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SOME MIXED WEAK TYPE INEQUALITIES

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Abstract. We study mixed weak type (1,1) weighted inequalities for the Hardy averaging operator, $T_c f(x) = \chi_{(c,\infty)}(x)\frac{1}{x-c}\int_c^x f(y)dy$. This type of inequalities have the form

$$\int_{\{x\in\mathbb{R}:|T_c f(x)|>v(x)\}} uv \leqslant C\int_{\mathbb{R}} |f|u,$$

where C is independent of f and c. We improve the results in [Q. J. Math. 60 (2009), no. 1, 63–73] by giving a wider class of pairs of weights for which the inequality holds. In particular, and as a corollary, we prove that the inequality holds for $u \in A_1^-$ and $v \in A_\infty^+$.

1. Introduction

It is well known that a sublinear operator is of weak type $(1,1)$ with respect to the measures $u(x)\,dx$ and $w(x)\,dx$ if

$$\int_{\{x\in\mathbb{R}^n:|Tf(x)|>\lambda\}} u \leqslant \frac{C}{\lambda}\int_{\mathbb{R}^n} |f|w, \tag{1.1}$$

where u and w are nonnegative measurable functions and C is independent of f and $\lambda > 0$. The pairs of weights which satisfy this inequality for classical operators have been studied in the last 50 years. Among these operators we have the following:

1. The Hardy averaging operator

$$Tf(x) = \chi_{(0,\infty)}(x)\frac{1}{x}\int_0^x f(y)\,dy$$

and its adjoint

$$T^* f(x) = \chi_{(0,\infty)}(x)\int_x^\infty \frac{f(y)}{y}\,dy.$$

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2. The one–sided Hardy-Litlewood maximal operators

$$M^- f(x) = \sup_{c<x} \frac{1}{x-c}\int_c^x |f| \quad \text{and} \quad M^+ f(x) = \sup_{b>x} \frac{1}{b-x}\int_x^b |f|.$$

3. The Hardy-Litlewood maximal operator

$$Mf(x) = \sup_Q \chi_Q(x)\frac{1}{|Q|}\int_Q |f|,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ with sides parallel to the axes.

4. The Hilbert transform

$$Hf(x) = \lim_{\varepsilon\to 0^+}\int_{\{y:|x-y|>\varepsilon\}} \frac{f(y)}{x-y}\,dy$$

or, with more generality, the Calderón-Zygmund operators.

Let T be any sublinear operator and, for $1<p<\infty$, let $A_p(T)$ be the class of weights u such that $T : L^p(u) \to L^p(u)$ is bounded and let $A_\infty(T) = \cup_{p>1}A_p(T)$ (we simply write A_p and A_∞ when T is the Hardy-Littlewood maximal operator). These classes of weights have been extensively studied and characterised for the above operators (see [13], [16], [9]). Now, take a positive measurable function v and consider the modification of the above operator given by

$$T_v f = \frac{1}{v}T(fv).$$

Which is the class of weights $A_p(T_v)$?, that is, which are the weights u such that

$$\int |T_v f|^p u \leqslant C\int |f|^p u$$

for all functions f? Clearly, the last inequality is equivalent to

$$\int |Tf|^p uv^{-p} \leqslant C\int |f|^p uv^{-p}.$$

Thus, $u \in A_p(T_v)$ if and only if $uv^{-p} \in A_p(T)$. Therefore, the “good weights” for T_v are obtained from the “good weights” for T.

Now we wonder about the weighted weak type inequalities. Let weak-$A_p(T)$ be the class of weights u such that $T : L^p(u) \to L^{p,\infty}(u)$ is bounded (as before, we omit the T when T is the Hardy-Littlewood maximal operator; for p=1, we usually write A_1 instead of weak-A_1). Which is the class of weights weak-$A_p(T_v)$?, that is, which are the weights u such that

$$\int_{\{|T_v f|>\lambda\}} u \leqslant \frac{C}{\lambda^p}\int |f|^p u$$

for all functions f and all $\lambda > 0$? The last inequality is equivalent to

$$\int_{\{|Tf|>\lambda v\}} u \leqslant \frac{C}{\lambda^p} \int |f|^p u v^{-p}.$$

As we can see, it is not possible to reduce the study of the weak type inequalities for T_v to the corresponding weak-type inequalities for T. For convenience, we will replace u by uv in such a way that the last inequality can be written as

$$\int_{\{x\in\mathbb{R}^n:|Tf(x)|>\lambda v(x)\}} uv \leqslant \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |f|^p u v^{1-p}. \tag{1.2}$$

In this way, u belongs to weak-$A_p(T_v)$ if and only if the pair $(u/v, v)$ satisfies (1.2) for all functions f and all $\lambda > 0$. These inequalities are of different nature and they are called mixed weak type inequalities. As far as we know, the expression mixed weak type inequality for a sublinear operator T was coined in [1]. We are specially interested in the case $p = 1$, that is, in this paper we will search mixed weak type $(1,1)$ inequalities of the form

$$\int_{\{x\in\mathbb{R}^n:|Tf(x)|>\lambda v(x)\}} uv \leqslant \frac{C}{\lambda} \int_{\mathbb{R}^n} |f| u, \tag{1.3}$$

where C is independent of f and $\lambda > 0$; clearly, this inequality is equivalent to saying that the operator T_v is of weak type $(1,1)$ with respect to the measure $u(x)v(x)\,dx$.

Obviously, if $v = 1$ then (1.3) is the same as (1.1) with $w = u$. For that reason, a natural approach is to start with a function u such that (1.1) holds, i.e. $u \in$weak-$A_1(T)$, and to ask about which kind of weights v allow to get mixed weak type inequalities of the form (1.3). In [1] the authors obtain mixed weak type $(1,1)$ inequalities for some classical operators. We collect their result in the following theorem.

THEOREM 1.1. [1] *Let u satisfy the Muckenhoupt A_1 condition: $Mu(x) \leqslant Cu(x)$ a.e.. Let $v(x) = |x|^{-d}$, $x \in \mathbb{R}$, where d is a real number, $d \neq 1$. If T denotes the Hilbert transform or the Hardy-Littlewood maximal operator (in one dimension) then there is a constant C dependent on d such that for all $\lambda > 0$ and all $f \in L^1(u)$*

$$\int_{\{x\in\mathbb{R}:|Tf(x)|>\lambda v(x)\}} uv \leqslant \frac{C}{\lambda} \int_{\mathbb{R}} |f| u. \tag{1.4}$$

REMARK 1.2. If $d < 1$ then v is a weight in the Muckenhoupt A_∞ class; if $d > 1$ then v is not a Muckenhoupt weight (it is not locally integrable); if $d = 1$ then v is not a Muckenhoupt weight and the result is false [1].

However, this is not the first mixed weak type $(1,1)$ inequality which appeared in this setting. In fact, we can found the following result in the previous paper [14].

THEOREM 1.3. *Let u satisfy the Muckenhoupt A_1 condition and $v = u^{-1}$. If T denotes the Hilbert transform or the Hardy-Littlewood maximal operator (in one* *dimension) then there is a constant C such that* (1.4) *holds for all $\lambda > 0$ and all $f \in L^1(u)$, that is*

$$|\{x \in \mathbb{R} : |Tf(x)| > \lambda v(x)\}| \leqslant \frac{C}{\lambda} \int_{\mathbb{R}} |f| u. \tag{1.5}$$

REMARK 1.4. Notice that v is not necessarily a weight in the Muckenhoupt A_1 class but v is in A_2 and, therefore, it is in A_∞.

Eric Sawyer used some ideas of the proof of the above result and established that (1.4) holds if v is also in the Muckenhoupt A_1 class.

THEOREM 1.5. [16] *Let u and v satisfy the Muckenhoupt A_1 condition in the real line. If T denotes the Hardy-Littlewood maximal operator (in one dimension) then there is a constant C such that* (1.4) *holds for all $\lambda > 0$ and all $f \in L^1(u)$.*

Sawyer conjectured that the result should be true for the Hilbert transform. This was proved in [5] where, in fact, the result was extended to the n dimensional setting and also for any Calderón-Zygmund operator T. In a recent paper, [8], Li, Ombrosi and Pérez have proved (among other many things) that the result is true with a weaker condition on v, that is, v is an A_∞ weight.

THEOREM 1.6. [8] *Let u satisfy the Muckenhoupt A_1 condition in $\mathbb{R}^n$ and let v satisfy the Muckenhoupt A_∞ condition. If T denotes the Hardy-Littlewood maximal operator or a Calderón-Zygmund operator on $\mathbb{R}^n$ then there is a constant C such that* (1.4) *holds for all $\lambda > 0$ and all $f \in L^1(u)$.*

REMARK 1.7. Notice that the last theorem generalizes Theorems 1.3 and 1.5 (see Remark 1.4). However it is not a generalization of Theorem 1.1 because the functions $v(x) = |x|^{-d}$ satisfy A_∞ condition if and only if $d < 1$.

Other interesting mixed weak type $(1,1)$ inequalities appear in [11].

THEOREM 1.8. [11, Theorems 1.7 and 1.8] *Let u satisfy the Muckenhoupt A_1 condition on $\mathbb{R}^n$ and let v be a radial function on $\mathbb{R}^n$ which is essentially constant on the sets $\{x \in \mathbb{R}^n : 2^k \leqslant |x| < 2^{k+1}\}$. Assume also that one of the following conditions is satisfied: for some $\varepsilon > 0$, the function $v(x)|x|^{n+\varepsilon}$ is radially decreasing or the function $v(x)|x|^{n-\varepsilon}$ is radially increasing. If T denotes a Calderón-Zygmund operator on $\mathbb{R}^n$ (or the Hardy-Littlewood maximal operator) then there is a constant C such that* (1.4) *holds for all $\lambda > 0$ and all $f \in L^1(u)$.*

REMARK 1.9. Notice that in the last theorem we can take $v(x) = |x|^{-n-\varepsilon}(\log(1+|x|))^{-1}$ or $v(x) = |x|^{\varepsilon-n}\log(1+|x|)$. Therefore, Theorem 1.8 generalizes Theorem 1.1 and the weight v is not necessarily in A_∞. Therefore, it is not a consequence of Theorem 1.6.

The mixed weak type two-weight estimates were also investigated in the monograph [6]. In that monograph (see Theorems 8.2.8 and 8.2.9, pp. 536–537) mixed weak type inequalities are derived for the Calderón-Zygmund operators K defined on spaces of homogeneous type. Here we formulate those statements for Euclidean spaces.

THEOREM 1.10. [6] *Let $\alpha \leqslant 0$ and let σ and u be increasing functions on $(0,\infty)$, provided that σ is continuous. Suppose that ρ is a weight on $\mathbb{R}^n$ such that it belongs to the Muckenhoupt class A_1. We set $v(x) = \sigma(|x|)\rho(x)$ and $w(x) = u(|x|)\rho(x)$, where $|x|$ is the norm of x. If*

$$\sup_{\tau>t>0}\left(\frac{1}{\tau^n}\int_{t<|x|<\tau} v(x)dx\right)\operatorname{ess\,sup}_{|x|<t}\frac{1}{w(x)} < \infty,$$

then the two-weight weak type inequality holds:

$$\int_{\{x\in\mathbb{R}^n:|x|^{\alpha n}|Kf(x)|>\lambda\}} v(x)|x|^{-\alpha n}dx \leqslant \frac{C}{\lambda}\int_{\mathbb{R}^n}|f(x)|w(x)dx. \tag{1.6}$$

THEOREM 1.11. [6] *Let $\alpha \leqslant 0$ and let σ and u be decreasing functions on $(0,\infty)$, provided that σ is continuous. Suppose that ρ is a weight on $\mathbb{R}^n$ such that it belongs to the Muckenhoupt class A_1. We set $v(x) = \sigma(|x|)\rho(x)$ and $w(x) = u(|x|)\rho(x)$, where $|x|$ is the norm of x. If*

$$\sup_{t>0}\left(\int_{|x|<2t} v(x)dx\right)\operatorname{ess\,sup}_{|x|>t}\frac{1}{|x|^n w(x)} < \infty,$$

then the inequality (1.6) holds.

REMARK 1.12. As in Remark 1.9, we notice that, even for $\rho = 1$, it can be constructed a pair of weights (v,w) such that v and w are out of the Muckenhoupt A_∞ class but the two-weight inequality (1.6) holds.

Other interesting results about mixed weak estimates of Sawyer type can be found in [2, 3, 4, 15].

It is interesting to ask whether or not the corresponding results for the one-sided Hardy-Littlewood maximal operators hold. As far as we know there are no analogous results for M^- (M^+). Then, one wonders about a more modest and probably easier question. Take the Hardy averaging-operator defined for $c \in \mathbb{R}$ by

$$T_c f(x) = \chi_{(c,\infty)}(x)\frac{1}{x-c}\int_c^x f(y)dy.$$

It is clear that $|T_c f| \leqslant M^- f$ and that T_c is apparently easier than M^-. Then we ask the same question for T_c: which kind of mixed weak type $(1,1)$ inequality can be obtained for T_c? In fact, the mixed weak type $(1,1)$ inequality for T_c were characterised in [11] (see [10] for a more clarifying statement and proof).

THEOREM 1.13. [11, 10] *Let u and v be non-negative measurable functions defined on $\mathbb{R}$. Let $c \in \mathbb{R}$. The following statements are equivalent.*

(a) There exists a constant C such that

$$\int_{\{x:|T_c f(x)|>v(x)\}} uv \leqslant C \int_{\mathbb{R}} |f|u$$

for all measurable functions.

(b) There exists a constant C such that for all $a > c$

$$\sup_{\lambda>0} \lambda \int_{\{x>a:\frac{1}{x-c}>\lambda v(x)\}} uv \leqslant Cu(x) \quad \text{for a.e. } x \in (c,a).$$

As a corollary of the last result, the characterisation of the weak type $(1,1)$ inequality for T_c is obtained. It is convenient to state the characterisation not only for T_c but also for its formal adjoint

$$T_c^* f(x) = \chi_{(c,\infty)}(x) \int_x^\infty \frac{f(y)}{y-c} dy.$$

THEOREM 1.14. [1, 10]

1. *The operator T_c is of weak type $(1,1)$ with respect to the measure $u(x)dx$ if and only if $u \in A_1(T_c)$, that is, there exists $C > 0$ such that for all $a > c$*

$$\sup_{y>a} \frac{1}{y-c} \int_a^y u \leqslant Cu(x), \qquad \text{for a.e. } x \in (c,a).$$

2. *The operator T_c^* is of weak type $(1,1)$ with respect to the measure $v(x)dx$ if and only if $v \in A_1(T_c^*)$, that is, there exists $C > 0$ such that*

$$\frac{1}{x-c} \int_c^x v \leqslant Cv(x), \qquad \text{for a.e. } x > c.$$

Now, a first question for T_c appears: if $u \in A_1(T_c)$ and $v \in A_1(T_c^*)$ does the mixed weak type inequality (1.3) hold for T_c? We know that the answer is affirmative but with a little bit stronger conditions on u and v.

THEOREM 1.15. [10] *Let $c \in \mathbb{R}$ and let u, v be two weights such that $u^{1+\varepsilon} \in A_1(T_c)$ and $v^{1+\varepsilon} \in A_1(T_c^*)$, for some $\varepsilon > 0$. Let $T = T_c$. Then, there exists $C > 0$ independent on c such that (1.4) holds for all $\lambda > 0$ and all $f \in L^1(u)$.*

REMARK 1.16. Notice that there exist weights such that $u \in A_1(T_c)$ and $u^{1+\varepsilon} \notin A_1(T_c)$ for all $\varepsilon > 0$ (see [10]).

As a consequence they obtained the following corollary (see [17] and [12] for the definitions of the one-sided Muckenhoupt classes).

COROLLARY 1.17. [10] *Let u,v be two weights such that $u \in A_1^-$ ($M^+u \leqslant Cu$ a.e.) and $v \in A_1^+$ ($M^-v \leqslant Cv$ a.e.). Let $T = T_c$. Then, there exists $C > 0$ independent on c such that* (1.4) *holds for all $\lambda > 0$ and all $f \in L^1(u)$.*

Then, taking into account Theorem 1.5 and Corollary 1.17, it is natural to conjecture that the one-sided version of Theorem 1.5 holds, that is, if $u \in A_1^-$ and $v \in A_1^+$ then

$$\int_{\{x:M^- f(x)>\lambda v(x)\}} uv \leqslant \frac{C}{\lambda}\int_{\mathbb{R}} |f|u.$$

In view of the results in [8] one may conjecture an even more general result: that the last inequality holds for $u \in A_1^-$ and $v \in A_\infty^+ = \cup_{p\geqslant 1} A_p^+$. So far, it has not been possible to prove these conjectures. By the moment, we shall continue to deepen the study of the behavior of T_c. In this paper we are going to give other mixed weak type inequalities for T_c and for some generalization of these operators, that will allow us to improve some of the results in [10].

2. Results

In order to simplify, we are going to establish all the results for $c = 0$, although they hold for $c \in \mathbb{R}$, and we will write T or T^* instead of T_0 or T_0^*. Let us start with some definitions.

DEFINITION 2.1. Let $1 < p < \infty$. We say that a weight w belongs to the class $A_p(T^*)$ if

$$||w||_{A_p(T^*)} = \sup_{t>0}\left(\int_0^t w\right)^{1/p}\left(\int_t^\infty \frac{w(x)^{1-p'}}{x^{p'}}dx\right)^{1/p'} < \infty.$$

The operator T^* is bounded in $L^p(w)$ if and only if $w \in A_p(T^*)$ and $A_p(T^*) \supset A_1(T^*)$.

DEFINITION 2.2. Let $1 < p < \infty$. We say that a weight w belongs to the class C_p if for some $\gamma \in (0,1)$

$$||w^{1+\gamma}||_{C_p} = \sup_{0<b} b^\gamma \left(\int_{(0,b)\cap E} w^{1+\gamma}\right)^{1/p}\left(\int_{(b,\infty)\cap E} w^{1+\gamma}\right)^{1/p'} < \infty, \tag{2.1}$$

where $E = \{x > 0 : w(x) < 1/x\}$.

PROPOSITION 2.3. *Suppose that $v^{1+\gamma} \in A_p(T^*)$ for some $0 < \gamma < 1 < p$. Then $\lambda v \in C_p$ for all positive λ and $||(\lambda v)^{1+\gamma}||_{C_p} \leqslant ||v^{1+\gamma}||_{A_p(T^*)}$.*

Proof. Since $v^{1+\gamma} \in A_p(T^*)$ implies that $(\lambda v)^{1+\gamma} \in A_p(T^*)$ and $||(\lambda v)^{1+\gamma}||_{A_p(T^*)} = ||v^{1+\gamma}||_{A_p(T^*)}$ then it suffices to prove it only for $\lambda = 1$.

For $x\in E$, $1<\dfrac{v(x)^{-(1+\gamma)p'}}{x^{(1+\gamma)p'}}$. This implies that

$$\begin{aligned} b^{\gamma p'}\int_{(b,\infty)\cap E} v^{1+\gamma}(x)dx &\leqslant b^{\gamma p'}\int_{(b,\infty)\cap E} v^{1+\gamma}(x)\frac{v(x)^{-(1+\gamma)p'}}{x^{p'}x^{\gamma p'}}dx \\ &\leqslant \int_{(b,\infty)}\frac{v(x)^{(1+\gamma)(1-p')}}{x^{p'}}dx. \end{aligned}$$

Therefore,

$$\begin{aligned} &b^{\gamma}\left(\int_{(0,b)\cap E} v^{1+\gamma}\right)^{1/p}\left(\int_{(b,\infty)\cap E} v^{1+\gamma}\right)^{1/p'} \\ &\qquad\leqslant \left(\int_0^b v^{1+\gamma}\right)^{1/p}\left(\int_b^\infty \frac{v(x)^{(1+\gamma)(1-p')}}{x^{p'}}dx\right)^{1/p'} \leqslant C \end{aligned}$$

and the constant C is $||v^{1+\gamma}||_{A_p(T^*)}$. $\square$

THEOREM 2.4. *Let u,v be two weights such that $u^{1+\varepsilon}\in A_1(T)$ and λv satisfies C_p, with constant independent on $\lambda>0$, for some ε,γ,p such that $0<\gamma<\varepsilon<1<p$. Then, there exists $C>0$ such that*

$$\int_{\{x:|Tf(x)|>\lambda v(x)\}} uv \leqslant \frac{C}{\lambda}\int_{\mathbb{R}}|f|u,$$

for all $\lambda>0$ and $f\in L^1(u)$.

Proof. By Theorem 1.13, we only have to prove that there exists $C>0$ such that for all $a>0$ and all $\lambda>0$

$$\lambda\int_{\{x>a:\,\lambda v(x)<1/x\}} uv \leqslant C\operatorname{ess\,inf}\{u(x):x\in(0,a)\}. \tag{2.2}$$

Since λv satisfies C_p, with constant independent on $\lambda>0$, it is enough to prove the above inequality with $\lambda=1$ and a constant depending only on the uniform constant of the condition C_p.

Fix $a>0$ and let $E_a=\{x>a:\,0<v(x)<1/x\}$. If $|E_a|=0$ there is nothing to prove. Then we can suppose that $|E_a|>0$. Denote by z the essencial supremum of E_a. Then $a<z$ and

$$\int_{E_a\cap(a,z)} v(x)^{1+\gamma}dx \leqslant \int_a^\infty \frac{1}{x^{1+\gamma}}dx<\infty.$$

This allows us to define the sequence $\{z_n\}_{n=0}^\infty$ in the following way: let $z_0=a$ and

$$\int_{(z_{k+1},z)\cap E_a} v^{1+\gamma} = \int_{(z_k,z_{k+1})\cap E_a} v^{1+\gamma}.$$

(We point out that the existence of that sequence follows easily considering the function

$$F(t)=\int_{E_a\cap(t,z)}v(x)^{1+\gamma}dx=\int_t^z v(x)^{1+\gamma}\chi_{E_a}(x)dx$$

because $F:[a,z)\to\mathbb{R}$ is well defined, continuous, decreasing and $F(t)>0$ for all $t\in[a,z)$ (z is the essential supremum of E_a.) Then it is immediate to prove that for all $k\in\mathbb{N}\cup\{0\}$,

$$\int_{(a,z_1)\cap E_a}v^{1+\gamma}=2^k\int_{(z_k,z_{k+1})\cap E_a}v^{1+\gamma}=2^k\int_{(z_{k+1},z)\cap E_a}v^{1+\gamma} \tag{2.3}$$

and that $\lim_{n\to\infty}z_n=z$.

Then, the hypothesis in v gives us that

$$z_{k+1}^{\gamma}\left(\int_{(a,z_1)\cap E_a}v^{1+\gamma}\right)^{1/p}\left(\int_{(z_{k+1},z)\cap E_a}v^{1+\gamma}\right)^{1/p'}\leqslant C. \tag{2.4}$$

Taking into account (2.3) and (2.4) we get

$$2^{k/p}z_{k+1}^{\gamma}\int_{(z_{k+1},z)\cap E_a}v^{1+\gamma}\leqslant C,$$

and by the definition of z_k,

$$\int_{(z_k,z_{k+1})\cap E_a}v^{1+\gamma}\leqslant\frac{C}{2^{k/p}z_{k+1}^{\gamma}}. \tag{2.5}$$

Now we proceed to prove inequality (2.2). Observe that

$$\int_{\{x>a:v(x)<1/x\}}uv=\int_{(a,z)\cap E_a}uv=\sum_{k=0}^{\infty}\int_{(z_k,z_{k+1})\cap E_a}uv. \tag{2.6}$$

For $x\in E_a$, $v(x)<1/x$ then, since $1-\gamma\varepsilon>0$, we have that $v(x)^{\frac{1-\gamma\varepsilon}{1+\varepsilon}}<\frac{1}{x^{\frac{1-\gamma\varepsilon}{1+\varepsilon}}}$. Therefore, by Hölder's inequality with exponents $(1+\varepsilon,\frac{1+\varepsilon}{\varepsilon})$ and inequality (2.5) we obtain

$$\begin{aligned}\int_{(z_k,z_{k+1})\cap E_a}uv&\leqslant\int_{(z_k,z_{k+1})\cap E_a}\frac{u(x)v(x)^{\frac{(1+\gamma)\varepsilon}{1+\varepsilon}}}{x^{\frac{1-\gamma\varepsilon}{1+\varepsilon}}}dx\\&\leqslant\left(\int_{(z_k,z_{k+1})\cap E_a}\frac{u(x)^{1+\varepsilon}}{x^{1-\gamma\varepsilon}}dx\right)^{\frac{1}{1+\varepsilon}}\left(\int_{(z_k,z_{k+1})\cap E_a}v^{1+\gamma}\right)^{\frac{\varepsilon}{1+\varepsilon}}\\&\leqslant C\left(\int_{(a,z_{k+1})}\frac{u(x)^{1+\varepsilon}}{x^{1-\gamma\varepsilon}}dx\right)^{\frac{1}{1+\varepsilon}}\left(\frac{1}{2^{k/p}z_{k+1}^{\gamma}}\right)^{\frac{\varepsilon}{1+\varepsilon}}.\end{aligned} \tag{2.7}$$

Since $0<\gamma<\varepsilon<1$, we get that $1-\varepsilon<\frac{1-\gamma\varepsilon}{1+\varepsilon}<\frac{1}{1+\varepsilon}$. Then, following the same steps than in the proof of Theorem 2.4 in [10], taking $\alpha=\frac{1-\gamma\varepsilon}{1+\varepsilon}$, we obtain that

$$\left(\int_{(a,z_{k+1})}\frac{u(x)^{1+\varepsilon}}{x^{1-\gamma\varepsilon}}dx\right)^{\frac{1}{1+\varepsilon}}\leqslant Cz_{k+1}^{\frac{\gamma\varepsilon}{1+\varepsilon}}\operatorname{ess\,inf}\{u(x):x\in(0,a)\}$$

This fact together with inequalities (2.7) and (2.6) give us

$$\int_{\{x>a:v(x)<1/x\}} uv \leqslant C\operatorname{ess\,inf}\{u(x): x\in(0,a)\}\sum_{k=0}^{\infty}\left(\frac{1}{2^{\frac{\varepsilon}{p(1+\varepsilon)}}}\right)^k$$
$$\leqslant C\operatorname{ess\,inf}\{u(x): x\in(0,a)\}. \quad \square$$

As a direct consequence of Proposition 2.3 we obtain the following corollary.

COROLLARY 2.5. *Let u,v be two weights such that $u^{1+\varepsilon}\in A_1(T)$ and $v^{1+\gamma}\in A_p(T^*)$, for some ε,γ,p such that $0<\gamma<\varepsilon<1<p$. Then, there exists $C>0$ such that*

$$\int_{\{x:|Tf(x)|>\lambda v(x)\}} uv \leqslant \frac{C}{\lambda}\int_{\mathbb{R}}|f|u,$$

for all $\lambda>0$ and $f\in L^1(u)$.

If $u\in A_1^-$ then, for $\varepsilon>0$ small enough, $u^{1+\varepsilon}\in A_1^-$ and if $v\in A_p^+$ then, for $\gamma>0$ small enough, $v^{1+\gamma}\in A_p^+$ (see [12]). We also have that $A_1^-\subset A_1(T)$ and $A_p^+\subset A_p(T^*)$ (see [10] and [17]). Then, the following result holds.

COROLLARY 2.6. *Let u,v be two weights such that $u\in A_1^-$ and $v\in A_\infty^+=\cup_{p\geqslant 1}A_p^+$. Then, there exists $C>0$ such that*

$$\int_{\{x:|Tf(x)|>\lambda v(x)\}} uv \leqslant \frac{C}{\lambda}\int_{\mathbb{R}}|f|u,$$

for all $\lambda>0$ and $f\in L^1(u)$.

In section 3 we provide examples of weights v such that λv satisfies the C_p condition, for all $\lambda>0$, with constant not depending on λ and some of them do not satisfy $A_p(T^*)$.

We can consider a more general operator (see [7], [11] and [1])

$$T_{g,\varphi}f(x)=g(x)\int_0^x f(t)\varphi(t)dt,$$

where g and φ are positive measurable functions. The following result is proved in [7] but we include the proof for the sake of clarity.

PROPOSITION 2.7. *Let u and v be two weights. Then the following assertions are equivalent:*

1. There exists $C>0$ such that

$$\int_{\{x:|T_{g,\varphi}f(x)|>v(x)\}} uv \leqslant C\int_{\mathbb{R}}|f|u.$$

2. *There exists* $C > 0$ *such that for all* $a > 0$,

$$\sup_{\lambda>0} \lambda \int_{\{x>a: g(x)>\lambda v(x)\}} uv \leqslant C \operatorname{ess\,inf}\{u(x)(\varphi(x))^{-1} : x \in (0,a)\}.$$

Proof. $(1) \Rightarrow (2)$ It is sufficient to prove it for $\lambda = 1$.

Let $a > 0$ and let $E \subset (0,a)$ be measurable. Consider $f = \frac{\chi_E}{\varphi(E)}$, where $\varphi(E) = \int_E \varphi(x)dx$. Then, for all $x > a$,

$$T_{g,\varphi} f(x) = g(x) \int_0^x \frac{\chi_E(t)}{\varphi(E)} \varphi(t)dt = g(x).$$

Therefore,

$$\int_{\{x>a: g(x)>v(x)\}} uv = \int_{\{x: |T_{g,\varphi} f(x)|>v(x)\}} uv \leqslant \frac{C}{\varphi(E)} \int_E u = C \frac{|E|}{\varphi(E)} \frac{1}{|E|} \int_E u.$$

Since $E \subset (0,a)$ is any measurable set, by the Lebesgue's differenciation theorem, we get

$$\int_{\{x>a: g(x)>v(x)\}} uv \leqslant C \operatorname{ess\,inf}\{u(x)(\varphi(x))^{-1} : x \in (0,a)\}.$$

$(2) \Rightarrow (1)$ Without lost of generality, we can suppose that $f \geqslant 0$, $f\varphi \in L^1$ and $\int_0^a f\varphi > 0$, for all $a > 0$.

Let us define the nonincreasing sequence $\{x_n\}_{n=0}^{\infty}$ by $x_0 = \infty$ and $\int_0^{x_{n+1}} f\varphi = \int_{x_{n+1}}^{x_n} f\varphi$. Then, $\lim_{n\to\infty} x_n = 0$.

Observe that for all $x \in [x_{n+1}, x_n)$,

$$T_{g,\varphi} f(x) \leqslant g(x) \int_0^{x_n} f\varphi = 4g(x) \int_{x_{n+2}}^{x_{n+1}} f\varphi.$$

Then

$$\{x : |T_{g,\varphi} f(x)| > v(x)\} \subset \bigcup_{n=1}^{\infty} \left\{ x \in [x_{n+1}, x_n) : g(x) > \frac{v(x)}{4\int_{x_{n+2}}^{x_{n+1}} f\varphi} \right\}.$$

Let $\beta_n = \operatorname{ess\,inf}\{u(x)(\varphi(x))^{-1} : x \in (0, x_{n+1})\}$. Then, by (2),

$$\int_{\left\{x \in [x_{n+1}, x_n): g(x) > \frac{v(x)}{4\int_{x_{n+2}}^{x_{n+1}} f\varphi}\right\}} uv \leqslant C\beta_n 4 \int_{x_{n+2}}^{x_{n+1}} f\varphi$$

$$\leqslant 4C \int_{x_{n+2}}^{x_{n+1}} f\varphi u(\varphi)^{-1} = 4C \int_{x_{n+2}}^{x_{n+1}} fu.$$

As a consequence,

$$\int_{\{x: |T_{g,\varphi} f(x)|>v(x)\}} uv \leqslant C \sum_{n=1}^{\infty} \int_{x_{n+2}}^{x_{n+1}} fu \leqslant C \int_{\mathbb{R}} fu. \quad \square$$

REMARK 2.8. For the particular case of $v(x) = 1$ we obtain that $T_{g,\varphi}$ is of weak type $(1,1)$ respect to the weight u if and only if

$$\sup_{a>0} ||g\chi_{(a,\infty)}||_{L^{1,\infty}(u)}||\chi_{(0,a)}u^{-1}\varphi||_{L^\infty} < \infty.$$

If we also take $\varphi = 1$, then $T_{g,\varphi} = T_g$ is of weak type $(1,1)$ respect to the weight u if and only if

$$\sup_{a>0} ||g\chi_{(a,\infty)}||_{L^{1,\infty}(u)}||\chi_{(0,a)}u^{-1}||_{L^\infty} < \infty.$$

And, if furthermore, g is nonincreasing, this is equivalent to

$$\sup_{y>a} g(y)\int_a^y u \leqslant C \operatorname{ess\,inf}\{u(x) : x \in (0,a)\}.$$

DEFINITION 2.9. Let $1 < p < \infty$ and g a positive, nonincreasing measurable function in $(0,\infty)$. We say that a weight w belongs to the class $A_p(T_g^*)$ if

$$\sup_{t>0}\left(\int_0^t w\right)^{1/p}\left(\int_t^\infty w(x)^{1-p'}g(x)^{p'}dx\right)^{1/p'} < \infty.$$

We say that w belongs to the class $A_1(T_g)$ if there exists $C > 0$ such that, for all $a > 0$,

$$\sup_{y>a} g(y)\int_a^y w \leqslant C \operatorname{ess\,inf}\{w(x) : x \in (0,a)\}.$$

We say that w belongs to the class $A_1(T_g^*)$ if there exists $C > 0$ such that,

$$g(x)\int_0^x w \leqslant Cw(x), \qquad \text{a.e. } x > 0.$$

Following the same steps as in Proposition 2.3 we have that if g is nonincreasing and $v^{1+\gamma} \in A_p(T_g^*)$ then

$$\sup_{0<b} g(b)^{-\gamma}\left(\int_{(0,b)\cap E} v^{1+\gamma}\right)^{1/p}\left(\int_{(b,\infty)\cap E} v^{1+\gamma}\right)^{1/p'} < \infty, \qquad (C_p(g))$$

where now $E = \{x > 0 : v(x) < g(x)\}$.

Then, changing the function $1/x$ by $g(x)$ in Theorem 2.4, we obtain the following result.

THEOREM 2.10. *Let g be a positive, nonincreasing measurable function in $(0,\infty)$ and let u,v be two weights such that $u^{1+\varepsilon} \in A_1(T_g)$ and λv satisfies $C_p(g)$ with constant not depending on $\lambda > 0$, for some ε,γ,p such that $0 < \gamma < \varepsilon < 1 < p$. Then, there exists $C > 0$ such that*

$$\int_{\{x:|T_g f(x)|>\lambda v(x)\}} uv \leqslant \frac{C}{\lambda}\int_{\mathbb{R}} |f|u,$$

for all $\lambda > 0$ and $f \in L^1(u)$.

COROLLARY 2.11. *Let g be a positive, nonincreasing measurable function in $(0,\infty)$ and let u,v be two weights such that $u^{1+\varepsilon} \in A_1(T_g)$ and $v^{1+\gamma} \in A_p(T_g^*)$, for some ε,γ,p such that $0<\gamma<\varepsilon<1<p$. Then, there exists $C>0$ such that*

$$\int_{\{x:|T_g f(x)|>\lambda v(x)\}} uv \leqslant \frac{C}{\lambda}\int_{\mathbb{R}} |f|u,$$

for all $\lambda>0$ and $f\in L^1(u)$.

3. Remarks on the condition C_p

PROPOSITION 3.1. *If $1<p<q$ and $v\in C_p$ then $v\in C_q$.*

Proof. We are going to prove that, for the same $\gamma\in(0,1)$ that gives $v\in C_p$,

$$\sup_{b\in(0,\infty)} b^\gamma \int_{(0,b)\cap E} v^{1+\gamma}\left(\gamma b^\gamma \int_{(b,\infty)\cap E} v^{1+\gamma}\right)^{q-1} <\infty.$$

By Jensen's inequality with exponent $(q-1)/(p-1)$ we get

$$\begin{aligned}
\gamma b^\gamma \int_{(b,\infty)\cap E} v^{1+\gamma}(x)\,dx &= \gamma b^\gamma \int_{(b,\infty)\cap E} x^{1+\gamma} v^{1+\gamma}(x) x^{-1-\gamma}\,dx \\
&\leqslant \left(\gamma b^\gamma \int_{(b,\infty)\cap E} x^{\frac{(1+\gamma)(q-1)}{p-1}} v^{\frac{(1+\gamma)(q-1)}{p-1}}(x) x^{-1-\gamma}\,dx\right)^{\frac{p-1}{q-1}} \\
&\leqslant \left(\gamma b^\gamma \int_{(b,\infty)\cap E} v^{1+\gamma}(x) x^{\frac{(1+\gamma)(q-1)}{p-1}} v^{\frac{(1+\gamma)(q-p)}{p-1}}(x) x^{-1-\gamma}\,dx\right)^{\frac{p-1}{q-1}} \\
&\leqslant \left(\gamma b^\gamma \int_{(b,\infty)\cap E} v^{1+\gamma}(x) x^{\frac{(1+\gamma)(q-1)}{p-1}} x^{\frac{(1+\gamma)(p-q)}{p-1}} x^{-1-\gamma}\,dx\right)^{\frac{p-1}{q-1}} \\
&= \left(\gamma b^\gamma \int_{(b,\infty)\cap E} v^{1+\gamma}(x)\,dx\right)^{\frac{p-1}{q-1}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sup_{b\in(0,\infty)} b^\gamma \int_{(0,b)\cap E} v^{1+\gamma}\left(\gamma b^\gamma \int_{(b,\infty)\cap E} v^{1+\gamma}\right)^{q-1} \\
&\qquad\leqslant \sup_{b\in(0,\infty)} b^\gamma \int_{(0,b)\cap E} v^{1+\gamma}\left(\gamma b^\gamma \int_{(b,\infty)\cap E} v^{1+\gamma}\right)^{p-1} <\infty. \qquad \square
\end{aligned}$$

The following two results provide examples of weights v such that $\lambda v\in C_p$ for all positive λ. Notice that the examples in Proposition 3.2 and some of the examples in Proposition 3.3 show that there are weights v satisfying the assumptions in Proposition 2.4 which are not included in Corollary 2.5.

PROPOSITION 3.2. *Let $\beta > 1$ and $v(x) = \frac{1}{x^\beta}$. Then, for all $\lambda > 0$, the weight λv satisfies C_p with constant independent of λ.*

Proof. We have to prove that there exists $\gamma \in (0,1)$ such that

$$\sup_{b>0,\lambda>0} b^\gamma \lambda^{1+\gamma} \left(\int_{(0,b)\cap E} \frac{1}{x^{\beta(1+\gamma)}} dx \right)^{1/p} \left(\int_{(b,\infty)\cap E} \frac{1}{x^{\beta(1+\gamma)}} dx \right)^{1/p'} < \infty, \tag{3.1}$$

where $E = \{x > 0 : \frac{\lambda}{x^\beta} < \frac{1}{x}\}$. Observe that $x \in E$ if and only if $x^{\beta-1} > \lambda$, therefore $E = (\lambda^{\frac{1}{\beta-1}}, \infty)$. This implies that, if $b \leqslant \lambda^{\frac{1}{\beta-1}}$, then $\int_{(0,b)\cap E} \frac{1}{x^{\beta(1+\gamma)}} dx = 0$. As a consequence we only have to consider those $b > 0$ and $\lambda > 0$ such that $\lambda^{\frac{1}{\beta-1}} < b$ which is equivalent to $\lambda < b^{\beta-1}$. In these cases,

$$\int_{(0,b)\cap E} \frac{1}{x^{\beta(1+\gamma)}} dx = \int_{\lambda^{\frac{1}{\beta-1}}}^{b} \frac{1}{x^{\beta(1+\gamma)}} dx.$$

Therefore,

$$\begin{aligned} & b^\gamma \lambda^{1+\gamma} \left(\int_{(0,b)\cap E} \frac{1}{x^{\beta(1+\gamma)}} dx \right)^{1/p} \left(\int_{(b,\infty)\cap E} \frac{1}{x^{\beta(1+\gamma)}} dx \right)^{1/p'} \\ &= b^\gamma \lambda^{1+\gamma} \frac{1}{\beta - 1 + \beta\gamma} \left(\frac{1}{\lambda^{1+\beta'\gamma}} - \frac{1}{b^{\beta-1+\beta\gamma}} \right)^{1/p} \left(\frac{1}{b^{\beta-1+\beta\gamma}} \right)^{1/p'} \\ &\leqslant b^\gamma \lambda^{1+\gamma} \frac{1}{\beta - 1 + \beta\gamma} \left(\frac{1}{\lambda^{1+\beta'\gamma}} \right)^{1/p} \left(\frac{1}{b^{\beta-1+\beta\gamma}} \right)^{1/p'} \\ &= \frac{1}{\beta - 1 + \beta\gamma} \frac{\lambda^{\frac{1}{p'} + \gamma\left(\frac{p-\beta'}{p}\right)}}{b^{\frac{\beta-1+\beta\gamma}{p'} - \gamma}} = \frac{1}{\beta - 1 + \beta\gamma} \left(\frac{\lambda}{b^{\beta-1}} \right)^{\frac{1}{p'} + \gamma\left(\frac{p-\beta'}{p}\right)}. \end{aligned} \tag{3.2}$$

If $p \geqslant \beta'$ then $\frac{1}{p'} + \gamma\left(\frac{p-\beta'}{p}\right) \geqslant 0$ for all $\gamma \in (0,1)$. If $p < \beta'$ then we choose $\gamma \in (0,1)$ small enough to have $\frac{1}{p'} + \gamma\left(\frac{p-\beta'}{p}\right) \geqslant 0$, i.e., $0 < \gamma \leqslant \frac{p-1}{\beta'-p}$. Now we use that $\lambda < b^{\beta-1}$ to obtain that

$$\left(\frac{\lambda}{b^{\beta-1}} \right)^{\frac{1}{p'} + \gamma\left(\frac{p-\beta'}{p}\right)} \leqslant 1. \quad \square$$

PROPOSITION 3.3. *Let $\lambda > 0$, $\beta \in \mathbb{R}$ and $v(x) = h(x)x^{-\beta}$ defined in $(0,\infty)$.*

(a) If h is increasing and $\beta < 1$ then λv satisfies C_p for all $p \in (1,\infty)$

(b) If h is decreasing and $\beta > 1$ then λv satisfies C_p for all $p \in (1,\infty)$.

(c) If $h(x) = 1$, $\beta = 1$, $\lambda < 1$ and $p \in (1,\infty)$ then λv does not satisfy C_p.

Proof. The proof of (a) is straightforward. Since $\beta < 1$ then $x^{-\beta} \in A_1^+$. Therefore, $v \in A_1^+$ because h is increasing. Consequently, $\lambda v \in A_p^+$ for all $p \in (1, +\infty)$, and (a) follows from Proposition 2.3.

We proceed to prove (b). Let $\beta > 1$. We have to prove that there exists $\gamma \in (0,1)$ such that

$$\sup_{b>0,\lambda>0} b^{\gamma}\lambda^{1+\gamma}\left(\int_{(0,b)\cap E}\left(\frac{h(x)}{x^{\beta}}\right)^{(1+\gamma)}dx\right)^{1/p}\left(\int_{(b,\infty)\cap E}\left(\frac{h(x)}{x^{\beta}}\right)^{(1+\gamma)}dx\right)^{1/p'} < \infty, \tag{3.3}$$

where $E = \{x > 0 : h(x) < \frac{x^{\beta-1}}{\lambda}\}$. We may assume that $(0,b) \cap E \neq \emptyset$. Then $E = (\alpha, \infty)$ with $0 \leqslant \alpha < b$. If $\alpha = 0$ then $h = 0$ and we have nothing to prove. Assume $\alpha > 0$. Then for all $x \in (\alpha, \infty)$ we get

$$h(x) \leqslant \frac{\alpha^{\beta-1}}{\lambda}.$$

Therefore,

$$\begin{aligned} &b^{\gamma}\lambda^{1+\gamma}\left(\int_{(0,b)\cap E}\left(\frac{h(x)}{x^{\beta}}\right)^{(1+\gamma)}dx\right)^{1/p}\left(\int_{(b,\infty)\cap E}\left(\frac{h(x)}{x^{\beta}}\right)^{(1+\gamma)}dx\right)^{1/p'} \\ &\leqslant b^{\gamma}\frac{\alpha^{(\beta-1)(1+\gamma)}}{\beta-1+\beta\gamma}\left(\frac{1}{\alpha^{\beta-1+\beta\gamma}}\right)^{1/p}\left(\frac{1}{b^{\beta-1+\beta\gamma}}\right)^{1/p'} \\ &= \frac{1}{\beta-1+\beta\gamma}\left(\frac{\alpha}{b}\right)^{\frac{\beta-1+\gamma(\beta-p')}{p'}}. \end{aligned} \tag{3.4}$$

If $p' \leqslant \beta$ then $\beta - 1 + \gamma(\beta - p') > 0$ for all $\gamma \in (0,1)$. If $p' < \beta$ then we choose $\gamma \in (0,1)$ small enough to get $\beta - 1 + \gamma(\beta - p') \geqslant 0$, i.e., $0 < \gamma \leqslant \frac{\beta-1}{p'-\beta}$. Now we use that $\alpha < b$ to obtain that

$$\left(\frac{\alpha}{b}\right)^{\frac{\beta-1+\gamma(\beta-p')}{p'}} \leqslant 1.$$

Finally, (c) is obvious since, under the assumptions of (c), the set E equals $(0,\infty)$. □

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