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MAXIMAL MOMENT INEQUALITY FOR PARTIAL SUMS OF ρ–MIXING SEQUENCES AND ITS APPLICATIONS

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Abstract. A maximal moment inequality for partial sums of ρ-mixing random variable sequences is established, which uses some moment summations as upper bound. As its applications, we discuss the strong law of large numbers for weighted sums and the Berry-Esseen bound of nonparametric regression estimate.

1. Introduction and inequality

Suppose that $\{X_i : i \geqslant 1\}$ is a real-valued random variable sequence on a probability space $(\Omega, \mathscr{F}, P)$. Let $\mathscr{F}_m^n$ denote the σ-field generated by $\{X_i : m \leqslant i \leqslant n\}$ and $||X||_r = (E|X|^r)^{1/r}$. Let

$$\rho(n) = \sup\left\{\frac{|\mathrm{Cov}(X,Y)|}{||X||_2||Y||_2} : X \in L_2(\mathscr{F}_1^m), Y \in L_2(\mathscr{F}_{m+n}^\infty), m \geqslant 1\right\}, \tag{1.1}$$

where L_2 represents a space whose second-order norm is finite. The sequence $\{X_i : i \geqslant 1\}$ is said to be ρ-mixing if $\rho(n) \to 0$ as $n \to \infty$.

It is well known that the moment inequalities for partial sums $S_n = \sum_{i=1}^n X_i$ play an important role in various proofs of limit theorems. For example, the Rosenthal inequality for independent random variables, which is

$$E \max_{1\leqslant j\leqslant n} |S_j|^r \leqslant C\left\{\sum_{i=1}^n E|X_i|^r + \left(\sum_{i=1}^n E|X_i|^2\right)^{r/2}\right\}, \tag{1.2}$$

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the Marcinkiewicz-Zygmund inequality for independent random sequences and the Burkholder inequality for martingales. For dependent random variables, many scholars have also been trying to develop these inequalities. One can refer to Billingsley (1968), Peligrad (1982,1985,1987), Roussas and Ioannides (1987), Shao (1988,1989,1995) and Yang (1997) for ϕ-mixing or ρ-mixing sequences, Yokoyama (1980), Shao and Yu (1996), Yang (2000), Yang (2007) and Xing et al. (2009) for α-mixing sequences, Birkel (1988), Shao and Yu (1996) for associated sequences, Shao and Su (1999), Shao (2000) and Yang (2001) for negatively associated sequences, Wang et al. (2014) for negatively superadditive-dependent sequences, Ding et al. (2017) for widely orthant-dependent sequences, Wang et al. (2019) for m-extended negatively dependent sequences.

Motivated by the above scholars, we try our best to give the following maximal moment inequality for partial sums of ρ-mixing random variables, which is similar to (1.2) and uses some moment summations as upper bound.

THEOREM 1.1. *Let $r > 2$ and $\{X_i, i \geqslant 1\}$ be a ρ-mixing sequence of random variables with $EX_i = 0$, $E|X_i|^r < \infty$ and $\rho(n) \leqslant Cn^{-\theta}$ for some $\theta > 1$ and $C > 0$. Then, for any $\varepsilon > 0$, there exists a positive constant $K = K(\varepsilon, r, \theta, C) < \infty$ such that*

$$E \max_{1 \leqslant j \leqslant n} |S_j|^r \leqslant K \left\{ n^{\varepsilon} \sum_{i=1}^{n} E|X_i|^r + \left(\sum_{i=1}^{n} EX_i^2 \right)^{r/2} \right\}. \tag{1.3}$$

REMARK.

(i) The inequality (1.3) is near to (1.2) for independent sequences by taking a sufficiently small ε.

(ii) Since the upper bound of the inequality (1.3) contains the information of moment summations, it plays an important role in researching the asymptotical property of weight sums, which can be seen in the proofs of the theorems in sections 2 and 3. Indeed, there are many weighted estimates in statistics, such as least squares regression estimate, nonparametric regression estimate and nonparametric density estimate. So Theorem 1.1 is a useful result.

To show the applications of the inequality (1.3), we apply it to discuss the strong law of large numbers for weighted sums and the Berry-Esseen bound of nonparametric regression estimate.

The explicit applications are described in sections 2 and 3, respectively. The proof of the inequality (1.3) is given in section 4. Throughout this paper, it is supposed that C denotes a constant which only depends on some given numbers, $[x]$ denotes the integer part of x and $a \wedge b := \min\{a, b\}$.

2. Strong law of large numbers of weighted sums

In this section, we will show the applications of the inequality (1.3) in researching the strong law of large numbers for weighted sums.

THEOREM 2.1. *Let $p > 1$ and $\{X_i, i \geqslant 1\}$ be a ρ-mixing sequence of random variables with $EX_i = 0$, $\sup_{i\geqslant 1} E|X_i|^p < \infty$ and $\rho(n) \leqslant Cn^{-\theta}$ for some $\theta > 1$ and $C > 0$. And let $\{a_{ni} : 1 \leqslant i \leqslant n, n \geqslant 1\}$ be a triangular array of real numbers satisfying*

$$\max_{1\leqslant i\leqslant n} |a_{ni}| \leqslant Cn^{-\delta} \ \text{and} \ \sum_{i=1}^{n} |a_{ni}| \leqslant C, \tag{2.1}$$

where $\delta > 1/p$. Then

$$\sum_{i=1}^{n} a_{ni}X_i \to 0, \ a.s. \tag{2.2}$$

Proof. Let $X_{ni} = X_i I(|X_i| < n^{1/p}\log n)$, $X'_{ni} = X_i I(|X_i| \geqslant n^{1/p}\log n)$, $S_{n1} = \sum_{i=1}^{n} a_{ni}X_{ni}$ and $S_{n2} = \sum_{i=1}^{n} a_{ni}X'_{ni}$. Then $\sum_{i=1}^{n} a_{ni}X_i = [S_{n1} - ES_{n1}] + [S_{n2} - ES_{n2}]$. Hence, it is sufficient to show that $S_{n1} - ES_{n1} \to 0$, $a.s.$ and $S_{n2} - ES_{n2} \to 0$ $a.s.$

Take $r > \max\{2, p\}$. For any $\varepsilon > 0$, we obtain by Theorem 1.1

$$\begin{aligned}
& P(|S_{n1} - ES_{n1}| > \varepsilon) \\
\leqslant\ & CE\left|\sum_{i=1}^{n} a_{ni}[X_{ni} - EX_{ni}]\right|^r \\
\leqslant\ & C\left\{n^{\varepsilon}\sum_{i=1}^{n} |a_{ni}|^r E\,|X_{ni} - EX_{ni}|^r + \left(\sum_{i=1}^{n} a_{ni}^2 E[(X_{ni} - EX_{ni})^2]\right)^{r/2}\right\} \\
\leqslant\ & C\left\{n^{\varepsilon}(n^{-\delta+1/p}\log n)^{r-1}\sum_{i=1}^{n} |a_{ni}|E|X_{ni}| + \left(n^{-\delta+1/p}\log n\sum_{i=1}^{n} |a_{ni}|E|X_{ni}|\right)^{r/2}\right\} \\
\leqslant\ & C\left\{n^{\varepsilon}n^{-(\delta-1/p)(r-1)}(\log n)^{r-1} + n^{-(\delta-1/p)r/2}(\log n)^{r/2}\right\}.
\end{aligned} \tag{2.3}$$

Hence, $\sum_{n=1}^{\infty} P(|S_{n1} - S_{n1}| > \varepsilon) < \infty$ for sufficiently large r and sufficiently small ε. Thus $S_{n1} - ES_{n1} \to 0$, $a.s.$

Next, we will prove that $S_{n2} - ES_{n2} \to 0$ $a.s.$ It is obvious that

$$\begin{aligned}
|ES_{n2}| &\leqslant \sum_{i=1}^{n} |a_{ni}|E|X'_{ni}| \leqslant n^{-(p-1)/p}(\log n)^{-(p-1)}\sum_{i=1}^{n} |a_{ni}|E|X'_{ni}|^p \\
&\leqslant Cn^{-(p-1)/p}(\log n)^{-(p-1)} \to 0.
\end{aligned} \tag{2.4}$$

Note that $\sum_{i=1}^{\infty} P(|X_i| \geqslant i^{1/p}\log i) \leqslant C\sum_{i=1}^{\infty} i^{-1}(\log i)^{-p} < \infty$. By Borel-Cantelli lemma, we have

$$\sum_{i=1}^{\infty} i^{-\delta}|X_i|I(|X_i| \geqslant i^{1/p}\log i) < \infty, a.s.$$

From Kronecker lemma, it follows that $n^{-\delta}\sum_{i=1}^{n}|X_i|I(|X_i|\geqslant i^{1/p}\log i)\to 0\ a.s.$ Thus

$$|S_{n2}|\leqslant\sum_{i=1}^{n}|a_{ni}||X'_{ni}|\leqslant Cn^{-\delta}\sum_{i=1}^{n}|X_i|I(|X_i|\geqslant n^{1/p}\log n)$$
$$\leqslant Cn^{-\delta}\sum_{i=1}^{n}|X_i|I(|X_i|\geqslant i^{1/p}\log i)\to 0, a.s. \tag{2.5}$$

From (2.4) and (2.5), $S_{n2}-ES_{n2}\to 0, a.s.$ follows. The proof is completed. □

The result of Theorem 2.1 may be applied to nonparametric regression estimate, which is defined as follows.

Let d be a natural number and A be a compact set in R^d. Consider observations

$$Y_i=g(x_i)+\varepsilon_i,\ \ i=1,2,\cdots,n,$$

where $x_1,x_2,\cdots,x_n\in A$ are fixed design points, g is a bounded real valued function on A and $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ are random errors with $E\varepsilon_i=0$, $i=1,2,\cdots,n$. The general linear smooth estimate of the function $g(x)$ is defined by the formula

$$g_n(x)=\sum_{i=1}^{n}w_{ni}(x)Y_i,\ \ x\in A\subset R^d, \tag{2.6}$$

where weight functions w_{ni}, $i=1,2,\cdots,n$, depend on the fixed design points $x_1,x_2,\cdots,x_n$ and the number of observations n.

In order to make $g_n(x)$ be asymptotically unbiased, i.e. $Eg_n(x)\to g(x)$ as $n\to\infty$, we suppose that

$$\sum_{i=1}^{n}|w_{ni}(x)|\leqslant C\ \text{ for all } n\geqslant 1 \tag{2.7}$$

and

$$\sum_{i=1}^{n}w_{ni}(x)\to 1,\ \ \sum_{i=1}^{n}|w_{ni}(x)|I(||x_i-x||>a)\to 0\ \text{ for all } a>0 \tag{2.8}$$

as $n\to\infty$.

Since $g_n(x)-Eg_n(x)=\sum_{i=1}^{n}w_{ni}(x)\varepsilon_i$, we have immediately the following consequence by Theorem 2.1.

COROLLARY 2.2. *Let* $p>1$ *and* $\{\varepsilon_i,i\geqslant 1\}$ *be a* ρ*-mixing sequence of random variables with* $E\varepsilon_i=0$, $\sup_{i\geqslant 1}E|\varepsilon_i|^p<\infty$ *and* $\rho(n)\leqslant Cn^{-\theta}$ *for some* $\theta>1$ *and* $C>0$. *If the conditions* (2.7) *and* (2.8) *hold and for some* $\delta>1/p$,

$$\max_{1\leqslant i\leqslant n}|w_{ni}|\leqslant Cn^{-\delta}, \tag{2.9}$$

then at every continuity point $x\in A$ *of the function* g, *we have*

$$g_n(x)\to g(x), a.s. \tag{2.10}$$

REMARK. Corollary 2.2 improves Theorem 4 of Georgiev (1988), which uses the following more restrictive conditions.

(i) $\{\varepsilon_i, i \geqslant 1\}$ is a sequence of independent random variables with $\sup_{i\geqslant 1} E|\varepsilon_i|^p < \infty$ for $p > 2$.

(ii) $\max_{1\leqslant i\leqslant n} w_{ni}^2(x) n \log\log n \to 0$ as $n \to \infty$. However, $\max_{1\leqslant i\leqslant n} w_{ni}^2(x) n \log\log n$ does not converge necessarily to zero as $\max_{1\leqslant i\leqslant n} |w_{ni}| \leqslant Cn^{-\delta}$ for $\delta > 1/p$.

THEOREM 2.3. *Let $p \geqslant 2$ and $\{X_i, i \geqslant 1\}$ be a ρ-mixing sequence of random variables with $EX_i = 0$, $\sup_{i\geqslant 1} E|X_i|^p < \infty$ and $\rho(n) \leqslant Cn^{-\theta}$ for some $\theta > 1$ and $C > 0$. And let $\{a_{ni} : 1 \leqslant i \leqslant n, n \geqslant 1\}$ be a triangular array of numbers satisfying*

$$\max_{1\leqslant i\leqslant n} |a_{ni}| \leqslant Cn^{-\delta} \quad \text{and} \quad \sum_{i=1}^{n} a_{ni}^2 \leqslant Cn^{-\beta}, \tag{2.11}$$

where $\delta > 1/p$ and $\beta > 0$. Then (2.2) holds.

Proof. We can obtain the desired result (2.2) by modifying (2.3) and (2.4) in the proof of Theorem 2.1. They are replaced by the following two inequalities.

$$\begin{aligned} &P(|S_{n1} - ES_{n1}| > \varepsilon) \\ &\leqslant C\left\{ n^{\varepsilon} n^{-(\delta-1/p)(r-2)} (\log n)^{r-2} \sum_{i=1}^{n} a_{ni}^2 E|X_{ni}|^2 + \left(\sum_{i=1}^{n} a_{ni}^2 E|X_{ni}|^2 \right)^{r/2} \right\} \\ &\leqslant C\left\{ n^{\varepsilon} n^{-(\delta-1/p)(r-2)} (\log n)^{r-2} + n^{-\beta r/2} \right\} \end{aligned} \tag{2.12}$$

and

$$|ES_{n2}| \leqslant \left(E|S_{n2}|^2\right)^{1/2} \leqslant C\left(\sum_{i=1}^{n} a_{ni}^2 E|X_{ni}'|^2 \right)^{1/2} \leqslant Cn^{-\beta/2} \to 0, \tag{2.13}$$

which can be obtained by the proof of Theorem 1.1, Lemma 4.2 in Section 4 and the condition (2.11), respectively. The proof is completed. □

REMARK. From the proofs of Theorems 2.1 and 2.3, it can be seen that the conditions $\sum_{i=1}^{n} |a_{ni}| \leqslant C$ and $\sum_{i=1}^{n} a_{ni}^2 \leqslant Cn^{-\beta}$ are applied in the expressions (2.3), (2.12) and (2.13), and play an important role. This is because of the applications of Theorem 1.1 playing a critical role.

3. Berry-Esseen bound of nonparametric regression estimate

In this section, we will give the Berry-Esseen bound of nonparametric regression estimate (2.6) for ρ-mixing samples by using Theorem (1.1). Define $w_n(x) := \max_{1\leqslant i\leqslant n} |w_{ni}(x)|$, $\sigma_n^2(x) := \mathrm{Var}(g_n(x))$ and $u(n) = \sum_{i=n}^{\infty} \rho(i)$. To formulate the main result obtained, we need the following assumptions.

(A1) (i) Assume that $\{\varepsilon_i, i \geqslant 1\}$ is a sequence of identically distributed and ρ-mixing random variables with zero mean; (ii) There exists $\delta > 0$ such that $E(|X_1|^{2+\delta}) < \infty$; (iii) Suppose $\rho(n) \leqslant Cn^{-\theta}$ for some $\theta > 1$.

(A2) (i) $\sum_{i=1}^n |w_{ni}(x)| \leqslant C$ for all $n \geqslant 1$; (ii) $w_n(x) = O(\sigma_n^2(x))$ and $\sigma_n^2(x) > 0$.

(A3) There exist positive integers $p_1 := p_1(n)$, $p_2 := p_2(n)$ and a positive constant c such that

$$p_1 + p_2 \leqslant n,\ p_2 p_1^{-1} < c < \infty \tag{3.1}$$

for sufficiently large n and as $n \to \infty$,

$$\gamma_{1n} \to 0,\ \gamma_{2n} \to 0,\ \gamma_{3n} \to 0, \tag{3.2}$$

where $\gamma_{1n} := np_2 p_1^{-1} w_n(x)$, $\gamma_{2n} := p_1 w_n(x)$ and $\gamma_{3n} := np_1^{-1}\rho^2(p_2)$.

Let $S_n(x) = \sigma_n^{-1}(x)\big(g_n(x) - Eg_n(x)\big) =: \sum_{i=1}^n Z_{ni}$ in which $Z_{ni} = \sigma_n^{-1}(x) w_{ni}(x)\varepsilon_i$, $F_n(u) = P\big(S_n(x) < u\big)$ and $\Phi(u)$ be the distribution function of the standard normal random variable. Then, we obtain

THEOREM 3.1. *If the assumptions (A1)-(A3) hold, then*

$$\sup_u |F_n(u) - \Phi(u)| \leqslant C\Big\{\gamma_{1n}^{1/3} + \gamma_{2n}^{1/3} + \gamma_{2n}^{\delta/2} + \gamma_{3n}^{1/4} + u(p_2)\Big\}. \tag{3.3}$$

Proof. For convenience, we omit everywhere the argument x and set $k = [n/(p_1 + p_2)]$. Then S_n may be split as

$$S_n = S_n' + S_n'' + S_n''', \tag{3.4}$$

where

$$S_n' = \sum_{m=1}^k y_{nm},\ S_n'' = \sum_{m=1}^k y_{nm}',\quad S_n''' = y_{nk+1}',$$

$$y_{nm} = \sum_{i=k_m}^{k_m+p_1-1} Z_{ni},\ y_{nm}' = \sum_{i=l_m}^{l_m+p_2-1} Z_{ni},\ y_{nk+1}' = \sum_{i=k(p_1+p_2)+1}^{n} Z_{ni},$$

$k_m = (m-1)(p_1+p_2)+1,\ l_m = (m-1)(p_1+p_2)+p_1+1,\ m = 1,\cdots,k.$ Set $s_n^2 = \sum_{m=1}^k \mathrm{Var}(y_{nm})$ and assume that $\{\eta_{nm} : m = 1,\cdots,k\}$ are independent random variables and, the distribution of η_{nm} is the same as that of y_{nm} for $m = 1,\cdots,k$. Let $T_n = \sum_{m=1}^k \eta_{nm}, B_n = \sum_{m=1}^k \mathrm{Var}(\eta_{nm})$, $\tilde{F}_n(u), G_n(u)$ and $\tilde{G}_n(u)$ be the distributions of $S_n', T_n/\sqrt{B_n}$ and T_n, respectively. Obviously,

$$B_n = s_n^2,\ \ \tilde{G}_n(u) = G_n(u/s_n). \tag{3.5}$$

Noticing Lemma A.2 in Section 4, the assumptions (A2) and (A3), we have

$$\begin{aligned} E\big(S_n''\big)^2 &\leqslant C\sum_{m=1}^k \sum_{i=k_m}^{k_m+p_2-1} \sigma_n^{-2} w_{ni}^2 \leqslant Ckp_2\sigma_n^{-2}w_n^2 \leqslant C\frac{n}{p_1+p_2}p_2 w_n \\ &\leqslant C\big(1 + p_2 p_1^{-1}\big)^{-1} np_2 p_1^{-1} w_n \leqslant C\gamma_{1n}. \end{aligned} \tag{3.6}$$

By the same way, we have

$$E\left(S_n'''\right)^2 \leqslant C \sum_{i=k(p_1+p_2)+1}^{n} \sigma_n^{-2} w_{ni}^2 \leqslant C(n-k(p_1+p_2))\sigma_n^{-2} w_n^2$$
$$\leqslant C\left(\frac{n}{p_1+p_2}-k\right)(p_1+p_2)w_n \leqslant C(1+p_2p_1^{-1})p_1w_n \leqslant C\gamma_{2n}, \quad (3.7)$$

Combining the two results (3.6) and (3.7) yields that

$$P\left(|S_n''| \geqslant \gamma_{1n}^{1/3}\right) \leqslant C\gamma_{1n}^{1/3}, \;\; P\left(|S_n'''| \geqslant \gamma_{2n}^{1/3}\right) \leqslant C\gamma_{2n}^{1/3}, \quad (3.8)$$

which, together with (3.4), the following result

$$\sup_u \left|\tilde{F}_n(u) - \Phi(u)\right| \leqslant C\left\{\gamma_{1n}^{1/2} + \gamma_{2n}^{1/2} + \gamma_{2n}^{\delta/2} + \gamma_{3n}^{1/4} + u(p_2)\right\} \quad (3.9)$$

and Lemma 3.7 in Yang (2003) conclude the desired result (3.3). Hence, it suffices to prove (3.9). It is easy to observe that

$$\sup_u \left|\tilde{F}_n(u) - \Phi(u)\right|$$
$$\leqslant \sup_u |\Phi(u/s_n) - \Phi(u)| + \sup_u \left|\tilde{G}_n(u) - \Phi(u/s_n)\right| + \sup_u \left|\tilde{F}_n(u) - \tilde{G}_n(u)\right|$$
$$=: J_{1n} + J_{2n} + J_{3n}.$$

Next, we will give the bounds of J_{1n}, J_{2n} and J_{3n} in order to obtain (3.9), respectively.

(i) Let $\tau_n = \Sigma_{1\leqslant i<j\leqslant k}\mathrm{Cov}(y_{ni}, y_{nj})$. Obviously, $E(S_n)^2 = \mathrm{Var}(S_n) = 1$ and $s_n^2 = E\left(S_n'\right)^2 - 2\tau_n$. Hence, we obtain

$$E\left(S_n'\right)^2 = E\left[S_n - \left(S_n'' + S_n'''\right)\right]^2 = 1 + E\left(S_n'' + S_n'''\right)^2 - 2E\left[S_n\left(S_n'' + S_n'''\right)\right],$$

which together (3.6) and (3.7), concludes that

$$\left|E\left(S_n'\right)^2 - 1\right| = \left|E\left(S_n'' + S_n'''\right)^2 - 2E\left[S_n\left(S_n'' + S_n'''\right)\right]\right|$$
$$\leqslant E\left|S_n'' + S_n'''\right|^2 + 2E\left|S_n\left(S_n'' + S_n'''\right)\right|$$
$$\leqslant 2\left(E|S_n''|^2 + E|S_n'''|^2\right) + 2\left(E|S_n|^2\right)^{1/2}\left(E\left(S_n'' + S_n'''\right)^2\right)^{1/2}$$
$$\leqslant C\left(E|S_n''|^2 + E|S_n'''|^2 + \left(E|S_n''|^2\right)^{1/2} + \left(E|S_n'''|^2\right)^{1/2}\right)$$
$$\leqslant C\left(\gamma_{1n}^{1/2} + \gamma_{2n}^{1/2}\right). \quad (3.10)$$

On the other hand,

$$|\tau_n| \leqslant \sum_{1\leqslant i<j\leqslant k} |\mathrm{Cov}(y_{ni}, y_{nj})|$$
$$\leqslant \sum_{1\leqslant i<j\leqslant k} \sum_{s=k_i}^{k_i+p_1-1} \sum_{t=k_j}^{k_j+p_1-1} |\mathrm{Cov}(Z_{ns}, Z_{nt})|$$

$$\begin{aligned}
&\leqslant \sum_{1\leqslant i<j\leqslant k} \sum_{s=k_i}^{k_i+p_1-1} \sum_{t=k_j}^{k_j+p_1-1} \sigma_n^{-2}|w_{ns}w_{nt}|\cdot|\mathrm{Cov}(\varepsilon_s,\varepsilon_t)| \\
&\leqslant C \sum_{1\leqslant i<j\leqslant k} \sum_{s=k_i}^{k_i+p_1-1} \sum_{t=k_j}^{k_j+p_1-1} \sigma_n^{-2}|w_{ns}w_{nt}|\cdot\rho(t-s)\sqrt{\mathrm{Var}(\varepsilon_s)\mathrm{Var}(\varepsilon_t)} \\
&\leqslant C\sum_{i=1}^{k-1} \sum_{s=k_i}^{k_i+p_1-1} |w_{ns}| \sum_{j=i+1}^{k} \sum_{t=k_j}^{k_j+p_1-1} \rho(t-s) \\
&\leqslant C\sum_{s=1}^{n}|w_{ns}| \sum_{j=p_2}^{\infty}\rho(j) \\
&\leqslant Cu(p_2). \qquad (3.11)
\end{aligned}$$

From (3.10) and (3.11), it follows that

$$J_{1n} \leqslant C\left(\gamma_{1n}^{1/2}+\gamma_{2n}^{1/2}+u(p_2)\right). \qquad (3.12)$$

(ii) By Theorem 1.1 with $\varepsilon=\delta/2$, we have

$$\begin{aligned}
\sum_{m=1}^{k} E|y_{nm}|^{2+\delta} &\leqslant C\sum_{m=1}^{k}\left\{p_1^{\delta/2}\sum_{i=k_m}^{k_m+p_1-1}E|Z_{ni}|^{2+\delta}+\left(\sum_{i=k_m}^{k_m+p_1-1}E|Z_{ni}|^2\right)^{1+\delta/2}\right\} \\
&\leqslant C\sum_{m=1}^{k}\left\{p_1^{\delta/2}\sum_{i=k_m}^{k_m+p_1-1}|w_{ni}|^{1+\delta/2}+\left(\sum_{i=k_m}^{k_m+p_1-1}|w_{ni}|\right)^{1+\delta/2}\right\} \\
&\leqslant Cp_1^{\delta/2}\sum_{i=1}^{n}|w_{ni}|^{1+\delta/2} \\
&\leqslant Cp_1^{\delta/2}w_n^{\delta/2} \\
&= C\gamma_{2n}^{\delta/2}. \qquad (3.13)
\end{aligned}$$

Also, from (3.12), it follows that $B_n^2=s_n^2\to 1$. Thus,

$$\frac{1}{B_n^{1+\delta/2}}\sum_{m=1}^{k}E|\eta_{nm}|^{2+\delta}\leqslant C\gamma_{2n}^{\delta/2}.$$

By Berry-Esseen theorem and the results stated earlier, we obtain

$$\sup_u |G_n(u)-\Phi(u)| \leqslant C\gamma_{2n}^{\delta/2}, \qquad (3.14)$$

which implies

$$J_{2n}\leqslant C\gamma_{2n}^{\delta/2}. \qquad (3.15)$$

(iii) Suppose that $\varphi(t)$ and $\psi(t)$ are the characteristic functions of $S_n^{'}$ and T_n, respectively. Noticing that

$$\psi(t)=E(\exp\{itT_n\})=\Pi_{m=1}^{k}E\exp\{it\eta_{nm}\}=\Pi_{m=1}^{k}E\exp\{ity_{nm}\},$$

we have by Lemmas 4.1 and 4.2 in Section 4,

$$\begin{aligned}
|\varphi(t)-\psi(t)| &= \left|E\exp\{it\sum_{m=1}^{k} y_{nm}\} - \Pi_{m=1}^{k} E\exp\{ity_{nm}\}\right| \\
&\leqslant C|t|\rho(p_2)\sum_{m=1}^{k}\|y_{nm}\|_2 \\
&\leqslant C|t|\rho(p_2)\sum_{m=1}^{k}\left\{\sum_{i=k_m}^{k_m+p_1-1}\sigma_n^{-2}w_{ni}^2\right\}^{1/2} \\
&= C|t|\rho(p_2)\left\{\sum_{m=1}^{k}\left\{\sum_{i=k_m}^{k_m+p_1-1}\sigma_n^{-2}w_{ni}^2\right\}^{1/2}\right\}^{2\cdot\frac{1}{2}} \\
&\leqslant C|t|\rho(p_2)\left\{k\sum_{m=1}^{k}\sum_{i=k_m}^{k_m+p_1-1}|w_{ni}|\right\}^{1/2} \\
&\leqslant C|t|(k\rho^2(p_2))^{1/2} \\
&\leqslant C|t|\gamma_{3n}^{1/2}.
\end{aligned}$$

Hence,

$$\int_{-T}^{T}\left|\frac{\varphi(t)-\psi(t)}{t}\right|dt \leqslant C\gamma_{3n}^{1/2}T. \tag{3.16}$$

By $\tilde{G}_n(u) = G_n(u/s_n)$ and (3.14), it follows that

$$\begin{aligned}
&\sup_u\left|\tilde{G}_n(u+y)-\tilde{G}_n(u)\right| \\
&= \sup_u|G_n((u+y)/s_n)-G_n(u/s_n)| \\
&\leqslant \sup_u|G_n((u+y)/s_n)-\Phi((u+y)/s_n)| + \sup_u|\Phi((u+y)/s_n)-\Phi(u/s_n)| \\
&\quad + \sup_u|G_n(u/s_n)-\Phi(u/s_n)| \\
&\leqslant 2\sup_u|G_n(u)-\Phi(u)| + \sup_u|\Phi((u+y)/s_n)-\Phi(u/s_n)| \\
&\leqslant C\{\gamma_{2n}^{\delta/2}+|y|/s_n\} \\
&\leqslant C\{\gamma_{2n}^{\delta/2}+|y|\}.
\end{aligned}$$

Thus, we have

$$T\sup_u\int_{|y|\leqslant c/T}\left|\tilde{G}_n(u+y)-\tilde{G}_n(u)\right|dy \leqslant CT\int_{|y|\leqslant c/T}\{\gamma_{2n}^{\delta/2}+|y|\}dy \leqslant C\{\gamma_{2n}^{\delta/2}+1/T\}. \tag{3.17}$$

Setting $T=\gamma_{3n}^{-1/4}$ and applying Esseen inequality, (3.16) and (3.17), we obtain

$$\begin{aligned}
&\sup_u \left|\tilde{F}_n(u)-\tilde{G}_n(u)\right| \\
&\leqslant CT\int_{-T}^{T}\left|\frac{\varphi(t)-\psi(t)}{t}\right|dy+T\sup_u\int_{|y|\leqslant c/T}\left|\tilde{G}_n(u+y)-\tilde{G}_n(u)\right|dy \\
&\leqslant C\{\gamma_{3n}^{1/2}T+\gamma_{2n}^{\delta/2}+1/T\} \\
&\leqslant C\{\gamma_{2n}^{\delta/2}+\gamma_{3n}^{1/4}\},
\end{aligned}$$

i.e.,

$$J_{3n}\leqslant C\{\gamma_{2n}^{\delta/2}+\gamma_{3n}^{1/4}\}. \tag{3.18}$$

Finally, combining (3.12), (3.15) and (3.18) yields (3.9). Thus, the proof of Theorem 3.1 is completed. □

By Theorem 3.1, we can obtain easily the following corollaries.

COROLLARY 3.2. *If the assumptions (A1)-(A3) hold, then*

$$\sup_u |F_n(u)-\Phi(u)|=o(1). \tag{3.19}$$

COROLLARY 3.3. *If the assumptions (A1)-(A3) hold for* $\delta\geqslant 2/3$ *and* $\rho(n)=O(n^{-\lambda})$ *for* $\lambda\geqslant 7/6$, *then*

$$\sup_u |F_n(u)-\Phi(u)|\leqslant C\left\{n^{-\lambda/(6\lambda+7)}\right\}. \tag{3.20}$$

Proof. Let $p_1=[n^\tau],\ \ p_2=[n^{2\tau-1}]$, where $\tau=\frac{1}{2}+\frac{7}{2(6\lambda+7)}$. Then, we have from $\delta\geqslant 2/3$,

$$\begin{aligned}
\gamma_{1n}^{1/3}&=O(n^{-(1-\tau)/3})=O(n^{-\lambda/(6\lambda+7)}),\\
\gamma_{2n}^{\delta/2}\leqslant\gamma_{2n}^{1/3}&=O(n^{-(1-\tau)/3})=O(n^{-\lambda/(6\lambda+7)}),\\
\gamma_{3n}^{1/4}&=O(n^{-(\lambda+7/2)/(6\lambda+7)})=O(n^{-\lambda/(6\lambda+7)})
\end{aligned}$$

and

$$u(p_2)=O(\sum_{i=p_2}^{\infty}i^{-\lambda})=O(p_2^{-\lambda+1})=O(n^{-(2\tau-1)(\lambda-1)})=O(n^{-(7\lambda-7)/(6\lambda+7)}).$$

Also, $\lambda\geqslant 7/6$ implies $7\lambda-7\geqslant\lambda$. Hence, $u(p_2)=O(n^{-\lambda/(6\lambda+7)})$. Thus by (3.3), (3.20) follows. The proof is completed. □

REMARK. In view of (3.20), we obtain that the convergence rate of the uniformly asymptotic normality of the nonparametric estimate is near to $n^{-1/6}$ as λ is sufficiently large.

4. Proof of Theorem 1.1

To prove Theorem 1.1, We need the following lemmas.

LEMMA 4.1. (Shao 1995, Lemma 2.1) *Suppose that* $\{X_i, i \geqslant 1\}$ *is a* ρ*-mixing sequence. Let* $p,q > 1$ *with* $1/p + 1/q = 1$*,* $\xi \in L_p(\mathscr{F}_1^k)$ *and* $\eta \in L_q(\mathscr{F}_{k+n}^{\infty})$*. Then*

$$|E(\xi\eta) - (E\xi)(E\eta)| \leqslant 10\rho^{2(1/p \wedge 1/q)}(n)||\xi||_p \cdot ||\eta||_q. \tag{4.1}$$

LEMMA 4.2. *If* $\{X_i, i \geqslant 1\}$ *is a* ρ*-mixing sequence with zero mean and* $\sum_{n=1}^{\infty}\rho(n) < \infty$*, then we have*

$$E\left(\sum_{i=1}^{n} X_i\right)^2 \leqslant C\sum_{i=1}^{n} EX_i^2. \tag{4.2}$$

Proof. Taking $p = q = 2$ in Lemma 4.1, we have

$$\begin{aligned}
E\left(\sum_{i=1}^{n} X_i\right)^2 &= \sum_{i=1}^{n} EX_i^2 + 2\sum_{1\leqslant i<j\leqslant n} E(X_iX_j) \\
&\leqslant \sum_{i=1}^{n} EX_i^2 + 20\sum_{1\leqslant i<j\leqslant n} \rho(j-i)\left(EX_i^2\right)^{1/2}\left(EX_j^2\right)^{1/2} \\
&\leqslant \sum_{i=1}^{n} EX_i^2 + 10\sum_{k=1}^{n-1}\sum_{i=1}^{n-k} \rho(k)\left(EX_i^2 + EX_{k+i}^2\right) \\
&\leqslant \left(1 + 20\sum_{k=1}^{\infty}\rho(k)\right)\sum_{i=1}^{n} EX_i^2 \\
&\leqslant C\sum_{i=1}^{n} EX_i^2,
\end{aligned}$$

which completes the proof of the lemma. □

LEMMA 4.3. *For* $r > 2$ *and any* $x, y \in R^1$*, we have*

$$|x+y|^r \leqslant |y|^r + d_1|x|^r + rx|y|^{r-1}\operatorname{sgn}(y) + d_2x^2|y|^{r-2}, \tag{4.3}$$

where $d_1 = 2^r, d_2 = 2^r \cdot r^2$.

Proof. For $r > 2$ and $t \in R^1$, it is easy to show that $|1+t|^r \leqslant 1 + d_1|t|^r + rt + d_2t^2$. From this result, we have (4.3) by taking $t = x/y$ as $y \neq 0$. It is clear as $y = 0$. □

Since $0 < \frac{r-1}{r-1+2\theta} < 1$ and $0 < \frac{r-2}{r-2+2\theta(2\wedge(r-2))} < 1$ for $r > 2$, we can take λ which satisfies

$$\max\left\{\frac{r-1}{r-1+2\theta}, \frac{r-2}{r-2+2\theta(2\wedge(r-2))}\right\} < \lambda < 1 \text{ for } r > 2. \tag{4.4}$$

On the other hand, let $k=[(n/2)^{\lambda}]$ and $m=[(n/2)^{1-\lambda}]$. Clearly,

$$n<2(m+1)k,\ \ Cn^{\lambda}<k<n^{\lambda},\ \ m<n^{1-\lambda}. \tag{4.5}$$

For convenience, we fix n and redefine X_i as $X_i=X_i$ for $1\leqslant i\leqslant n$ and $X_i=0$ for $i>n$. For $j=1,2,\cdots,m+1$, set

$$Y_j=\sum_{i=2(j-1)k+1}^{(2j-1)k}X_i,\quad Z_j=\sum_{i=(2j-1)k+1}^{2jk}X_i$$

and $S_{1,j}=\sum_{i=1}^{j}Y_i$, $S_{2,j}=\sum_{i=1}^{j}Z_i$. Then $S_n=\sum_{j=1}^{m+1}Y_j+\sum_{j=1}^{m+1}Z_j$.

LEMMA 4.4. *If $r>2$, then*

$$E\max_{1\leqslant j\leqslant n}|S_j|^r\leqslant C\left\{E\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^r+E\max_{1\leqslant j\leqslant m+1}|S_{2,j}|^r+\sum_{j=1}^{2(m+1)}E\max_{1\leqslant l<k}\left|\sum_{i=(j-1)k+1}^{(j-1)k+l}X_i\right|^r\right\}.$$

Proof. Note that $S_j=\sum_{i=1}^{[j/k]k}X_i+\sum_{i=[j/k]k+1}^{j}X_i$, we have

$$\max_{1\leqslant j\leqslant n}|S_j|^r\leqslant 2^{r-1}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{[j/k]k}X_i\right|^r+2^{r-1}\max_{1\leqslant j\leqslant n}\left|\sum_{i=[j/k]k+1}^{j}X_i\right|^r:=I_1+I_2. \tag{4.6}$$

Since

$$I_1\leqslant 2^{2(r-1)}\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^r+2^{2(r-1)}\max_{1\leqslant j\leqslant m+1}|S_{2,j}|^r \tag{4.7}$$

and

$$I_2\leqslant 2^{r-1}\max_{1\leqslant j\leqslant 2(m+1)}\max_{1\leqslant l<k}\left|\sum_{i=(j-1)k+1}^{(j-1)k+l}X_i\right|^r\leqslant 2^{r-1}\sum_{j=1}^{2(m+1)}\max_{1\leqslant l<k}\left|\sum_{i=(j-1)k+1}^{(j-1)k+l}X_i\right|^r, \tag{4.8}$$

Combining (4.6)–(4.8) yields the desired result. The proof is completed. □

Clearly

$$\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^r\leqslant\left|\max_{1\leqslant j\leqslant m+1}S_{1,j}\right|^r+\left|\max_{1\leqslant j\leqslant m+1}(-S_{1,j})\right|^r. \tag{4.9}$$

Denote

$$M_j=\max\{0,Y_{j+1},Y_{j+1}+Y_{j+2},\cdots,Y_{j+1}+Y_{j+2}+\cdots+Y_{m+1}\},$$
$$N_j=\max\{Y_{j+1},Y_{j+1}+Y_{j+2},\cdots,Y_{j+1}+Y_{j+2}+\cdots+Y_{m+1}\},$$
$$\widetilde{M}_j=\max\{0,-Y_{j+1},-Y_{j+1}-Y_{j+2},\cdots,-Y_{j+1}-Y_{j+2}-\cdots-Y_{m+1}\}$$

and

$$\widetilde{N}_j=\max\{-Y_{j+1},-Y_{j+1}-Y_{j+2},\cdots,-Y_{j+1}-Y_{j+2}-\cdots-Y_{m+1}\}.$$

Then

$$\max_{1\leqslant j\leqslant m+1} S_{1,j} = N_0,\ N_j = Y_{j+1} + M_{j+1},\ 0 \leqslant M_j \leqslant |N_j|, \tag{4.10}$$

$$\max_{1\leqslant j\leqslant m+1} (-S_{1,j}) = \widetilde{N}_0,\ \widetilde{N}_j = -Y_{j+1} + \widetilde{M}_{j+1},\ 0 \leqslant \widetilde{M}_j \leqslant |\widetilde{N}_j|, \tag{4.11}$$

and

$$\begin{aligned} M_j &= \max\{S_{1,j}, S_{1,j+1}, \cdots, S_{1,m+1}\} - S_{1,j} \leqslant \max_{j\leqslant i\leqslant m+1} |S_{1,i}| + |S_{1,j}| \\ &\leqslant 2 \max_{1\leqslant j\leqslant m+1} |S_{1,j}|, \end{aligned} \tag{4.12}$$

$$\begin{aligned} \widetilde{M}_j &= \max\{-S_{1,j}, -S_{1,j+1}, \cdots, -S_{1,m+1}\} + S_{1,j} \\ &\leqslant \max_{j\leqslant i\leqslant m+1} |S_{1,i}| + |S_{1,j}| \leqslant 2 \max_{1\leqslant j\leqslant m+1} |S_{1,j}|. \end{aligned} \tag{4.13}$$

LEMMA 4.5. *Suppose that the sequence $\{X_i, i \geqslant 1\}$ satisfies the conditions in Theorem 1.1. If $r > 2$, then for any $\tau > 0$, there exist positive constants $C_\tau = C(\tau, r, \theta) < \infty$ and $C_r = C(r) < \infty$ such that*

$$\sum_{j=1}^{m} E\left(Y_j M_j^{r-1}\right) \leqslant C_\tau \sum_{j=1}^{m} E|Y_j|^r + \tau C_r E \max_{1\leqslant j\leqslant m+1} |S_{1,j}|^r, \tag{4.14}$$

$$\sum_{j=1}^{m} E\left(Y_j \widetilde{M}_j^{r-1}\right) \leqslant C_\tau \sum_{j=1}^{m} E|Y_j|^r + \tau C_r E \max_{1\leqslant j\leqslant m+1} |S_{1,j}|^r. \tag{4.15}$$

Proof. For $r > 2$, we have from (4.4) and (4.5)

$$\begin{aligned} m^{r-1}\rho^{2(1\wedge(r-1))}(k) &= m^{r-1}\rho^2(k) \leqslant C m^{r-1} k^{-2\theta} \leqslant C n^{(1-\lambda)(r-1)-2\lambda\theta} \\ &\leqslant C n^{r-1-\lambda(r-1+2\theta)} \leqslant C. \end{aligned}$$

By the above result, Lemma 4.1 with $p = r$ and $q = r/(r-1)$ and (4.12), we obtain

$$\begin{aligned} \sum_{j=1}^{m} E\left(Y_j M_j^{r-1}\right) &\leqslant 10\rho^{2(1\wedge(r-1))/r}(k) \sum_{j=1}^{m} \| Y_j \|_r \cdot \| M_j \|_r^{r-1} \\ &\leqslant 10 \cdot 2^{r-1} \rho^{2(1\wedge(r-1))/r}(k) \sum_{j=1}^{m} \| Y_j \|_r \cdot \left(E \max_{1\leqslant j\leqslant m+1} |S_{1,j}|^r\right)^{(r-1)/r} \\ &\leqslant 2^{r+3} \tau^{-(r-1)/r} \rho^{2(1\wedge(r-1))/r}(k) \sum_{j=1}^{m} \| Y_j \|_r \cdot \left(\tau E \max_{1\leqslant j\leqslant m+1} |S_{1,j}|^r\right)^{(r-1)/r} \\ &\leqslant \frac{2^{r(r+3)}\rho^{2(1\wedge(r-1))}(k)}{r\tau^{(r-1)}} \left(\sum_{j=1}^{m} \| Y_j \|_r\right)^r + \frac{\tau(r-1)}{r} E \max_{1\leqslant j\leqslant m+1} |S_{1,j}|^r \end{aligned}$$

$$\leqslant \frac{2^{r(r+3)}m^{r-1}\rho^{2(1\wedge(r-1))}(k)}{r\tau^{(r-1)}}\sum_{j=1}^{m}E|Y_j|^r+\frac{\tau(r-1)}{r}E\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^r$$
$$\leqslant C_\tau\sum_{j=1}^{m}E|Y_j|^r+\tau C_r E\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^r,$$

which implies (4.14). Similarly, we can get (4.15). □

LEMMA 4.6. *Suppose that the sequence* $\{X_i, i\geqslant 1\}$ *satisfies the conditions in Theorem* 1.1. *If* $r>2$, *then for any* $\tau>0$, *there exist positive constants* $C_\tau=C(\tau,r,\theta)<\infty$ *and* $C_r=C(r)$ $<\infty$ *such that*

$$\sum_{j=1}^{m}E\left(Y_j^2M_j^{r-2}\right)\leqslant C_\tau\left(\sum_{i=1}^{n}EX_i^2\right)^{r/2}+C_\tau\sum_{j=1}^{m}E|Y_j|^r+\tau C_\tau E\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^r, \quad (4.16)$$

$$\sum_{j=1}^{m}E\left(Y_j^2\widetilde{M}_j^{r-2}\right)\leqslant C_\tau\left(\sum_{i=1}^{n}EX_i^2\right)^{r/2}+C_\tau\sum_{j=1}^{m}E|Y_j|^r+\tau C_r E\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^r. \quad (4.17)$$

Proof. From (4.4) and (4.5), we have

$$\begin{aligned} m^{r/2-1}\rho^{2\wedge(r-2)}(k) &\leqslant Cm^{r/2-1}k^{-\theta(2\wedge(r-2))}\leqslant Cn^{(1-\lambda)(r/2-1)-\lambda\theta(2\wedge(r-2))}\\ &\leqslant Cn^{r/2-1-\lambda\{r-2+2\theta(2\wedge(r-2))\}/2}\leqslant C. \end{aligned} \quad (4.18)$$

By (4.18) and Lemma 4.1 with $p=r/(r-2)$ and $q=r/2$, we obtain that

$$\begin{aligned} \sum_{j=1}^{m}E(Y_j^2M_j^{r-2}) &= \sum_{j=1}^{m}E(Y_j^2)E(M_j^{r-2})+\sum_{j=1}^{m}\mathrm{Cov}\left(Y_j^2,M_j^{r-2}\right)\\ &\leqslant \sum_{j=1}^{m}E(Y_j^2)E(M_j^{r-2})+10\rho^{2(2\wedge(r-2))/r}(k)\sum_{j=1}^{m}\|Y_j\|_r^2\|M_j\|_r^{r-2}\\ &=: II_1+II_2. \end{aligned} \quad (4.19)$$

Noting (4.12), Lemma 4.2 and Hölder inequality, we have

$$\begin{aligned} II_1 &\leqslant C\sum_{j=1}^{m}E(Y_j^2)E\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^{r-2}\\ &\leqslant C\sum_{i=1}^{n}EX_i^2\left(E\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^r\right)^{(r-2)/r}\\ &\leqslant \frac{C}{\tau^{r(r-2)/4}}\left(\sum_{i=1}^{n}EX_i^2\right)^{r/2}+\frac{\tau(r-2)}{r}E\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^r. \end{aligned} \quad (4.20)$$

Also, it follows by (4.12), Hölder inequality and (4.18) that

$$\begin{aligned} II_2 &\leqslant C\rho^{2(2\wedge(r-2))/r}(k)\sum_{j=1}^{m}\|Y_j\|_r^2\left(E\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^r\right)^{(r-2)/r} \\ &\leqslant \frac{C}{\tau^{r(r-2)/4}}\rho^{2\wedge(r-2)}(k)\left(\sum_{j=1}^{m}\|Y_j\|_r^2\right)^{r/2}+\frac{\tau(r-2)}{r}E\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^r \\ &\leqslant \frac{C}{\tau^{r(r-2)/4}}m^{r/2-1}\rho^{2\wedge(r-2)}(k)\sum_{j=1}^{m}E|Y_j|^r+\frac{\tau(r-2)}{r}E\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^r \\ &\leqslant \frac{C}{\tau^{r(r-2)/4}}\sum_{j=1}^{m}E|Y_j|^r+\frac{\tau(r-2)}{r}E\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^r. \end{aligned} \tag{4.21}$$

Combining (4.19)–(4.21) yields (4.16). Similarly, we have (4.17). The proof is completed. □

LEMMA 4.7. *Suppose that the sequence $\{X_i : i \geqslant 1\}$ satisfies the conditions in Theorem 1.1. If $r > 2$, then for any $\varepsilon > 0$,*

$$E\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^r \leqslant C\left\{\sum_{j=1}^{m+1}E|Y_j|^r+\left(\sum_{i=1}^{n}EX_i^2\right)^{r/2}\right\}, \tag{4.22}$$

$$E\max_{1\leqslant j\leqslant m+1}|S_{2,j}|^r \leqslant C\left\{\sum_{j=1}^{m+1}E|Z_j|^r+\left(\sum_{i=1}^{n}EX_i^2\right)^{r/2}\right\}. \tag{4.23}$$

Proof. For $r > 2$, we have by (4.10) and Lemma 4.3,

$$\begin{aligned} \left|\max_{1\leqslant j\leqslant m+1}S_{1,j}\right|^r &= |N_0|^r = |Y_1+M_1|^r \leqslant d_1|Y_1|^r + rY_1M_1^{r-1}+d_2Y_1^2M_1^{r-2}+M_1^r \\ &\leqslant d_1|Y_1|^r + rY_1M_1^{r-1}+d_2Y_1^2M_1^{r-2}+|N_1|^r \leqslant \cdots \\ &\leqslant d_1\sum_{j=1}^{m+1}|Y_j|^r + r\sum_{j=1}^{m}Y_jM_j^{r-1}+d_2\sum_{j=1}^{m}Y_j^2M_j^{r-2}. \end{aligned} \tag{4.24}$$

In the same way,

$$\left|\max_{1\leqslant j\leqslant m+1}(-S_{1,j})\right|^r \leqslant d_1\sum_{j=1}^{m+1}|Y_j|^r + r\sum_{j=1}^{m}Y_j\widetilde{M}_j^{r-1}+d_2\sum_{j=1}^{m}Y_j^2\widetilde{M}_j^{r-2}. \tag{4.25}$$

Thus we have

$$E\left|\max_{1\leqslant j\leqslant m+1}S_{1,j}\right|^r \leqslant C_\tau\sum_{j=1}^{m+1}E|Y_j|^r + C_\tau\left(\sum_{i=1}^{n}EX_i^2\right)^{r/2} + \tau C_r E\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^r$$

by combining (4.24) with (4.14) and (4.16), and

$$E\left|\max_{1\leqslant j\leqslant m+1}(-S_{1,j})\right|^r \leqslant C_\tau\sum_{j=1}^{m+1}E|Y_j|^r + C_\tau\left(\sum_{i=1}^{n}EX_i^2\right)^{r/2} + \tau C_r E\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^r$$

by combining (4.25) with (4.15) and (4.17). Hence, from (4.9) and the two inequalities mentioned above, it follows that

$$E\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^r \leqslant C_\tau\sum_{j=1}^{m+1}E|Y_j|^r + C_\tau\left(\sum_{i=1}^{n}EX_i^2\right)^{r/2} + \tau C_r E\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^r.$$

Thus, we have

$$(1-\tau C_r)E\max_{1\leqslant j\leqslant m+1}|S_{1,j}|^r \leqslant C_\tau\sum_{j=1}^{m+1}E|Y_j|^r + C_\tau n^{\varepsilon}\left(\sum_{i=1}^{n}EX_i^2\right)^{r/2},$$

which concludes (4.22) by taking a sufficiently small τ. Similarly, we obtain (4.23). The proof is completed. □

Next, we will give the proof of Theorem 1.1, as follows.

Proof of Theorem 1.1. By Lemmas 4.4 and 4.7, we have

$$E\max_{1\leqslant j\leqslant n}|S_j|^r \leqslant C\left\{\sum_{i=1}^{m+1}(E|Y_i|^r + E|Z_i|^r) + \sum_{j=1}^{2(m+1)}E\max_{1\leqslant l<k}\left|\sum_{i=(j-1)k+1}^{(j-1)k+l}X_i\right|^r + \left(\sum_{i=1}^{n}EX_i^2\right)^{r/2}\right\}. \tag{4.26}$$

Using Minkowski inequality to $E|Y_i|^r$, $E|Z_i|^r$ and $E\max_{1\leqslant l<k}\left|\sum_{i=(j-1)k+1}^{(j-1)k+l}X_i\right|^r$ in (4.26), and noting (4.5) and $X_i=0$ for $i>n$, we have

$$\begin{aligned}E\max_{1\leqslant j\leqslant n}|S_j|^r &\leqslant C\left\{k^{r-1}\sum_{i=1}^{n}E|X_i|^r + \left(\sum_{i=1}^{n}EX_i^2\right)^{r/2}\right\}\\ &\leqslant C\left\{n^{\lambda(r-1)}\sum_{i=1}^{n}E|X_i|^r + \left(\sum_{i=1}^{n}EX_i^2\right)^{r/2}\right\}.\end{aligned}$$

Applying the above inequality to $E|Y_i|^r$, $E|Z_i|^r$ and $E\max_{1\leqslant l<k}\left|\sum_{i=(j-1)k+1}^{(j-1)k+l}X_i\right|^r$ in (4.26), we obtain

$$\begin{aligned}E\max_{1\leqslant j\leqslant n}|S_j|^r &\leqslant C\left\{k^{\lambda(r-1)}\sum_{i=1}^{n}E|X_i|^r + \left(\sum_{i=1}^{n}EX_i^2\right)^{r/2}\right\}\\ &\leqslant C\left\{n^{\lambda^2(r-1)}\sum_{i=1}^{n}E|X_i|^r + \left(\sum_{i=1}^{n}EX_i^2\right)^{r/2}\right\}.\end{aligned}$$

Again, applying the inequality above to $E|Y_i|^r$, $E|Z_i|^r$ and $E\max_{1\leqslant l<k}\left|\sum_{i=(j-1)k+1}^{(j-1)k+l} X_i\right|^r$ in (4.26), and repeating t times in this way, we have

$$E\max_{1\leqslant j\leqslant n}|S_j|^r \leqslant C\left\{n^{\lambda^t(r-1)}\sum_{i=1}^n E|X_i|^r + \left(\sum_{i=1}^n EX_i^2\right)^{r/2}\right\}$$

for positive integer $t \geqslant 1$. Since $0<\lambda<1$, $\lambda^t(r-1)<\varepsilon$ for sufficiently large t. Thus (1.3) holds. The proof is completed. □

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REFERENCES

[1] P. BILLINGSLEY, *Convergence of Probability Measures*, Wiley, New York, 1968.

[2] T. BIRKEL, *Moment Bounds for associated sequences*, Ann. Probab., 16: 1184–1193, 1988.

[3] A. A. GEORGIEV, *Consistent nonparametric multilpe regression: the fixed design case*, J. Multivariate Anal., 25: 100–110, 1988.

[4] M. PELIGRAD, *Invariance principles for mixing sequencs of random variales*, Ann. Probab., 10: 968–981, 1982.

[5] M. PELIGRAD, *Convergence rates of the strong law for stationary mixing sequences*, Z. Wahrsch. Verw. Gebirte., 70: 307–314, 1985.

[6] M. PELIGRAD, *On the central limit theorem for ρ-mixing sequences of random variables*, Ann. Probab., 15: 1387–1394, 1987.

[7] G. G. ROUSSAS AND D. A. IOANNIDES, *Moment inequalities for mixing sequences of random variables*, Stochastic Anal. Appl., 5: 61–120, 1987.

[8] SHAO QIMAN, *A moment inequality and its applications*, Acta Mathematica Sinica (in chinese), 31: 736–747, 1988.

[9] SHAO QIMAN, *Complete convergence for ρ-mixing sequences*, Acta Mathematica Sinica (in chinese), 32: 377–393, 1989.

[10] SHAO QIMAN, *Maximal inequalities for partial sums of ρ-mixing sequences*, Ann. Probab., 23: 948–965, 1995.

[11] SHAO QIMAN AND YU HAO, *Weak convergence for weighed emprirical processes of dependent sequences*, Ann. Probab., 24: 2098–2127, 1996.

[12] SHAO QIMAN, *A comparison theorem on maximal inequalities between negatively associated and independent random variables*, J. Theor. Probab., 13 (2): 343–356, 2000.

[13] SHAO QIMAN AND SU CHUN, *The law of the iterated logarithm for negatively associated random variables*, Stochastic Process. Appl., 83: 139–148, 1999.

[14] YANG SHANCHAO, *Moment inequality for mixing sequences and nonparametric estimation*, Acta Mathematica Sinica (in chinese), 40: 271–279, 1997.

[15] YANG SHANCHAO, *Moment Bounds for strong mixing sequences and their application*, J. Math. Research and Exposition, 20: 349–359, 2000.

[16] YANG SHANCHAO, *Moment inequalities for partial sums of random variables*, Science in China (series A), 44: 1–6, 2001.

[17] YANG SHANCHAO, *Uniformly asymptotic normality of the regression weighted estimator for negatively associated samples*, Statist. Probab. Lett., 62: 101–110, 2003.

[18] YANG SHANCHAO, *Maximal Moment Inequality for Partial Sum of Strong Mixing Sequences and Application*, Acta Mathematica Sinica (English Series), 23 (6): 1013–1024, 2007.

[19] XING GUODONG, YANG SHANCHAO AND CHEN AIWU, *A maximal moment inequality for α-mixing sequences and its applications*, Statist. Probab. Lett., 79: 1429–1437, 2009.

[20] R. YOKOYAMA, *Moment bounds for stationary mixing sequences*, Z. Wahrsch. Verw. Gebiete., 52: 45–57, 1980.

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