligned}$$

where $C_2=\max_{1\leqslant k\leqslant n}\mathscr{H}^{n-1}(p_k(A))N6n$, and $p_k:\mathbb{R}^n\to\mathbb{R}^{n-1}$ is the projection missing the k-th coordinate.

Using the similar estimations for $k=1,2,\cdots$, we get

$$\left|\int_{B_{p,c,t}} e^{i\lambda f(x,p)}g(x)dx\right| \leqslant \sum_{k=1}^n\left|\int_{|\partial_k f_p|\geqslant\frac{t}{n}} e^{i\lambda f_p(x)}g(x)\chi_{A_{p,c}}(x)dx\right| \leqslant nC_2(\lambda t)^{-1}(\|g\|_\infty+\|\nabla g\|_1).$$

So, for each $p\in P$ and $c\in K_a(f_0)$, we have

$$\begin{aligned}\left|\int_{A_{p,c}} e^{i\lambda f(p,x)}g(x)dx\right| &\leqslant C_1(p)\varphi_1(t)\|g\|_\infty + nC_2(\lambda t)^{-1}(\|g\|_\infty+\|\nabla g\|_1)\\ &\leqslant (C_1(p)\varphi_1(t)+nC_2(\lambda t)^{-1})(\|g\|_\infty+\|\nabla g\|_1),\ \text{when } 0<t\leqslant\tau(p),\end{aligned}$$

To estimate the integral on A'_p, note that, by the definition of $K_a(f_p)$, the function $C_3: P\to\mathbb{R}$, defined by $C_3(p)=\inf_{x\in A'_p\cap C^1(f_p)}\|\nabla f_p(x)\|$, is a positive definable function. Therefore, applying the estimation for the integral on $B_{p,c,t}$ with $t=C_3(p)$, we get

$$\left|\int_{A'_p} e^{i\lambda f(p,x)}g(x)dx\right| \leqslant nC_2(\lambda C_3(p))^{-1}(\|g\|_\infty+\|\nabla g\|_1) = C_4(p)\lambda^{-1}(\|g\|_\infty+\|\nabla g\|_1).$$

Summing up, for each $p\in P$, using $\operatorname{card}K_a(f_p)\leqslant M$, when $0<t\leqslant\tau(p)$, we have

$$\begin{aligned}\left|\int_A e^{i\lambda f(x,p)}g(x)dx\right| &\leqslant \textstyle\sum_{c\in K_a(f_p)}\left|\int_{A_{p,c}} e^{i\lambda f(p,x)}g(x)dx\right| + \left|\int_{A'_p} e^{i\lambda f(p,x)}g(x)dx\right|\\ &\leqslant (MC_1(p)\varphi_1(t)+MnC_2(\lambda t)^{-1}+C_4(p)\lambda^{-1})(\|g\|_\infty+\|\nabla g\|_1)\\ &\leqslant C_5(p)(\varphi_1(t)+(\lambda t)^{-1}+\lambda^{-1})(\|g\|_\infty+\|\nabla g\|_1),\end{aligned}$$

where $C_5(p) = \max(MC_1(p), MnC_2, C_4(p))$.

Now we choose $t = \lambda^{-\frac{1}{2}}$, and take $\varphi \in \Phi$, defined by $\varphi(s) = \varphi_1(\sqrt{s}) + \sqrt{s} + s^2, s \geqslant 0$, to get

$$\left| \int_A e^{i\lambda f(p,x)} g(x) dx \right| \leqslant C_5(p) \varphi(\lambda^{-1})(\|g\|_\infty + \|\nabla g\|_1), \text{when } 0 < \lambda^{-1} \leqslant \tau^2(p).$$

Since $\left| \int_A e^{i\lambda f(p,x)} dx \right| \leqslant \mathrm{Vol}(A)$ for all $\lambda > 0$, taking $C(p) = \max(C_5(p), \frac{\mathrm{Vol}(A)}{\varphi(\tau^2(p))})$, we have

$$\left| \int_A e^{i\lambda f(p,x)} g(x) dx \right| \leqslant C(p) \varphi(\lambda^{-1})(\|g\|_\infty + \|\nabla g\|_1), \text{ for all } \lambda > 0.$$

The desired estimate is made.

When the structure is polynomially bounded, the last inequality of the theorem comes from Lemma 2.6. □

REMARK 3.9. (cf. [5, Remark 3.4]) In Theorem 1.2, the supposition $\mathrm{int}(\{x \in A : f_p(x) = t\}) = \emptyset$, for all t, is necessary. Since if there is some $c \in \mathbb{R}$ such that $C = \mathrm{int}(\{x \in A : f_p(x) = c\}) \neq \emptyset$, then $c \in K_a(f_p)$ and for $g = \chi_C$, we have

$$\left| \int_A e^{i\lambda f(p,x)} g(x) dx \right| = \left| \int_C e^{i\lambda c} dx \right| = \mathrm{Vol}(C) \not\to 0, \text{ when } \lambda \to \infty.$$

REFERENCES

[1] V. I. ARNOLD, S. M. GUSEIN-ZADE AND A. N. VARCHENKO, *Singularities of differentialble maps, Vol. II*, Monographs in Mathematics, vol. 83, Birkhäuser Boston Inc., MA, 1988.

[2] M. COSTE, *An Introduction to O-minimal Geometry*, Dottorato di Ricerca in Matematica, Dip. Mat. Pisa. Instituti, 2000.

[3] A. CARBERY, M. CHRIST AND J. WRIGHT, *Multidimensional van der Corput and sublevel set estimates*, J. Amer. Math. Soc. 12, no. 4, 1999, 981–1015.

[4] R. CLUCKERS, G. COMTE, D. J. MILLER, J.-P. ROLIN AND T. SERVI, *Intergration of oscillarory and subanalytic functions*, Duke Math. J. 165, no. 7, 2018, 1239–1309.

[5] R. CLUCKERS AND D. J. MILLER, *Bounding the decay of oscillatory integrals with a constructible amplitute function and a globally subanalytic phase functions*, J. Fourier Anal. Appl. 22, 2016, 215–236.

[6] L. VAN DEN DRIES, *Tame Topology and o-minimal Structures*, LMS Lecture Notes, Cambridge University Press, 1997.

[7] L. VAN DEN DRIES AND C. MILLER, *Geometric Categories and o-minimal Structures*, Duke Math. J. 84, no. 2, 1996, 497–540.

[8] H. FEDERER, *Geometric measures theory*, Springer-Verlag, 1969.

[9] R. M. HARDT, *Some analytic bounds for subanalytic sets*, Differential geometric control theory, (Houghton, Mich., 1982): 259–267, Progr. Math. 27, Birkhauser Boston, Boston, Mass., 1983.

[10] K. KURDYKA, *On the gradients of functions definable in o-minimal structures*, Ann. Inst. Fourier (**48**) 3, 1998, 769–783.

[11] T. L. LOI, *Łojasiewicz inequalities in o-minimal structures*, Manuscripta Math., 150, 2016, 59–72.

[12] T. L. LOI, *Hölder-Łojasiewicz inequalities for volumes of tame objects*, Math. Z., 2021, `https://doi.org/10.10007/s00209-021-02696-9`.

[13] T. L. LOI, *Stratification of families of functions definable in o-minimal structures*, Acta Mat. Vietnamica 27, no. 2, 2002, 239–244.

[14] T. L. LOI, *Verdier Stratifications and Thom Stratifications in O-minimal Structures*, Illinois J. Math. 42, no. 2, 1998, 347–356.
[15] T. L. LOI AND P. PHIEN, *Bounds of Hausdorf measures of tame sets*, Acta Math. Vietnam 39, 2014, 637–647, doi:10.1007/s40306-014-0090-z.
[16] M. SHIOTA, *Geometry of subanalytic and semialgebraic sets*, Progress in Mathematics vol. 150, Birkhäuser, 1997.
[17] P. SPEISSEGGE, *Fiberwise properties of definable sets and functions in o-minimal structures*, Manuscipta math. 86, 1995, 283–291.
[18] E. M. STEIN, *Harmonic Analysis*, Priceton U. Press, Priceton, 1993.

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