

APPROXIMATION BY α -BERNSTEIN-SCHURER-STANCU OPERATORS

NURSEL ÇETIN AND ANA-MARIA ACU

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Abstract. In this paper, we consider a new family of generalized Bernstein-Schurer-Stancu operators, depending on a non-negative real parameter α and study some approximation properties of these operators. We obtain a recurrence formula concerning calculation of moments by Schurer-Stancu operators. We prove a uniform approximation result using the well-known Korovkin theorem and obtain the rate of convergence in terms of modulus of continuity. Also, we present Voronovskaya and Grüss-Voronovskaya type results for these operators. Moreover, we give some numerical examples to illustrate approximation by the new operator.

1. Introduction

In 1912, Bernstein [4] introduced the classical Bernstein polynomials in order to give one of the simplest and most elegant proof of Weierstrass Approximation Theorem. Then, discovery of their various generalizations and modifications in different ways has been an intensive research area due to the advantages of their simple structures and many useful properties such as positivity, end-point interpolation, symmetry, degree raising, etc. More information concerning the state of the art can be found in [5, 13].

In 1962, considering a given non-negative integer p , Schurer [15] introduced and studied new generalization of Bernstein operators. In 1969, Stancu [17] constructed a linear positive operators known in literature as Bernstein-Stancu operators, which depend on two real parameters. On the other hand, in 2003, Barbosu [3] defined the Schurer-Stancu operators $\tilde{S}_{n,p}^{(\alpha,\beta)} : C[0, 1+p] \rightarrow C[0, 1]$ as

$$\tilde{S}_{n,p}^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n+p} f\left(\frac{k+\alpha}{n+\beta}\right) \binom{n+p}{k} x^k (1-x)^{n+p-k}, \quad x \in [0, 1] \quad (1.1)$$

where $n \in \mathbb{N}$, $f \in C[0, 1+p]$, p is a non-negative integer and α, β are real parameters satisfying the conditions $0 \leq \alpha \leq \beta$, and investigated some approximation properties of these operators.

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* Corresponding author.

In 2017, Chen et al. [10] introduced a new family of generalized Bernstein operators which is called as α -Bernstein operator, depending on a non-negative real parameter, as follows

$$T_{n,\alpha}(f;x) = \sum_{i=0}^n f\left(\frac{i}{n}\right) p_{n,i}^{(\alpha)}(x), \quad n \in \mathbb{N}, \alpha \in \mathbb{R}, \quad (1.2)$$

for any function f defined on $[0, 1]$. Here, for $i = 0, 1, \dots, n$, the α -Bernstein polynomial $p_{n,i}^{(\alpha)}(x)$ of degree n is defined by $p_{1,0}^{(\alpha)}(x) = 1 - x$, $p_{1,1}^{(\alpha)}(x) = x$ and

$$p_{n,i}^{(\alpha)}(x) = \left[\binom{n-2}{i} (1-\alpha)x + \binom{n-2}{i-2} (1-\alpha)(1-x) + \binom{n}{i} \alpha x(1-x) \right] \times x^{i-1} (1-x)^{n-i-1},$$

where $n \geq 2$, $x \in [0, 1]$ and the binomial coefficients $\binom{k}{l}$ are given by

$$\binom{k}{l} = \begin{cases} \frac{k!}{(k-l)! \cdot l!}, & \text{if } 0 \leq l \leq k, \\ 0, & \text{else.} \end{cases} \quad (1.3)$$

For $\alpha = 1$, the α -Bernstein operator becomes the classical Bernstein polynomial. Also, the α -Bernstein operators are linear positive operators for $\alpha \in [0, 1]$. In [10], the authors gave some elementary properties and proved the uniform convergence of the sequence of the α -Bernstein operators to $f \in C[0, 1]$ with the help of the well known Korovkin theorem. They obtained the rate of convergence and Voronovskaya-type theorem for the α -Bernstein operators. Also, they gave an upper bound for the approximation error by means of the modulus of continuity and proved that the α -Bernstein operators satisfy some shape preserving results. Very recently, Çetin [7] investigated some approximation properties of complex α -Bernstein operator in compact disks. The author obtained quantitative upper estimate for simultaneous approximation, a qualitative Voronovskaja type result and the exact order of approximation. Also, the author presented some shape preserving properties of the complex α -Bernstein operator such as univalence, starlikeness, convexity and spirallikeness. To mention some recent works concerning generalizations of α -Bernstein operator, we may refer to [1], [6], [9], [14], [16].

Inspired by the above works, in this paper we introduce a generalization of Bernstein-Schurer-Stancu operators given in (1.1), as follows:

$$T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(f;x) = \sum_{i=0}^{n+p} f\left(\frac{i+\alpha^*}{n+\beta^*}\right) \tilde{p}_{n,i}^{(\alpha)}(x), \quad n \in \mathbb{N}, p \in \mathbb{N} \cup \{0\}, \quad (1.4)$$

for any function f defined on $[0, 1+p]$, $x \in [0, 1]$, any fixed real α and α^*, β^* satisfying the condition $0 \leq \alpha^* \leq \beta^*$. Here, for $i = 0, 1, \dots, n$, the Schurer-type basis

functions are defined by $\tilde{p}_{1,0}^{(\alpha)}(x) = 1 - x$, $\tilde{p}_{1,1}^{(\alpha)}(x) = x$ and

$$\begin{aligned} \tilde{p}_{n,i}^{(\alpha)}(x) = & \left[\binom{n+p-2}{i} (1-\alpha)x + \binom{n+p-2}{i-2} (1-\alpha)(1-x) \right. \\ & \left. + \binom{n+p}{i} \alpha x (1-x) \right] x^{i-1} (1-x)^{n+p-i-1}, \end{aligned}$$

where $n + p \geq 2$, $x \in [0, 1]$ and the binomial coefficients $\binom{k}{i}$ are given as in (1.3). The operator (1.4) is called as α -Bernstein-Schurer-Stancu operators, which are a family of linear positive operators for $0 \leq \alpha \leq 1$. Note that for $\alpha = 1$, the operator $T_{n,\alpha,p}^{(\alpha^*,\beta^*)}$ becomes the Schurer-Stancu operators. If $\alpha = 1$ and $p = 0$, the operator (1.4) reduces to the Bernstein-Stancu operators. If $\alpha = 1$ and $\alpha^* = \beta^* = 0$, the operator (1.4) becomes the Bernstein-Schurer operators. The case $p = 0$ in (1.4) gives the α -Bernstein-Stancu operators defined in [9]. If $\alpha^* = \beta^* = 0$ in (1.4), the operator $T_{n,\alpha,p}^{(\alpha^*,\beta^*)}$ reduces to the α -Bernstein-Schurer operators defined in [8]. Also, in the special case $\alpha^* = \beta^* = p = 0$, the operator (1.4) reduces to α -Bernstein operator given by (1.2).

In the present paper, firstly we obtain a recurrence formula concerning calculation of moments by Schurer-Stancu operators given in (1.1). Then, we prove a uniform approximation result using the well-known Korovkin theorem and obtain the degree of approximation in terms of modulus of continuity. Also, we give some numerical examples to illustrate approximation by new constructed operator. Finally, we study Voronovskaya and Grüss-Voronovskaya type theorems for these operators.

2. Auxiliary results

In this section, we give some useful results that will be necessary in the proof of the main results. Let us denote by $e_k(x) = x^k$, $k \in \mathbb{N} \cup \{0\}$ the test functions and $\varphi_x^j(t) := (t - x)^j$, $j \in \mathbb{N}$.

The α -Bernstein-Schurer-Stancu operators defined by (1.4) have the following another representation.

THEOREM 2.1. *The α -Bernstein-Schurer-Stancu operators given by (1.4) can be stated as*

$$\begin{aligned} T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(f;x) = & (1-\alpha) \sum_{i=0}^{n+p-1} g_i^{(\alpha^*,\beta^*)} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} \\ & + \alpha \sum_{i=0}^{n+p} f_i^{(\alpha^*,\beta^*)} \binom{n+p}{i} x^i (1-x)^{n+p-i}, \end{aligned} \tag{2.1}$$

where $f \in C[0, 1 + p]$, $x \in [0, 1]$, $f_i^{(\alpha^*,\beta^*)} = f\left(\frac{i+\alpha^*}{n+\beta^*}\right)$ and

$$g_i^{(\alpha^*,\beta^*)} = \left(1 - \frac{i}{n+p-1}\right) f_i^{(\alpha^*,\beta^*)} + \frac{i}{n+p-1} f_{i+1}^{(\alpha^*,\beta^*)}, \quad n + p \geq 2.$$

Proof. It can be easily obtained by the method used for the α -Bernstein operator (see p. 247–248 in [10]). Therefore, the details are omitted. \square

LEMMA 2.1. (see [3]) *For the Schurer-Stancu operators given by (1.1), the following results hold*

$$\widetilde{S}_{n,p}^{(\alpha^*,\beta^*)}(e_0;x) = 1, \tag{2.2}$$

$$\widetilde{S}_{n,p}^{(\alpha^*,\beta^*)}(e_1;x) = \frac{n+p}{n+\beta^*}x + \frac{\alpha^*}{n+\beta^*}, \tag{2.3}$$

$$\begin{aligned} \widetilde{S}_{n,p}^{(\alpha^*,\beta^*)}(e_2;x) = \frac{1}{(n+\beta^*)^2} \{ & (n+p)^2x^2 + (n+p)x(1-x) \\ & + 2\alpha^*(n+p)x + \alpha^{*2} \}. \end{aligned} \tag{2.4}$$

Furthermore, by direct computation, we have the following moments.

LEMMA 2.2. *For the Schurer-Stancu operators given by (1.1), we have*

$$\begin{aligned} & \widetilde{S}_{n,p}^{(\alpha^*,\beta^*)}(e_3;x) \\ &= \frac{1}{(n+\beta^*)^3} \{ (n+p)(n+p-1)(n+p-2)x^3 + 3(n+p)(n+p-1)(1+\alpha^*)x^2 \\ & \quad + (n+p)(3\alpha^{*2} + 3\alpha^* + 1)x + \alpha^{*3} \} \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \widetilde{S}_{n,p}^{(\alpha^*,\beta^*)}(e_4;x) = \frac{1}{(n+\beta^*)^4} \{ & (n+p)(n+p-1)(n+p-2)(n+p-3)x^4 \\ & + 2(n+p)(n+p-1)(n+p-2)(3+2\alpha^*)x^3 \\ & + (n+p)(n+p-1)(6\alpha^{*2} + 12\alpha^* + 7)x^2 \\ & + (n+p)(1+2\alpha^*)(2\alpha^{*2} + 2\alpha^* + 1)x + \alpha^{*4} \}. \end{aligned} \tag{2.6}$$

In what follows, we will give a recurrence formula to calculate higher order moments of the new operator $T_{n,\alpha,p}^{(\alpha^*,\beta^*)}$ given by (1.4).

THEOREM 2.2. *For all $j, p \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$ and $x \in [0, 1]$, we have*

$$\begin{aligned} & T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_{j+1};x) \tag{2.7} \\ &= \frac{x(1-x)}{n+\beta^*} \left(T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_j;x) \right)' + \frac{[1+\alpha^*+(n+p-1)x]}{n+\beta^*} T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_j;x) \\ & \quad + \frac{(1-\alpha)}{(n+\beta^*)(n+p-1)} \left(\frac{n-1+\beta^*}{n+\beta^*} \right)^j \left[(n-1+\beta^*) \widetilde{S}_{n-1,p}^{(\alpha^*,\beta^*)}(e_{j+1};x) \right. \\ & \quad \left. - (n+p-1+\alpha^*) \widetilde{S}_{n-1,p}^{(\alpha^*,\beta^*)}(e_j;x) \right] - \frac{\alpha(1-x)}{n+\beta^*} \widetilde{S}_{n,p}^{(\alpha^*,\beta^*)}(e_j;x), \end{aligned}$$

where $\alpha \in [0, 1]$ and $\widetilde{S}_{n,p}^{(\alpha^*,\beta^*)}$ is the Schurer-Stancu operator given by (1.1).

Proof. Using (2.1), we can write

$$\begin{aligned} & T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_j;x) \\ &= (1-\alpha) \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} \left[\left(1 - \frac{i}{n+p-1}\right) \left(\frac{i+\alpha^*}{n+\beta^*}\right)^j \right. \\ & \quad \left. + \frac{i}{n+p-1} \left(\frac{i+1+\alpha^*}{n+\beta^*}\right)^j \right] + \alpha \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} \left(\frac{i+\alpha^*}{n+\beta^*}\right)^j. \end{aligned}$$

By some calculations, we get

$$\begin{aligned} & \left(T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_j;x)\right)' \\ &= (1-\alpha) \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} \left[ix^{i-1} (1-x)^{n+p-i-1} - (n+p-i-1)x^i (1-x)^{n+p-i-2} \right] \\ & \quad \times \left[\left(1 - \frac{i}{n+p-1}\right) \left(\frac{i+\alpha^*}{n+\beta^*}\right)^j + \frac{i}{n+p-1} \left(\frac{i+1+\alpha^*}{n+\beta^*}\right)^j \right] \\ & \quad + \alpha \sum_{i=0}^{n+p} \binom{n+p}{i} \left[ix^{i-1} (1-x)^{n+p-i} - (n+p-i)x^i (1-x)^{n+p-i-1} \right] \left(\frac{i+\alpha^*}{n+\beta^*}\right)^j \\ &= \frac{(1-\alpha)}{x(1-x)} \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} i \left[\left(1 - \frac{i}{n+p-1}\right) \left(\frac{i+\alpha^*}{n+\beta^*}\right)^j \right. \\ & \quad \left. + \frac{i}{n+p-1} \left(\frac{i+1+\alpha^*}{n+\beta^*}\right)^j \right] - \frac{(n+p-1)(1-\alpha)}{(1-x)} \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i \\ & \quad \times (1-x)^{n+p-i-1} \left[\left(1 - \frac{i}{n+p-1}\right) \left(\frac{i+\alpha^*}{n+\beta^*}\right)^j + \frac{i}{n+p-1} \left(\frac{i+1+\alpha^*}{n+\beta^*}\right)^j \right] \\ & \quad + \frac{\alpha}{x(1-x)} \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} i \left(\frac{i+\alpha^*}{n+\beta^*}\right)^j \\ & \quad - \frac{\alpha(n+p)}{(1-x)} \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} \left(\frac{i+\alpha^*}{n+\beta^*}\right)^j \\ &= \frac{(1-\alpha)}{x(1-x)} \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} i \left[\left(1 - \frac{i}{n+p-1}\right) \left(\frac{i+\alpha^*}{n+\beta^*}\right)^j \right] \\ & \quad + \frac{(1-\alpha)}{x(1-x)} \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} i \\ & \quad \times \left[\frac{i}{n+p-1} \left(\frac{i+1+\alpha^*}{n+\beta^*}\right)^j \right] - \frac{(n+p-1)(1-\alpha)}{(1-x)} \\ & \quad \times \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} \left[\left(1 - \frac{i}{n+p-1}\right) \left(\frac{i+\alpha^*}{n+\beta^*}\right)^j \right] \end{aligned}$$

$$\begin{aligned}
 & + \frac{i}{n+p-1} \left(\frac{i+1+\alpha^*}{n+\beta^*} \right)^j \Big] + \frac{\alpha}{x(1-x)} \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} i \left(\frac{i+\alpha^*}{n+\beta^*} \right)^j \\
 & - \frac{\alpha(n+p)}{(1-x)} \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} \left(\frac{i+\alpha^*}{n+\beta^*} \right)^j
 \end{aligned}$$

Considering that i can be written as $i + \alpha^* - \alpha^*$, we obtain

$$\begin{aligned}
 & \left(T_{n,\alpha,p}^{(\alpha^*,\beta^*)} (e_j; x) \right)' \\
 & = \frac{(n+\beta^*)(1-\alpha)}{x(1-x)} \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} \left[\left(1 - \frac{i}{n+p-1} \right) \left(\frac{i+\alpha^*}{n+\beta^*} \right)^{j+1} \right] \\
 & - \frac{\alpha^*(1-\alpha)}{x(1-x)} \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} \left[\left(1 - \frac{i}{n+p-1} \right) \left(\frac{i+\alpha^*}{n+\beta^*} \right)^j \right] \\
 & + \frac{(n+\beta^*)(1-\alpha)}{x(1-x)} \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} \left[\frac{i}{n+p-1} \left(\frac{i+1+\alpha^*}{n+\beta^*} \right)^{j+1} \right] \\
 & - \frac{(1+\alpha^*)(1-\alpha)}{x(1-x)} \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} \left[\frac{i}{n+p-1} \left(\frac{i+1+\alpha^*}{n+\beta^*} \right)^j \right] \\
 & - \frac{(n+p-1)(1-\alpha)}{(1-x)} \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} \left[\left(1 - \frac{i}{n+p-1} \right) \left(\frac{i+\alpha^*}{n+\beta^*} \right)^j \right] \\
 & + \frac{i}{n+p-1} \left(\frac{i+1+\alpha^*}{n+\beta^*} \right)^j \Big] + \frac{\alpha(n+\beta^*)}{x(1-x)} \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} \left(\frac{i+\alpha^*}{n+\beta^*} \right)^{j+1} \\
 & - \frac{\alpha\alpha^*}{x(1-x)} \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} \left(\frac{i+\alpha^*}{n+\beta^*} \right)^j \\
 & - \frac{\alpha(n+p)}{(1-x)} \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} \left(\frac{i+\alpha^*}{n+\beta^*} \right)^j \\
 & = \frac{(n+\beta^*)(1-\alpha)}{x(1-x)} \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} \left[\left(1 - \frac{i}{n+p-1} \right) \left(\frac{i+\alpha^*}{n+\beta^*} \right)^{j+1} \right] \\
 & + \frac{i}{n+p-1} \left(\frac{i+1+\alpha^*}{n+\beta^*} \right)^{j+1} \Big] - \frac{\alpha^*(1-\alpha)}{x(1-x)} \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} \\
 & \times \left[\left(1 - \frac{i}{n+p-1} \right) \left(\frac{i+\alpha^*}{n+\beta^*} \right)^j + \frac{i}{n+p-1} \left(\frac{i+1+\alpha^*}{n+\beta^*} \right)^j \right] \\
 & - \frac{(1-\alpha)}{x(1-x)} \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} \left[\frac{i}{n+p-1} \left(\frac{i+1+\alpha^*}{n+\beta^*} \right)^j \right] \\
 & - \frac{(n+p-1)(1-\alpha)}{(1-x)} \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} \left[\left(1 - \frac{i}{n+p-1} \right) \left(\frac{i+\alpha^*}{n+\beta^*} \right)^j \right]
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{i}{n+p-1} \left(\frac{i+1+\alpha^*}{n+\beta^*} \right)^j \Big] + \frac{\alpha(n+\beta^*)}{x(1-x)} \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} \left(\frac{i+\alpha^*}{n+\beta^*} \right)^{j+1} \\
 & - \frac{\alpha[(n+p)x+\alpha^*]}{x(1-x)} \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} \left(\frac{i+\alpha^*}{n+\beta^*} \right)^j
 \end{aligned}$$

Taking into account definition of the operators $T_{n,\alpha,p}^{(\alpha^*,\beta^*)}$ given by (2.1), we have

$$\begin{aligned}
 & \left(T_{n,\alpha,p}^{(\alpha^*,\beta^*)} (e_j;x) \right)' \\
 & = \frac{(n+\beta^*)}{x(1-x)} \left\{ T_{n,\alpha,p}^{(\alpha^*,\beta^*)} (e_{j+1};x) - \alpha \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} \left(\frac{i+\alpha^*}{n+\beta^*} \right)^{j+1} \right\} \\
 & - \frac{\alpha^*}{x(1-x)} \left\{ T_{n,\alpha,p}^{(\alpha^*,\beta^*)} (e_j;x) - \alpha \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} \left(\frac{i+\alpha^*}{n+\beta^*} \right)^j \right\} \\
 & - \frac{(1-\alpha)}{x(1-x)} \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} \left[\frac{i}{n+p-1} \left(\frac{i+1+\alpha^*}{n+\beta^*} \right)^j \right] \\
 & - \frac{(n+p-1)}{(1-x)} \left\{ T_{n,\alpha,p}^{(\alpha^*,\beta^*)} (e_j;x) - \alpha \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} \left(\frac{i+\alpha^*}{n+\beta^*} \right)^j \right\} \\
 & + \frac{\alpha(n+\beta^*)}{x(1-x)} \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} \left(\frac{i+\alpha^*}{n+\beta^*} \right)^{j+1} \\
 & - \frac{\alpha[(n+p)x+\alpha^*]}{x(1-x)} \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} \left(\frac{i+\alpha^*}{n+\beta^*} \right)^j \\
 & = \frac{(n+\beta^*)}{x(1-x)} T_{n,\alpha,p}^{(\alpha^*,\beta^*)} (e_{j+1};x) - \frac{[\alpha^*+(n+p-1)x]}{x(1-x)} T_{n,\alpha,p}^{(\alpha^*,\beta^*)} (e_j;x) \\
 & - \frac{(1-\alpha)}{x(1-x)} \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} \left[\frac{i}{n+p-1} \left(\frac{i+1+\alpha^*}{n+\beta^*} \right)^j \right] \\
 & - \frac{\alpha}{(1-x)} \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} \left(\frac{i+\alpha^*}{n+\beta^*} \right)^j \\
 & = \frac{(n+\beta^*)}{x(1-x)} T_{n,\alpha,p}^{(\alpha^*,\beta^*)} (e_{j+1};x) - \frac{[\alpha^*+(n+p-1)x]}{x(1-x)} T_{n,\alpha,p}^{(\alpha^*,\beta^*)} (e_j;x) \\
 & - \frac{1}{x(1-x)} \left\{ T_{n,\alpha,p}^{(\alpha^*,\beta^*)} (e_j;x) - (1-\alpha) \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} \right. \\
 & \times \left. \left(1 - \frac{i}{n+p-1} \right) \left(\frac{i+\alpha^*}{n+\beta^*} \right)^j - \alpha \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} \left(\frac{i+\alpha^*}{n+\beta^*} \right)^j \right\} \\
 & - \frac{\alpha}{(1-x)} \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} \left(\frac{i+\alpha^*}{n+\beta^*} \right)^j
 \end{aligned}$$

$$\begin{aligned}
&= \frac{(n+\beta^*)}{x(1-x)} T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_{j+1};x) - \frac{[1+\alpha^*+(n+p-1)x]}{x(1-x)} T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_j;x) \\
&+ \frac{(1-\alpha)}{x(1-x)} \left(1 + \frac{\alpha^*}{n+p-1}\right) \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} \left(\frac{i+\alpha^*}{n+\beta^*}\right)^j \\
&- \frac{(1-\alpha)}{x(1-x)} \frac{(n+\beta^*)}{(n+p-1)} \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} \left(\frac{i+\alpha^*}{n+\beta^*}\right)^{j+1} \\
&+ \frac{\alpha}{x} \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} \left(\frac{i+\alpha^*}{n+\beta^*}\right)^j.
\end{aligned}$$

In view of definition of Schurer-Stancu operator defined by (1.1),

$$\begin{aligned}
&\left(T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_j;x)\right)' \\
&= \frac{(n+\beta^*)}{x(1-x)} T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_{j+1};x) - \frac{[1+\alpha^*+(n+p-1)x]}{x(1-x)} T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_j;x) \\
&+ \frac{(1-\alpha)}{x(1-x)} \left(1 + \frac{\alpha^*}{n+p-1}\right) \left(\frac{n-1+\beta^*}{n+\beta^*}\right)^j \tilde{S}_{n-1,p}^{(\alpha^*,\beta^*)}(e_j;x) \\
&- \frac{(1-\alpha)}{x(1-x)} \frac{(n+\beta^*)}{(n+p-1)} \left(\frac{n-1+\beta^*}{n+\beta^*}\right)^{j+1} \tilde{S}_{n-1,p}^{(\alpha^*,\beta^*)}(e_{j+1};x) + \frac{\alpha}{x} \tilde{S}_{n,p}^{(\alpha^*,\beta^*)}(e_j;x),
\end{aligned}$$

which completes the proof. \square

Now, we give moments and central moments of the α -Bernstein-Schurer-Stancu operators, below.

LEMMA 2.3. For the operators $T_{n,\alpha,p}^{(\alpha^*,\beta^*)}$, one has

$$T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_0;x) = 1, \quad (2.8)$$

$$T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_1;x) = \frac{n+p}{n+\beta^*}x + \frac{\alpha^*}{n+\beta^*}, \quad (2.9)$$

$$\begin{aligned}
T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_2;x) &= \frac{1}{(n+\beta^*)^2} \{x^2[(n+p)(n+p-1) - 2(1-\alpha)] \\
&+ x[(n+p)(1+2\alpha^*) + 2(1-\alpha)] + \alpha^{*2}\}, \quad (2.10)
\end{aligned}$$

$$\begin{aligned}
&T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_3;x) \\
&= \frac{1}{(n+\beta^*)^3} \left\{ \left[(n+p)^3 - 3(n+p)^2 + 2(3\alpha-2)(n+p) + 12(1-\alpha) \right] x^3 \right. \\
&+ 3 \left[(1+\alpha^*)(n+p)^2 + (1-\alpha^*-2\alpha)(n+p) - 2(3+\alpha^*)(1-\alpha) \right] x^2 \\
&+ \left. \left[(n+p)(1+3\alpha^*+3\alpha^{*2}) + 6(1+\alpha^*)(1-\alpha) \right] x + \alpha^{*3} \right\}, \quad (2.11)
\end{aligned}$$

and

$$\begin{aligned}
 & T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_4;x) \\
 &= \frac{1}{(n+\beta^*)^4} \left\{ \left[(n+p)^4 - 6(n+p)^3 + (12\alpha-1)(n+p)^2 + 6(9-10\alpha)(n+p) - 72(1-\alpha) \right] x^4 \right. \\
 &+ 2 \left[(3+2\alpha^*)(n+p)^3 - 3(1+2\alpha+2\alpha^*)(n+p)^2 \right. \\
 &+ 2(24\alpha-4\alpha^*+6\alpha\alpha^*-21)(n+p) + 24(1-\alpha)(3+\alpha^*) \left. \right] x^3 \\
 &+ \left[(7+12\alpha^*+6\alpha^{*2})(n+p)^2 + (29+12\alpha^*-36\alpha-24\alpha\alpha^*-6\alpha^{*2}) \right. \\
 &\times (n+p) - 2(1-\alpha)(43+36\alpha^*+6\alpha^{*2}) \left. \right] x^2 \\
 &+ \left. \left[(1+4\alpha^*+6\alpha^{*2}+4\alpha^{*3})(n+p) + 2(1-\alpha)(7+12\alpha^*+6\alpha^{*2}) \right] x + \alpha^{*4} \right\}.
 \end{aligned} \tag{2.12}$$

Proof. Using the definition of the operators $T_{n,\alpha,p}^{(\alpha^*,\beta^*)}$ given by (2.1), we can write that if $f = e_0$, then $f_i^{(\alpha^*,\beta^*)} = g_i^{(\alpha^*,\beta^*)} = 1$ and

$$\begin{aligned}
 T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_0;x) &= (1-\alpha) \sum_{i=0}^{n+p-1} \binom{n+p-1}{i} x^i (1-x)^{n+p-i-1} \\
 &+ \alpha \sum_{i=0}^{n+p} \binom{n+p}{i} x^i (1-x)^{n+p-i} \\
 &= 1.
 \end{aligned}$$

On the other hand, by the recurrence formula obtained in Theorem 2.2, for $j = 0$ we have

$$\begin{aligned}
 & T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_1;x) \\
 &= \frac{x(1-x)}{n+\beta^*} \left(T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_0;x) \right)' + \frac{[1+\alpha^*+(n+p-1)x]}{n+\beta^*} T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_0;x) \\
 &+ \frac{(1-\alpha)}{(n+\beta^*)(n+p-1)} \left[(n-1+\beta^*) \tilde{S}_{n-1,p}^{(\alpha^*,\beta^*)}(e_1;x) - (n+p-1+\alpha^*) \tilde{S}_{n-1,p}^{(\alpha^*,\beta^*)}(e_0;x) \right] \\
 &- \frac{\alpha(1-x)}{n+\beta^*} \tilde{S}_{n,p}^{(\alpha^*,\beta^*)}(e_0;x).
 \end{aligned}$$

Using (2.8), (2.2) and (2.3), it follows that

$$\begin{aligned}
 T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_1;x) &= \frac{1+\alpha^*+(n+p-1)x}{n+\beta^*} + \frac{(1-\alpha)}{(n+\beta^*)(n+p-1)} \\
 &\times [(n+p-1)x + \alpha^* - (n+p-1+\alpha^*)] - \frac{\alpha(1-x)}{n+\beta^*}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1 + \alpha^* + (n + p - 1)x}{n + \beta^*} + \frac{(1 - \alpha)(x - 1)}{(n + \beta^*)} - \frac{\alpha(1 - x)}{n + \beta^*} \\
 &= \frac{n + p}{n + \beta^*}x + \frac{\alpha^*}{n + \beta^*}.
 \end{aligned}
 \tag{2.13}$$

Then, from (2.7), for $j = 1$ we get

$$\begin{aligned}
 T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_2;x) &= \frac{x(1-x)}{n+\beta^*} \left(T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_1;x) \right)' + \frac{[1 + \alpha^* + (n + p - 1)x]}{n + \beta^*} T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_1;x) \\
 &+ \frac{(1 - \alpha)}{(n + \beta^*)(n + p - 1)} \left(\frac{n - 1 + \beta^*}{n + \beta^*} \right) \left[(n - 1 + \beta^*) \widetilde{S}_{n-1,p}^{(\alpha^*,\beta^*)}(e_2;x) \right. \\
 &\left. - (n + p - 1 + \alpha^*) \widetilde{S}_{n-1,p}^{(\alpha^*,\beta^*)}(e_1;x) \right] - \frac{\alpha(1-x)}{n+\beta^*} \widetilde{S}_{n,p}^{(\alpha^*,\beta^*)}(e_1;x),
 \end{aligned}$$

By application of the equations (2.13), (2.3) and (2.4), we obtain the followings

$$\begin{aligned}
 &T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(e_2;x) \\
 &= \frac{x(1-x)}{n+\beta^*} \frac{(n+p)}{n+\beta^*} + \frac{[1 + \alpha^* + (n + p - 1)x]}{n + \beta^*} \frac{[(n+p)x + \alpha^*]}{n + \beta^*} \\
 &+ \frac{(1 - \alpha)}{(n + \beta^*)(n + p - 1)} \left(\frac{n - 1 + \beta^*}{n + \beta^*} \right) \left\{ \frac{1}{n - 1 + \beta^*} \left[(n + p - 1)^2 x^2 \right. \right. \\
 &\left. \left. + (n + p - 1)x(1 - x) + 2\alpha^*(n + p - 1)x + \alpha^{*2} \right] - (n + p - 1 + \alpha^*) \right. \\
 &\left. \times \frac{[(n + p - 1)x + \alpha^*]}{n - 1 + \beta^*} \right\} - \frac{\alpha(1-x)}{n+\beta^*} \frac{[(n+p)x + \alpha^*]}{n + \beta^*} \\
 &= \frac{1}{(n+\beta^*)^2} \left\{ x^2 [-(n+p) + (n+p)(n+p-1) + (1-\alpha)(n+p-2) + \alpha(n+p)] \right. \\
 &\left. + x[(n+p) + (1+\alpha^*)(n+p) + \alpha^*(n+p-1) + (1-\alpha)(1+2\alpha^*) \right. \\
 &\left. - (1-\alpha)(n+p-1+\alpha^*) + \alpha\alpha^* - \alpha(n+p)] + \alpha^*(1+\alpha^*) - (1-\alpha)\alpha^* - \alpha\alpha^* \right\} \\
 &= \frac{1}{(n+\beta^*)^2} \left\{ x^2 [(n+p)(n+p-1) - 2(1-\alpha)] + x[(n+p)(1+2\alpha^*) \right. \\
 &\left. + 2(1-\alpha)] + \alpha^{*2} \right\}.
 \end{aligned}
 \tag{2.14}$$

Writing for $j = 2$ and $j = 3$ in (2.7), by making use of (2.14), (2.4), (2.5) and then (2.11), (2.5), (2.6) respectively, the remain of the proof can be easily shown. So, we omit the details. \square

LEMMA 2.4. For the central moments of the operators $T_{n,\alpha,p}^{(\alpha^*,\beta^*)}$, one has

$$T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(\varphi_x^1;x) = \frac{(p - \beta^*)}{n + \beta^*}x + \frac{\alpha^*}{n + \beta^*},
 \tag{2.15}$$

$$\begin{aligned}
 T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(\varphi_x^2;x) &= \frac{1}{(n + \beta^*)^2} \left\{ (p - \beta^*)^2 x^2 + [n + p + 2(1 - \alpha)]x(1 - x) \right. \\
 &\left. + 2\alpha^*(p - \beta^*)x + \alpha^{*2} \right\} =: \left(\delta_{n,\alpha,p}^{(\alpha^*,\beta^*)}(x) \right)^2
 \end{aligned}
 \tag{2.16}$$

$$\begin{aligned}
 & T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(\varphi_x^4;x) \\
 &= \frac{1}{(n+\beta^*)^4} \left\{ \left[3n^2+2n(-3p^2+p(7+6\beta^*)+3-6\alpha-4\beta^*-3\beta^{*2})+p^4-2p^3(3+2\beta^*) \right. \right. \\
 &\quad + p^2(12\alpha-1+12\beta^*+6\beta^{*2})+2p(27-30\alpha+8\beta^*-12\alpha\beta^*-3\beta^{*2}-2\beta^{*3}) \\
 &\quad \left. \left. -12(1-\alpha)(6+4\beta^*+\beta^{*2})+\beta^{*4} \right] x^4 \right. \\
 &\quad + \left[-6n^2+2n(3p^2-6p(2+\alpha^*+\beta^*)+12\alpha-6+4\alpha^*+6\beta^*+6\alpha^*\beta^*+3\beta^{*2}) \right. \\
 &\quad + 2p^3(3+2\alpha^*)-6p^2(1+2\alpha+2\alpha^*+2\beta^*+2\alpha^*\beta^*) \\
 &\quad + 2p(48\alpha-42-8\alpha^*-6\beta^*+12\alpha\alpha^*+6\alpha^*\beta^*+12\alpha\beta^*+3\beta^{*2}+6\alpha^*\beta^{*2}) \\
 &\quad \left. \left. +12(1-\alpha)(12+4\alpha^*+6\beta^*+2\alpha^*\beta^*+\beta^{*2})-4\alpha^*\beta^{*3} \right] x^3 \right. \\
 &\quad + \left[3n^2+n(2p(5+6\alpha^*)+5-12\alpha-12\alpha^*-4\beta^*-12\alpha^*\beta^*-6\alpha^{*2}) \right. \\
 &\quad + p^2(7+12\alpha^*+6\alpha^{*2}) \\
 &\quad + p(29-36\alpha+12\alpha^*-4\beta^*-6\alpha^{*2}-24\alpha\alpha^*-12\alpha^*\beta^*-12\alpha^{*2}\beta^*) \\
 &\quad \left. \left. -2(1-\alpha)(43+36\alpha^*+12\beta^*+12\alpha^*\beta^*+6\alpha^{*2})+6\alpha^{*2}\beta^{*2} \right] x^2 \right. \\
 &\quad \left. + \left[(n+p)(1+4\alpha^*+6\alpha^{*2})+4p\alpha^{*3}+2(1-\alpha)(7+12\alpha^*+6\alpha^{*2})-4\alpha^{*3}\beta^* \right] x+\alpha^{*4} \right\}.
 \end{aligned}$$

LEMMA 2.5. For the operators $T_{n,\alpha,p}^{(\alpha^*,\beta^*)}$, the following expressions hold

$$\lim_{n \rightarrow \infty} nT_{n,\alpha,p}^{(\alpha^*,\beta^*)}(\varphi_x^1;x) = (p-\beta^*)x + \alpha^*, \tag{2.17}$$

$$\lim_{n \rightarrow \infty} nT_{n,\alpha,p}^{(\alpha^*,\beta^*)}(\varphi_x^2;x) = x(1-x), \tag{2.18}$$

$$\lim_{n \rightarrow \infty} nT_{n,\alpha,p}^{(\alpha^*,\beta^*)}(\varphi_x^4;x) = 0. \tag{2.19}$$

3. Main results

Applying the classical Korovkin Theorem to the sequence of linear positive operators $T_{n,\alpha,p}^{(\alpha^*,\beta^*)}$, from (2.8)–(2.10) we have the convergence theorem as follows.

THEOREM 3.1. For any $f \in C[0, 1+p]$ and $\alpha \in [0, 1]$, the sequence

$$\left\{ T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(f;x) \right\}$$

converges to f uniformly on $[0, 1]$.

In the next result, we will give the rate of convergence of the operator $T_{n,\alpha,p}^{(\alpha^*,\beta^*)}$ by means of the first modulus of continuity.

THEOREM 3.2. Let $f \in C[0, 1 + p]$ and $\alpha \in [0, 1]$. Then, for all $x \in [0, 1]$ we have

$$\left| T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(f;x) - f(x) \right| \leq 2\omega \left(f; \delta_{n,\alpha,p}^{(\alpha^*,\beta^*)}(x) \right)$$

where ω is the modulus of continuity and $\delta_{n,\alpha,p}^{(\alpha^*,\beta^*)}(x)$ is given as in (2.16).

Proof. Taking into account that the modulus of continuity of f has the following well known property

$$|f(t) - f(x)| \leq \omega(f; \delta) \left(1 + \frac{(t-x)^2}{\delta^2} \right)$$

for $x, t \in [0, 1]$ and $\delta > 0$, we can write

$$\begin{aligned} \left| T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(f;x) - f(x) \right| &\leq T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(|f(t) - f(x)|;x) \\ &\leq \omega(f; \delta) \left(1 + \frac{\left(\delta_{n,\alpha,p}^{(\alpha^*,\beta^*)}(x) \right)^2}{\delta^2} \right). \end{aligned}$$

If we choose $\delta = \delta_{n,\alpha,p}^{(\alpha^*,\beta^*)}(x)$, we arrive at desired result. \square

Now, with the help of Maple let us give some numerical examples to show the approximation process by these operators.

EXAMPLE 3.1. The convergence of $T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(f;x)$ to $f(x) = \cos(x^3)$ is shown in Figure 1 for $p = 1$, $\alpha^* = 1$, $\beta^* = 2$, $n = 2$ and different values of the parameter α .

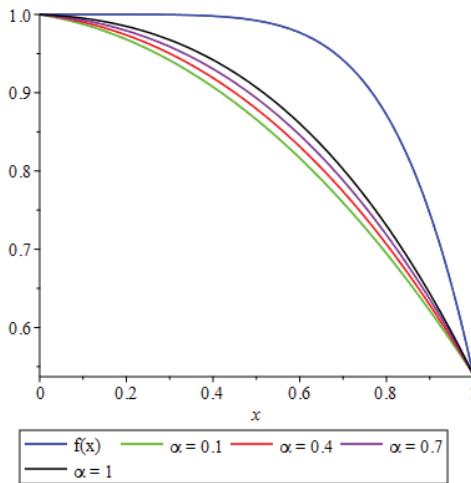


Figure 1: Approximation process of $T_{n,\alpha,p}^{(\alpha^*,\beta^*)}$, for $\alpha = 0.1, 0.4, 0.7, 1$.

EXAMPLE 3.2. The convergence of $T_{n,\alpha,p}(f;x)$ to $f(x) = xe^x$ is illustrated in Figure 2 for $p = 2$, $\alpha = 0.5$, $\alpha^* = 2$, $\beta^* = 10$ and different values of n .

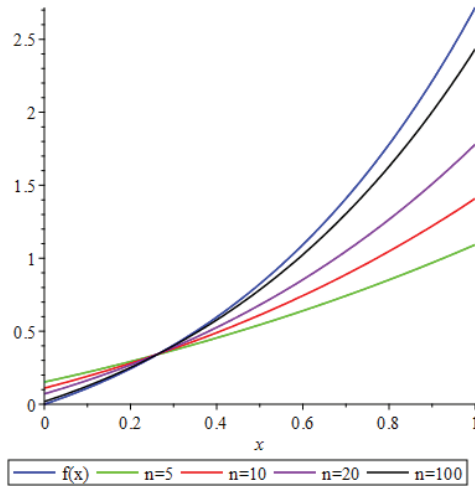


Figure 2: Approximation process of $T_{n,\alpha,p}^{(\alpha^*,\beta^*)}$, for $n = 5, 10, 20, 100$.

In what follows, we give a Voronovskaya-type result for the constructed operator $T_{n,\alpha,p}^{(\alpha^*,\beta^*)}$.

THEOREM 3.3. Suppose that $f \in C[0, 1 + p]$ and f has the second order derivative at $x \in [0, 1]$. Then we have

$$\lim_{n \rightarrow \infty} n \left[T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(f;x) - f(x) \right] = [(p - \beta^*)x + \alpha^*]f'(x) + \frac{x(1-x)}{2}f''(x),$$

where $0 \leq \alpha \leq 1$.

Proof. From Taylor’s formula, one has

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2}f''(x)(t-x)^2 + h(t-x)(t-x)^2, \tag{3.1}$$

at the fixed point $x \in [0, 1]$, where $h(t-x)$ is a continuous function on $[0, 1 + p]$ and $\lim_{t \rightarrow x} h(t-x) = 0$. Application of the operators $T_{n,\alpha,p}^{(\alpha^*,\beta^*)}$ to (3.1) implies

$$\begin{aligned} n \left[T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(f;x) - f(x) \right] &= f'(x) nT_{n,\alpha,p}^{(\alpha^*,\beta^*)}(t-x;x) + \frac{f''(x)}{2} nT_{n,\alpha,p}^{(\alpha^*,\beta^*)}((t-x)^2;x) \\ &\quad + nT_{n,\alpha,p}^{(\alpha^*,\beta^*)}(h(t-x)(t-x)^2;x). \end{aligned}$$

Using (2.17) and (2.18), we can write

$$\begin{aligned} \lim_{n \rightarrow \infty} n \left[T_{n, \alpha, p}^{(\alpha^*, \beta^*)}(f; x) - f(x) \right] &= [(p - \beta^*)x + \alpha^*] f'(x) + \frac{x(1-x)}{2} f''(x) \\ &\quad + \lim_{n \rightarrow \infty} n T_{n, \alpha, p}^{(\alpha^*, \beta^*)} \left(h(t-x)(t-x)^2; x \right). \end{aligned}$$

Then, it suffices to prove that $\lim_{n \rightarrow \infty} n T_{n, \alpha, p}^{(\alpha^*, \beta^*)} \left(h(t-x)(t-x)^2; x \right) = 0$.

Since $\lim_{t \rightarrow x} h(t-x) = 0$, for each $\varepsilon > 0$ there is a $\delta > 0$ such that $|h(t-x)| < \varepsilon$ for all t satisfying $|t-x| < \delta$. On the other hand, since $h(t-x)$ is bounded on $[0, 1+p]$, there is an $M > 0$ such that $|h(t-x)| \leq M$ for all t . Therefore, we may write $|h(t-x)| \leq M \frac{(t-x)^2}{\delta^2}$ when $|t-x| \geq \delta$. So, these arguments enable us to write $|h(t-x)| \leq \varepsilon + M \frac{(t-x)^2}{\delta^2}$ for all t .

Thus, we have

$$\begin{aligned} &n T_{n, \alpha, p}^{(\alpha^*, \beta^*)} \left(h(t-x)(t-x)^2; x \right) \\ &\leq \varepsilon n T_{n, \alpha, p}^{(\alpha^*, \beta^*)} \left((t-x)^2; x \right) + \frac{M}{\delta^2} n T_{n, \alpha, p}^{(\alpha^*, \beta^*)} \left((t-x)^4; x \right) \\ &= \varepsilon n T_{n, \alpha, p}^{(\alpha^*, \beta^*)} \left(\varphi_x^2; x \right) + \frac{M}{\delta^2} n T_{n, \alpha, p}^{(\alpha^*, \beta^*)} \left(\varphi_x^4; x \right). \end{aligned}$$

Making use of (2.18) and (2.19), we arrive at the desired result. \square

Recently, Acu et al. [2] initially studied the Grüss-type inequality for linear positive operators by using the least concave majorant of the modulus of continuity. In [12], Gonska and Tachev proved Grüss-type inequalities in terms of the second order modulus of continuity and the second order Ditizian-Totik modulus of smoothness. Recently, Gal and Gonska [11] obtained a Voronovskaya-type theorem with the help of Grüss inequality for Bernstein operators in both the real and the complex case and termed it as Grüss-Voronovskaya-type theorem.

In the next theorem, by using the approach in [11] we will give Grüss-Voronovskaya type theorem for the α -Bernstein-Schurer operator $T_{n, \alpha, p}^{(\alpha^*, \beta^*)}$.

THEOREM 3.4. *Let $f, g \in C^2[0, 1+p]$ and $\alpha \in [0, 1]$. Then, for each $x \in [0, 1]$ we have*

$$\lim_{n \rightarrow \infty} n \left[T_{n, \alpha, p}^{(\alpha^*, \beta^*)}(fg; x) - T_{n, \alpha, p}^{(\alpha^*, \beta^*)}(f; x) T_{n, \alpha, p}^{(\alpha^*, \beta^*)}(g; x) \right] = x(1-x) f'(x) g'(x).$$

Proof. Since

$$(fg)(x) = f(x)g(x), \quad (fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

and

$$(fg)''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x),$$

we can easily write

$$\begin{aligned}
 & T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(fg;x) - T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(f;x)T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(g;x) \\
 &= \left[T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(fg;x) - f(x)g(x) - (fg)'(x)T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(\varphi_x^1;x) - \frac{(fg)''(x)}{2}T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(\varphi_x^2;x) \right] \\
 &\quad - g(x) \left[T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(f;x) - f(x) - f'(x)T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(\varphi_x^1;x) - \frac{f''(x)}{2}T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(\varphi_x^2;x) \right] \\
 &\quad - T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(f;x) \left[T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(g;x) - g(x) - g'(x)T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(\varphi_x^1;x) - \frac{g''(x)}{2}T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(\varphi_x^2;x) \right] \\
 &\quad + \frac{1}{2}T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(\varphi_x^2;x) \left[f(x)g''(x) + 2f'(x)g'(x) - g''(x)T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(f;x) \right] \\
 &\quad + g'(x)T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(\varphi_x^1;x) \left[f(x) - T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(f;x) \right].
 \end{aligned}$$

By using (2.17) and (2.18), we get

$$\begin{aligned}
 & \lim_{n \rightarrow \infty} n \left[T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(fg;x) - T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(f;x)T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(g;x) \right] \\
 &= \lim_{n \rightarrow \infty} n \left[T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(fg;x) - f(x)g(x) \right] - (fg)'(x) \left[(p-\beta^*)x + \alpha^* \right] - \frac{(fg)''(x)}{2}x(1-x) \\
 &\quad - g(x) \left\{ \lim_{n \rightarrow \infty} n \left[T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(f;x) - f(x) \right] - f'(x) \left[(p-\beta^*)x + \alpha^* \right] - \frac{f''(x)}{2}x(1-x) \right\} \\
 &\quad - \lim_{n \rightarrow \infty} T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(f;x) \left\{ \lim_{n \rightarrow \infty} n \left[T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(g;x) - g(x) \right] \right. \\
 &\quad \left. - g'(x) \left[(p-\beta^*)x + \alpha^* \right] - \frac{g''(x)}{2}x(1-x) \right\} \\
 &\quad + \frac{x(1-x)}{2} \left\{ g''(x) \lim_{n \rightarrow \infty} \left[f(x) - T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(f;x) \right] + 2f'(x)g'(x) \right\} \\
 &\quad + g'(x) \left[(p-\beta^*)x + \alpha^* \right] \lim_{n \rightarrow \infty} \left[f(x) - T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(f;x) \right].
 \end{aligned}$$

Considering Theorem 3.1 and Theorem 3.3, we obtain

$$\lim_{n \rightarrow \infty} n \left[T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(fg;x) - T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(f;x)T_{n,\alpha,p}^{(\alpha^*,\beta^*)}(g;x) \right] = x(1-x)f'(x)g'(x),$$

which completes the proof. \square

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REFERENCES

- [1] T. ACAR, A. M. ACU, N. MANAV, *Approximation of functions by genuine Bernstein-Durrmeyer type operators*, Journal of Mathematical Inequalities 2018; 12 (4): 975–987.
- [2] A. M. ACU, H. GONSKA, I. RAŞA, *Grüss-type and Ostrowski-type inequalities in approximation theory*, Ukrainian Mathematical Journal 2011; 63 (6): 843–864.
- [3] D. BARBOSU, *Schurer-Stancu type operators*, Studia Universitatis Babeş-Bolyai Mathematica, 2003; XLVIII (3): 31–35.
- [4] S. N. BERNSTEIN, *Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités*, Communications Kharkov Mathematical Society 1912/1913; Vol. 13: 1–2.
- [5] J. BUSTAMANTE, *Bernstein Operators and Their Properties*, Springer International Publishing, Birkhäuser, 2017.
- [6] Q. CAI, X. XU, *Shape-preserving properties of a new family of generalized Bernstein operators*, Journal of Inequalities and Applications 2018; 2018:241.
- [7] N. ÇETIN, *Approximation and geometric properties of complex α -Bernstein operator*, Results in Mathematics 2019; 74: 40, <https://doi.org/10.1007/s00025-018-0953-z>.
- [8] N. ÇETIN, *Approximation by α -Bernstein-Schurer operator*, under review.
- [9] N. ÇETIN, V.A. RADU, *Approximation by generalized Bernstein–Stancu operators*, Turkish Journal of Mathematics 2019; 43: 2032–2048.
- [10] X. CHEN, J. TAN, Z. LIU, J. XIE, *Approximation of functions by a new family of generalized Bernstein operators*, Journal of Mathematical Analysis and Applications 2017; 450: 244–261.
- [11] S. GAL, H. GONSKA, *Grüss and Grüss-Voronovskaya-type estimates for some Bernstein-type polynomials of real and complex variables*, Jaen Journal on Approximation, 2015; 7 (1): 97–122.
- [12] H. GONSKA, G. TACHEV, *Grüss-type inequalities for positive linear operators with second order moduli*, Matematički Vesnik, 2011; 63 (4): 247–252.
- [13] V. GUPTA, T. M. RASSIAS, P. N. AGRAWAL, A. M. ACU, *Estimates for the Differences of Positive Linear Operators*, In: Recent Advances in Constructive Approximation Theory. Springer Optimization and Its Applications, vol 138, (2018), Springer, Cham.
- [14] S. A. MOHIUDDINE, T. ACAR, A. ALOTAIBI, *Construction of a new family of Bernstein-Kantorovich operators*, Mathematical Methods in the Applied Sciences 2017; 40: 7749–7759.
- [15] F. SCHURER, *Linear positive operators in approximation theory*, Math. Inst. Techn. Univ. Delft Report, 1962.
- [16] M. SMUC, *On a Chlodovsky variant of Bernstein operator*, Bulletin of the Transilvania University of Braşov, Series III: Mathematics, Informatics, Physics 2017; 10 (1): 165–178.
- [17] D. D. STANCU, *Asupra unei generalizări a polinoamelor lui Bernstein*, Studia Universitatis Babeş-Bolyai, Ser. Math.-Phys., 1969; 14 (2): 31–45 (in Romanian).

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Nursel Çetin

Ankara Hacı Bayram Veli University
Polatlı Faculty of Science and Letters, Department of Mathematics
06900, Turkey

e-mail: nurselcetin07@gmail.com

Ana-Maria Acu

Lucian Blaga University of Sibiu
Department of Mathematics and Informatics
Str. Dr. I. Ratiu, No.5-7, RO-550012 Sibiu

e-mail: anamaria.acu@ulbsibiu.ro