Journal of Mathematical Inequalities
Volume 15, Number 3 (2021), 923–940 doi:10.7153/jmi-2021-15-64

# COEFFICIENT ESTIMATES AND FEKETE–SZEGÖ INEQUALITY FOR NEW SUBCLASS OF BI–BAZILEVIČ FUNCTIONS BY $(s,t)$–DERIVATIVE OPERATOR AND QUASI–SUBORDINATION

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(*Communicated by M. Krnić*)

*Abstract.* In this paper we introduce and investigate a new generalized class of bi-bazilevič functions defined by using $(s,t)$-derivative operator and quasi-subordination in the open unit disk $\mathbb{D}$. We obtain two kinds of coefficient estimate by using Faber polynomial expansion and get Fekete–Szegö inequality for the new class and some of its subclasses.

## 1. Introduction

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Also let $\mathscr{S}$ denote the subclass of functions in $\mathscr{A}$ that are univalent in $\mathbb{D}$.

For two analytic functions $f$ and $g$, the function $f$ is subordinate to $g$ in $\mathbb{D}$, written as follows

$$f(z) \prec g(z), \quad z \in \mathbb{D},$$

if there exists an Schwarz function $\omega$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \mathbb{D}$ such that

$$f(z) = g(\omega(z)).$$

Furthermore, if the function $g$ is univalent in $\mathbb{D}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

In 1970, Robertson [1] introduced the concept of quasi-subordination. For two analytic functions $f$ and $g$, the function $f$ is quasi-subordinate to $g$ in $\mathbb{D}$, written as follows

$$f(z) \prec_q g(z), \quad z \in \mathbb{D},$$

*Mathematics subject classification* (2020): 30C45.
*Keywords and phrases*: bi-univalent function, bi-bazilevič function, $(s,t)$-derivative, quasi-subordination, coefficient estimate, Fekete–Szegö problem, Faber polynomial expansion.

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if there exists an analytic functions $h$ with $|h(z)| \leqslant 1$ such that $\frac{f(z)}{h(z)}$ analytic in $\mathbb{D}$ and

$$\frac{f(z)}{h(z)} \prec g(z), \quad z \in \mathbb{D}$$

that is, there exists a Schwarz function $\omega$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \mathbb{D}$ such that

$$f(z) = h(z)g(\omega(z)).$$

Observe that when $h(z) = 1$, then $f(z) = g(\omega(z))$, so that $f(z) \prec g(z)$ in $\mathbb{D}$. Also notice that if $\omega(z) = z$, then $f(z) = h(z)g(z)$ and it is said that $f$ is majorized by $g$ and written $f(z) \ll g(z)$ in $\mathbb{D}$. Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization. See [2–8] for works related to quasi-subordination.

A function $f \in \mathscr{A}$ is said to be bi-univalent in $\mathbb{D}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{D}$. It is a well known fact that every function $f \in \mathscr{S}$ has an inverse functions $f^{-1}$, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{D})$$

and

$$f(f^{-1}(\omega)) = \omega \quad (|\omega| < r_0(f), \ r_0(f) \geqslant \frac{1}{4}).$$

In fact, according to the Kobe One-Quarter Theorem [9], the inverse function $f^{-1}$ is given by

$$\begin{aligned} g(\omega) = f^{-1}(\omega) &= \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots \\ &= \omega + \sum_{n=2}^{\infty} b_n\omega^n. \end{aligned} \tag{1.2}$$

Let $\Sigma$ denote the class of all bi-univalent functions in $\mathbb{D}$ given by the Taylor-Maclaurin series expansion by (1.1). Coefficient estimate problem of bi-univalent function were widely researched in the literature. In 1967, Lewin [10] first introduced the class $\Sigma$ and studied the estimate for the coefficient $|a_2|$ of functions in $\Sigma$, and obtained that $|a_2| \leqslant 1.51$. Subsequently Branan and Clunie [11] improved Lewin's result to $|a_2| \leqslant \sqrt{2}$ and later Netanyahu [12] proved that $|a_2| \leqslant 4/3$. Kedzierawski [13] proved the Brannan–Clunie conjecture for bi-starlike functions. In 1984, Tan [14] obtained that $|a_2| < 1.485$, which is the best known estimate for bi-univalent functions in $\Sigma$. Brannan and Taha [15] also investigated certain subclasses of bi-univalent functions and found the non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$. In recent years, many researchers have been devoted various subclasses of the bi-univalent functions and obtained the estimates on the initial coefficients $|a_2|$ and $|a_3|$. The interest on estimates for the initial coefficients $|a_2|, |a_3|$ of the bi-univalent functions keep on by some researchers(see, for example, Srivastava et al. [16], Frasin and Aouf [17], Hayami and Owa [18], Xu et al. [19], and others [6, 7, 20–24]). Quite recently, only few works also determined the Fekete–Szegö problem(i.e. estimate for the upper bound of $|a_3 - \mu a_2^2|$) for some subclasses of bi-univalent functions, for example [25–29]. In the meantime, the estimate on the general coefficients $|a_n|$ $(n \geqslant 4)$ of bi-univalent functions has attracted the attention of some researchers. By using the Faber polynomial coefficient expansions Jahangiri and Hamidi [30] obtained bounds for the coefficient $|a_n|$ of bi-univalent functions in certain subclass of $\Sigma$ with a given gap series condition. Since then, some of authors considered and studied the bound of general coefficient $|a_n|$ for bi-univalent functions in certain subclasses of $\Sigma$, for example [31–36]. The estimate on the general coefficients $|a_n|$ $(n \geqslant 4)$ of bi-univalent functions is still an open problem.

Although many subclasses of bi-univalent functions have already been introduced and studied some coefficient estimates, our focus is not only to further extend the bi-univalent functions class, but also to study the above coefficient estimate problems and Fekete–Szegö problem of the new classes of bi-univalent functions.

We begin by recalling the definition details of the following $(s,t)$-derivative operator (defined by Chakrabarti and Jagannathan [37], see also [38]), which will be used in this paper.

DEFINITION 1.1. Let the function $f \in \mathscr{A}$ given by (1.1) and $0 < t < s \leqslant 1$, the $(s,t)$-derivative of the function $f$ is defined as

$$(D_{s,t}f)(z) = \begin{cases} \frac{f(sz)-f(tz)}{(s-t)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

According to the above definition, we have

$$(D_{s,t}f)(z) = 1 + \sum_{n=2}^{\infty} [n]_{s,t} a_n z^{n-1}$$

where the symbol $[n]_{s,t}$ denotes the $(s,t)$-number or twin-basic number

$$[n]_{s,t} = \frac{s^n - t^n}{s-t}.$$

Note that by putting $s = 1$, the $(s,t)$-derivative reduces to the Jackson t-derivative given by (see [39])

$$(D_t f)(z) = \begin{cases} \frac{f(z)-f(tz)}{(1-t)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

And, for $f \in \mathscr{A}$ given by (1.1), we have

$$(D_t f)(z) = 1 + \sum_{n=2}^{\infty} [n]_t a_n z^{n-1}$$

where

$$[n]_t = \frac{1-t^n}{1-t}.$$

Also, by taking $t \to 1^-$, we have $[n]_t \to n$. So $(D_t f)(z)$ reduces to $f'(z)$ for $f \in \mathscr{A}$.

Now by using $(s,t)$-derivative operator and quasi-subordination we introduce a generalization class of analytic and bi-Bazilevič functions.

DEFINITION 1.2. Let $\lambda \geqslant 1$, $\alpha \geqslant 0$. A function $f(z) \in \sum$ given by (1.1) is said to be in the class $\mathscr{B}_{\Sigma}^{q}(\lambda,\alpha,s,t,\varphi)$ if the following conditions are satisfied

$$(1-\lambda)\frac{f(z)}{z}+\lambda(D_{s,t}f)(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}-1 \prec_q \varphi(z)-1, \ \ z\in\mathbb{D} \tag{1.3}$$

and

$$(1-\lambda)\frac{g(\omega)}{\omega}+\lambda(D_{s,t}g)(\omega)\left(\frac{g(\omega)}{\omega}\right)^{\alpha-1}-1 \prec_q \varphi(\omega)-1, \ \ \omega\in\mathbb{D} \tag{1.4}$$

where $g(\omega)=f^{-1}(\omega)$ is defined by (1.2).

REMARK 1.3. There are some suitable choices of $\lambda,\alpha,s,t$ which would provide the following subclasses of the class $\mathscr{B}_{\Sigma}^{q}(\lambda,\alpha,s,t,\varphi)$.

(1) By taking $s=1$ in Definition 1.2, the class $\mathscr{B}_{\Sigma}^{q}(\lambda,\alpha,s,t,\varphi)$ reduces to the class $\mathscr{B}_{\Sigma}^{q}(\lambda,\alpha,t,\varphi)$ which is satisfied

$$(1-\lambda)\frac{f(z)}{z}+\lambda(D_{t}f)(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}-1 \prec_q \varphi(z)-1, \ \ z\in\mathbb{D}$$

and

$$(1-\lambda)\frac{g(\omega)}{\omega}+\lambda(D_{t}g)(\omega)\left(\frac{g(\omega)}{\omega}\right)^{\alpha-1}-1 \prec_q \varphi(\omega)-1, \ \ \omega\in\mathbb{D}.$$

(2) By taking $\alpha=0$ in Definition 1.2, the class $\mathscr{B}_{\Sigma}^{q}(\lambda,\alpha,s,t,\varphi)$ reduces to the class $\mathscr{H}_{\Sigma}^{q}(\lambda,s,t,\varphi)$ which is satisfied

$$(1-\lambda)\frac{f(z)}{z}+\lambda\frac{z(D_{s,t}f)(z)}{f(z)}-1 \prec_q \varphi(z)-1, \ \ z\in\mathbb{D}$$

and

$$(1-\lambda)\frac{g(\omega)}{\omega}+\lambda\frac{\omega(D_{s,t}g)(\omega)}{g(\omega)}-1 \prec_q \varphi(\omega)-1, \ \ \omega\in\mathbb{D}.$$

Specially, for $s=1$ and $t\to 1^{-}$, the class $\mathscr{H}_{\Sigma}^{q}(\lambda,s,t,\varphi)$ reduces to the class $\mathscr{H}_{\Sigma}^{q}(\lambda,\varphi)$ which is satisfied

$$(1-\lambda)\frac{f(z)}{z}+\lambda\frac{zf'(z)}{f(z)}-1 \prec_q \varphi(z)-1, \ \ z\in\mathbb{D}$$

and

$$(1-\lambda)\frac{g(\omega)}{\omega}+\lambda\frac{\omega g'(\omega)}{g(\omega)}-1 \prec_q \varphi(\omega)-1, \ \ \omega\in\mathbb{D}.$$

(3) By taking $\alpha=1$ in Definition 1.2, the class $\mathscr{B}_{\Sigma}^{q}(\lambda,\alpha,s,t,\varphi)$ reduces to the class $\mathscr{R}_{\Sigma}^{q}(\lambda,s,t,\varphi)$ which is satisfied

$$(1-\lambda)\frac{f(z)}{z}+\lambda(D_{s,t}f)(z)-1 \prec_q \varphi(z)-1, \ \ z\in\mathbb{D}$$

and

$$(1-\lambda)\frac{g(\omega)}{\omega}+\lambda(D_{s,t}g)(\omega)-1\prec_q \varphi(\omega)-1,\ \ \omega\in\mathbb{D}.$$

Specially, for $s=1$ and $t\to 1^-$, the class $\mathscr{R}_{\Sigma}^{q}(\lambda,s,t,\varphi)$ reduces the class $\mathscr{R}_{\Sigma}^{q}(\lambda,\varphi)$ introduced by Amol B. Patil1 and Uday H. Naik [6].

(4) By taking $\lambda=1$ in Definition 1.2, the class $\mathscr{B}_{\Sigma}^{q}(\lambda,\alpha,s,t,\varphi)$ reduces to the class $\mathscr{J}_{\Sigma}^{q}(\alpha,s,t,\varphi)$ which is satisfied

$$(D_{s,t}f)(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}-1\prec_q \varphi(z)-1,\ \ z\in\mathbb{D}$$

and

$$(D_{s,t}g)(\omega)\left(\frac{g(\omega)}{\omega}\right)^{\alpha-1}-1\prec_q \varphi(\omega)-1,\ \ \omega\in\mathbb{D}.$$

Specially, for $s=1$ and $t\to 1^-$, the class $\mathscr{J}_{\Sigma}^{q}(\alpha,s,t,\varphi)$ reduces the class $\mathscr{J}_{\Sigma}^{q}(\alpha,\varphi)$ introduced by S. P. Goyal, Onkar Singh and Rohit Mukherjee [7].

Our object of this paper is to study two kinds of coefficient estimate problems and Fekete–Szegö problem for the class $\mathscr{B}_{\Sigma}^{q}(\lambda,\alpha,s,t,\varphi)$ and some of its subclasses. Our results are new in this direction and they give birth to many corollaries.

## 2. Preliminary results

In order to derive our main results, we need the following lemmas.

For $\beta\in\mathbb{Z}:=\{0,\pm1,\pm2,\cdots\}$, let $E_{n-1}^{\beta}=E_{n-1}^{\beta}(a_2,a_3,\cdots,a_n)$ be homogeneous polynomial explicated in (see, for details, [40] and [41])

$$E_{n-1}^{\beta}(a_2,a_3,\cdots,a_n)=\sum_{n=2}^{\infty}\frac{\beta!}{j_1\cdots j_{n-1}}a_2^{j_1}\cdots a_n^{j_{n-1}}\quad\text{for } \beta\leqslant n-1, \tag{2.1}$$

and the sum is taken over all nonnegative integers $j_1,\cdots,j_{n-1}$ satisfying

$$\begin{cases} j_1+j_2+\cdots+j_{n-1}=\beta,\\ j_1+2j_2+\cdots+(n-1)j_{n-1}=n-1.\end{cases}$$

It is clear that $E_{n-1}^{n-1}(a_2,a_3,\cdots,a_n)=a_2^{n-1}$.

LEMMA 2.1. [40, 41] *Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{A}$, then for any $\beta\in\mathbb{Z}$*

$$\left(\frac{f(z)}{z}\right)^{\beta}=1+\sum_{n=2}^{\infty}K_{n-1}^{\beta}(a_2,a_3,\cdots,a_n)z^{n-1}, \tag{2.2}$$

*where*

$$K_{n-1}^{\beta}(a_2,a_3,\cdots,a_n)=\beta a_n+\frac{\beta(\beta-1)}{2}E_{n-1}^{2}+\frac{\beta!}{(\beta-3)!3!}E_{n-1}^{3}+\cdots+\frac{\beta!}{(\beta-n+1)!(n-1)!}E_{n-1}^{n-1},$$

*with* $E^{\beta}_{n-1}$ *is given by* (2.1). *In particular, the first three terms of* $K^{\beta}_{n-1}$ *are*

$$K^1_1 = 2a_2,\ \ K^2_2 = 2a_3 + a_2^2,\ \ K^2_3 = 2a_4 + 2a_2a_3,\ \ K^2_4 = 2a_5 + 2a_2a_4 + a_3^2.$$

LEMMA 2.2. *Let* $f(z) \in \mathscr{B}^q_{\Sigma}(\lambda,\alpha,s,t,\varphi)$, *then we have the following expansions*

$$\begin{aligned}(1-\lambda)\frac{f(z)}{z} + \lambda(D_{s,t}f)(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} - 1 \\ = \sum_{n=2}^{\infty}\left((1-\lambda)a_n + \lambda\sum_{i=1}^{n}[n+1-i]_{s,t}a_{n+1-i}A_i\right)z^{n-1}\end{aligned} \tag{2.3}$$

*where* $a_1 = A_1 = 1,\ A_i = K^{\alpha-1}_{i-1}(a_2,a_3,\cdots,a_i)\ \ (i \geqslant 2)$. *In particular, the coefficients of* $z$ *and* $z^2$ *on the right-hand side of the equation are*

$$[1+\lambda([2]_{s,t}-\alpha)]a_2$$

*and*

$$[1+\lambda([3]_{s,t}-\alpha)]a_3 + \lambda(1-\alpha)\left([2]_{s,t}-\frac{\alpha}{2}\right)a_2^2.$$

*Proof.* For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, by applying Lemma 2.1 we have

$$\left(\frac{f(z)}{z}\right)^{\alpha-1} = 1 + \sum_{n=2}^{\infty} A_n z^{n-1}, \tag{2.4}$$

where $A_n = K^{\alpha-1}_{n-1}(a_2,a_3,\cdots,a_n)$.

Since $(D_{s,t}f)(z) = 1 + \sum_{n=2}^{\infty}[n]_{s,t}a_n z^{n-1}$, we have

$$\begin{aligned}(D_{s,t}f)(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} &= \left(1+\sum_{n=2}^{\infty}[n]_{s,t}a_n z^{n-1}\right)\left(1+\sum_{n=2}^{\infty}A_n z^{n-1}\right) \\ &= 1 + ([2]_{s,t}a_2 + A_2)z + ([3]_{s,t}a_3 + [2]_{s,t}a_2A_2 + A_3)z^2 + \cdots \\ &\quad + ([n]_{s,t}a_n + [n-1]_{s,t}a_{n-1}A_2 + \cdots + [2]_{s,t}a_2A_{n-1} + A_n)z^{n-1} + \cdots.\end{aligned}$$

Let $a_1 = 1, A_1 = 1$, we have

$$(D_{s,t}f)(z)\left(\frac{f(z)}{z}\right)^{\alpha-1} = 1 + \sum_{n=2}^{\infty}\left(\sum_{i=1}^{n}[n+1-i]_{s,t}a_{n+1-i}A_i\right)z^{n-1}. \tag{2.5}$$

From (2.4) and (2.5), we can obtain (2.3). This evidently completes the proof of Lemma 2.2. □

LEMMA 2.3. [40] *Let* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathscr{A}$*, then the inverse map* $g=f^{-1}$ *of* $f$ *is given in terms of the Faber polynomials of* $f$ *with*

$$g(\omega)=f^{-1}(\omega)=\omega+\sum_{n=2}^{\infty}\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,\cdots,a_n)\omega^n, \tag{2.6}$$

*where*

$$\begin{aligned}K_{n-1}^{-n}(a_2,a_3,\cdots,a_n)&=\frac{(-n)!}{(-2n+1)!(n-1)!}a_2^{n-1}+\frac{(-n)!}{(-2n+2)!(n-3)!}a_2^{n-3}a_3\\&+\frac{(-n)!}{(-2n+3)!(n-4)!}a_2^{n-4}a_4+\frac{(-n)!}{(-2n+4)!(n-5)!}a_2^{n-5}[a_5+(n-2)a_3^2]\\&+\frac{(-n)!}{(-2n+5)!(n-6)!}a_2^{n-5}[a_6+(-2n+5)a_3a_4]+\sum_{j\geqslant 7}a_2^{n-j}V_j,\end{aligned}$$

*such that* $V_j$ $(7\leqslant j\leqslant n)$ *is a homogeneous polynomial in the variables* $a_2,a_3,\cdots,a_n$ *(see* [42]*). In particular, the first three terms of* $K_{n-1}^{-n}$ *are*

$$K_1^{-2}=-2a_2,\ \ K_2^{-3}=3(2a^2-a_3),\ \ K_3^{-4}=-4(5a_2^3-5a_2a_3+a_4).$$

LEMMA 2.4. *Let* $f(z)\in\mathscr{B}_{\Sigma}^{q}(\lambda,\alpha,s,t,\varphi)$*, then we have the following expansions*

$$\begin{aligned}&(1-\lambda)\left(\frac{g(\omega)}{\omega}\right)+\lambda(D_{s,t}g)(\omega)\left(\frac{g(\omega)}{\omega}\right)^{\alpha-1}-1\\&=\sum_{n=2}^{\infty}\left((1-\lambda)b_n+\lambda\sum_{i=1}^{n}[n+1-i]_{s,t}b_{n+1-i}B_i\right)\omega^{n-1},\end{aligned} \tag{2.7}$$

*where* $g(\omega)=f^{-1}(\omega)$*,* $b_1=1$*,* $b_n=\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,\cdots,a_n)$ $(n\geqslant 2)$*,* $B_1=1$*,* $B_i=K_{i-1}^{1-\alpha}(b_2,b_3,\cdots,b_i)$ $(i\geqslant 2)$*. In particular, the coefficients of* $\omega$ *and* $\omega^2$ *on the right-hand side of the equation are*

$$[1+\lambda([2]_{s,t}-\alpha)]b_2$$

*and*

$$[1+\lambda([3]_{s,t}-\alpha)]b_3+\lambda(1-\alpha)\left([2]_{s,t}-\frac{\alpha}{2}\right)b_2^2.$$

*Proof.* Suppose that $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ and $g(\omega)=f^{-1}(\omega)=\omega+\sum_{n=2}^{\infty}b_n\omega^n$. By applying Lemma 2.3 we have

$$g(\omega)=\omega+\sum_{n=2}^{\infty}\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,\cdots,a_n)\omega^n=\omega+\sum_{n=2}^{\infty}b_n\omega^n.$$

Similar to the proof of Lemma 2.2, we can also prove the result (2.7). This evidently completes the proof of Lemma 2.4. □

LEMMA 2.5. *Let analytic functions* $u(z) = c_1z + c_2z^2 + \cdots$, $v(\omega) = d_1\omega + d_2\omega^2 + \cdots$ *and* $\varphi(z) = 1 + \xi_1 z + \xi_2 z^2 + \cdots \in \mathscr{A}$, *then we have the following expansions*

$$\varphi(u(z)) - 1 = \sum_{n=1}^{\infty}\sum_{k=1}^{n} \xi_k E_n^k(c_1, c_2, \cdots, c_n) z^n \tag{2.8}$$

*and*

$$\varphi(v(\omega)) - 1 = \sum_{n=1}^{\infty}\sum_{k=1}^{n} \xi_k E_n^k(d_1, d_2, \cdots, d_n) \omega^n. \tag{2.9}$$

*Proof.* For $\varphi(z) = 1 + \xi_1 z + \xi_2 z^2 + \cdots$, $u(z) = c_1z + c_2z^2 + \cdots$, we have

$$\begin{aligned}\varphi(u(z)) - 1 &= \xi_1 u(z) + \xi_2 (u(z))^2 + \xi_3 (u(z))^3 + \cdots \\ &= \xi_1 c_1 z + (\xi_1 c_2 + \xi_2 c_1^2) z^2 + (\xi_1 c_3 + 2\xi_2 c_1 c_2 + \xi_3 c_1^3) z^3 + \cdots \\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} \xi_k E_n^k(c_1, c_2, \cdots, c_n) z^n.\end{aligned}$$

Similarly, for $\varphi(z) = 1 + \xi_1 z + \xi_2 z^2 + \cdots, v(\omega) = d_1\omega + d_2\omega^2 + \cdots$, we can get (2.9). This evidently completes the proof of Lemma 2.5. □

LEMMA 2.6. [43] *If* $p \in \mathscr{P}$, *then* $|p_n| \leqslant 2$ *for each* $n$, *where* $\mathscr{P}$ *is the family of all function* $p$ *analytic in* $\mathbb{D}$ *for which* Re $p(z) > 0$, $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ *for* $z \in \mathbb{D}$.

## 3. Main results

In the sequel, it is assumed that $\varphi(z)$ is an analytic function with positive real part in $\mathbb{D}$, $\varphi(\mathbb{D})$ is symmetric with respect to the real axis and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. And function $\varphi(z)$ has the Taylor series expansion of the form

$$\varphi(z) = 1 + \xi_1 z + \xi_2 z^2 + \cdots, \quad \xi_1 > 0. \tag{3.1}$$

Suppose that $\psi(z)$ and $\phi(z)$ are analytic in the unit disk $\mathbb{D}$ with $|\psi(z)| < 1$, $|\phi(\omega)| < 1$, and suppose that

$$\psi(z) = h_0 + h_1 z + h_2 z^2 + \cdots, \ \ \phi(\omega) = l_0 + l_1\omega + l_2\omega^2 + \cdots. \tag{3.2}$$

### 3.1. Coefficient estimates problem

In this section, we obtain the coefficient estimates for the function class $\mathscr{B}_{\Sigma}^{q}(\lambda, \alpha, s, t, \varphi)$.

By using Faber polynomial expansions we prove our first main result which provides an estimates for the general coefficients $|a_n|$ of functions in $\mathscr{B}_{\Sigma}^{q}(\lambda, \alpha, s, t, \varphi)$ subject to a given gap series condition.

THEOREM 2.1. *Let the function* $f(z) \in \mathscr{B}_{\Sigma}^{q}(\lambda, \alpha, s, t, \varphi)$ *be given by* (1.1). *If* $a_m = 0$ $(2 \leqslant m \leqslant n-1)$, *then*

$$|a_n| \leqslant \frac{2}{|1+\lambda([n]_{s,t}-\alpha)|} \min\left\{ \sum_{i=0}^{n-1} |h_i| \left( \sum_{k=1}^{n-i-1} |E_{n-i-1}^{k}(c_1, c_2, \cdots, c_{n-i-1})| \right), \right.$$
$$\left. \sum_{i=0}^{n-1} |l_i| \left( \sum_{k=1}^{n-i-1} |E_{n-i-1}^{k}(d_1, d_2, \cdots, d_{n-i-1})| \right) \right\} \tag{3.3}$$

*Proof.* Since $f \in \mathscr{B}_{\Sigma}^{q}(\lambda, \alpha, s, t, \varphi)$, then there exist two Schwarz functions $u(z) = c_1 z + c_2 z^2 + \cdots$, $v(\omega) = d_1 \omega + d_2 \omega^2 + \cdots$ and analytic functions $\psi, \phi$ defined by (3.2) such that

$$(1-\lambda)\frac{f(z)}{z} + \lambda (D_{s,t} f)(z) \left(\frac{f(z)}{z}\right)^{\alpha-1} - 1 = \psi(z)[\varphi(u(z)) - 1]$$

and

$$(1-\lambda)\frac{g(\omega)}{\omega} + \lambda (D_{s,t} g)(\omega) \left(\frac{g(\omega)}{\omega}\right)^{\alpha-1} - 1 = \phi(\omega)\,[\varphi(v(\omega)) - 1].$$

By using Lemma 2.5 we have

$$\psi(z)[\varphi(u(z)) - 1] = \sum_{n=1}^{\infty} \left[ \sum_{i=0}^{n} h_i \left( \sum_{k=1}^{n-i} \xi_k E_{n-i}^{k}(c_1, c_2, \cdots, c_{n-i}) \right) \right] z^n \tag{3.4}$$

and

$$\phi(\omega)\,[\varphi(v(\omega)) - 1] = \sum_{n=1}^{\infty} \left[ \sum_{i=0}^{n} l_i \left( \sum_{k=1}^{n-i} \xi_k E_{n-i}^{k}(d_1, d_2, \cdots, d_{n-i}) \right) \right] \omega^n. \tag{3.5}$$

By using Lemma 2.2 and comparing the corresponding coefficients of (2.3) and (3.4), for any $n \geqslant 2$ we have

$$(1-\lambda)a_n + \lambda \sum_{i=1}^{n} [n+1-i]_{s,t} a_{n+1-i} K_{i-1}^{\alpha-1}(a_2, a_3, \cdots, a_i)$$
$$= \sum_{i=0}^{n-1} h_i \left( \sum_{k=1}^{n-i-1} \xi_k E_{n-i-1}^{k}(c_1, c_2, \cdots, c_{n-i-1}) \right) \tag{3.6}$$

and similarly, by using Lemma 2.4 and comparing the corresponding coefficients of (2.7) and (3.4) we have

$$(1-\lambda)b_n + \lambda \sum_{i=1}^{n} [n+1-i]_{s,t} b_{n+1-i} K_{i-1}^{\alpha-1}(b_2, b_3, \cdots, b_i)$$
$$= \sum_{i=0}^{n-1} l_i \left( \sum_{k=1}^{n-i-1} \xi_k E_{n-i-1}^{k}(d_1, d_2, \cdots, d_{n-i-1}) \right). \tag{3.7}$$

For $a_m = 0\ (2 \leqslant m \leqslant n-1)$, we get $b_m = 0\ (2 \leqslant m \leqslant n-1)$ and $b_n = -a_n$. Hence

$$[1+\lambda([n]_{s,t}-\alpha)]a_n = \sum_{i=0}^{n-1} h_i \left( \sum_{k=1}^{n-i-1} \xi_k E_{n-i-1}^{k}(c_1, c_2, \cdots, c_{n-i-1}) \right) \tag{3.8}$$

and

$$-[1+\lambda([n]_{s,t}-\alpha)]a_n = \sum_{i=0}^{n-1} l_i \left( \sum_{k=1}^{n-i-1} \xi_k E_{n-i-1}^{k}(d_1, d_2, \cdots, d_{n-i-1}) \right). \tag{3.9}$$

Finally, by taking the moduli in both sides of (3.8) and (3.9) and using Lemma 2.6, we get the desired estimate on $|a_n|$ as asserted in (3.3). This evidently completes the proof of Theorem 3.1. $\square$

By taking special values of parameters $\lambda, \alpha, s, t$ in Theorem 3.1, we easily obtain the following results.

COROLLARY 3.2. *Let the function* $f(z) \in \mathscr{B}_{\Sigma}^{q}(\lambda, \alpha, t, \varphi)$ *be given by* (1.1). *If* $a_m = 0\ (2 \leqslant m \leqslant n-1)$, *then*

$$|a_n| \leqslant \frac{2}{|1+\lambda([n]_t-\alpha)|} \min\left\{ \sum_{i=0}^{n-1} |h_i| \left( \sum_{k=1}^{n-i-1} |E_{n-i-1}^{k}(c_1, c_2, \cdots, c_{n-i-1})| \right), \right.$$
$$\left. \sum_{i=0}^{n-1} |l_i| \left( \sum_{k=1}^{n-i-1} |E_{n-i-1}^{k}(d_1, d_2, \cdots, d_{n-i-1})| \right) \right\}$$

COROLLARY 3.3. *Let the function* $f(z) \in \mathscr{H}_{\Sigma}^{q}(\lambda, s, t, \varphi)$ *be given by* (1.1). *If* $a_m = 0\ (2 \leqslant m \leqslant n-1)$, *then*

$$|a_n| \leqslant \frac{2}{|1+\lambda[n]_{s,t}|} \min\left\{ \sum_{i=0}^{n-1} |h_i| \left( \sum_{k=1}^{n-i-1} |E_{n-i-1}^{k}(c_1, c_2, \cdots, c_{n-i-1})| \right), \right.$$
$$\left. \sum_{i=0}^{n-1} |l_i| \left( \sum_{k=1}^{n-i-1} |E_{n-i-1}^{k}(d_1, d_2, \cdots, d_{n-i-1})| \right) \right\}$$

COROLLARY 3.4. *Let the function* $f(z) \in \mathscr{R}_{\Sigma}^{q}(\lambda, s, t, \varphi)$ *be given by* (1.1). *If* $a_m = 0\ (2 \leqslant m \leqslant n-1)$, *then*

$$|a_n| \leqslant \frac{2}{|1+\lambda([n]_{s,t}-1)|} \min\left\{ \sum_{i=0}^{n-1} |h_i| \left( \sum_{k=1}^{n-i-1} |E_{n-i-1}^{k}(c_1, c_2, \cdots, c_{n-i-1})| \right), \right.$$
$$\left. \sum_{i=0}^{n-1} |l_i| \left( \sum_{k=1}^{n-i-1} |E_{n-i-1}^{k}(d_1, d_2, \cdots, d_{n-i-1})| \right) \right\}$$

COROLLARY 3.5. *Let the function* $f(z) \in \mathscr{J}_{\Sigma}^{q}(\alpha,s,t,\varphi)$ *be given by* (1.1). *If* $a_m = 0 \ (2 \leqslant m \leqslant n-1)$*, then*

$$|a_n| \leqslant \frac{2}{|[n]_{s,t}+1-\alpha|} \min\left\{ \sum_{i=0}^{n-1} |h_i| \left( \sum_{k=1}^{n-i-1} |E_{n-i-1}^{k}(c_1,c_2,\cdots,c_{n-i-1})| \right), \right.$$

$$\left. \sum_{i=0}^{n-1} |l_i| \left( \sum_{k=1}^{n-i-1} |E_{n-i-1}^{k}(d_1,d_2,\cdots,d_{n-i-1})| \right) \right\}$$

REMARK 3.6. For $\lambda = 1,\ s = 1,\ \psi(z) = \phi(\omega) = 1$ and $t \to 1^-$ in Theorem 3.6, we obtain the bounds on $|a_n|$ which are the improved results ([33], Theorem 1).

Our next main result provide estimates for the initial coefficients $|a_2|$ and $|a_3|$ of functions in $\mathscr{B}_{\Sigma}^{q}(\lambda,\alpha,s,t,\varphi)$ with no gap series restrictions imposed.

THEOREM 3.7. *Let the function* $f(z) \in \mathscr{B}_{\Sigma}^{q}(\lambda,\alpha,s,t,\varphi)$ *be given by* (1.1)*, then*

$$|a_2| \leqslant \min\left\{ \frac{\sqrt{2(h_0^2+l_0^2)}}{|1+\lambda([2]_{s,t}-\alpha)|}, \sqrt{\frac{2(|h_0|+|l_0|)+(|h_1|+|l_1|)}{|1+\lambda([3]_{s,t}-\alpha)+\frac{\lambda(1-\alpha)}{2}(2[2]_{s,t}-\alpha)|}} \right\} \tag{3.10}$$

$$|a_3| \leqslant \min\left\{ \frac{2(h_0^2+l_0^2)}{[1+\lambda([2]_{s,t}-\alpha)]^2} + \frac{2(|h_0|+|l_0|)+(|h_1|+|l_1|)}{|1+\lambda([3]_{s,t}-\alpha)|}, \right.$$
$$\left. \frac{|4+\lambda[4([3]_{s,t}-\alpha)+(1-\alpha)(2[2]_{s,t}-\alpha)]|(2|h_0|+|h_1|)+|\lambda(1-\alpha)(2[2]_{s,t}-\alpha)|(2|l_0|+|l_1|)}{|[1+\lambda([3]_{s,t}-\alpha)](2+\lambda[2([3]_{s,t}-\alpha)+(1-\alpha)(2[2]_{s,t}-\alpha)])|} \right\}. \tag{3.11}$$

*Proof.* Putting $n=2$ and $n=3$ in (3.6) and (3.7) respectively, we obtain

$$[1+\lambda([2]_{s,t}-\alpha)]a_2 = h_0\xi_1 c_1 \tag{3.12}$$

$$[1+\lambda([3]_{s,t}-\alpha)]a_3 + \lambda(1-\alpha)\left([2]_{s,t}-\frac{\alpha}{2}\right)a_2^2 = (h_0c_2+h_1c_1)\xi_1 + h_0c_1^2\xi_2 \tag{3.13}$$

and

$$-([1+\lambda([2]_{s,t}-\alpha)]b_2 = l_0\xi_1 d_1$$

$$[1+\lambda([3]_{s,t}-\alpha)]b_3 + \lambda(1-\alpha)\left([2]_{s,t}-\frac{\alpha}{2}\right)b_2^2 = (l_0d_2+l_1d_1)\xi_1 + l_0d_1^2\xi_2.$$

According to Lemma 2.3, we have $b_2 = -a_2, b_3 = 2a_2^2 - a_3$. Hence

$$-([1+\lambda([2]_{s,t}-\alpha)]a_2 = l_0\xi_1 d_1 \tag{3.14}$$

$$[1+\lambda([3]_{s,t}-\alpha)](2a_2^2-a_3) + \lambda(1-\alpha)\left([2]_{s,t}-\frac{\alpha}{2}\right)a_2^2 = (l_0d_2+l_1d_1)\xi_1 + l_0d_1^2\xi_2. \tag{3.15}$$

From (3.12) and (3.14), we obtain

$$a_2^2 = \frac{\xi_1^2(h_0^2c_1^2+l_0^2d_1^2)}{2[1+\lambda([2]_{s,t}-\alpha)]^2}. \tag{3.16}$$

Also, from (3.13) and (3.15), we find

$$a_2^2 = \frac{\xi_1(h_0c_2 + h_1c_1 + l_0d_2 + l_1d_1) + \xi_2(h_0c_1^2 + l_0d_1^2)}{2[1+\lambda([3]_{s,t}-\alpha)] + \lambda(1-\alpha)(2[2]_{s,t}-\alpha)}. \tag{3.17}$$

For the coefficients of the Schwarz functions $u(z)$ and $v(\omega)$ we have $|c_n| \leqslant 1$ and $|d_n| \leqslant 1$ (see [9]). Taking the moduli in both sides of (3.16) and (3.17), and applying Lemma 2.6 we get

$$|a_2| \leqslant \frac{\sqrt{2(h_0^2+l_0^2)}}{|1+\lambda([2]_{s,t}-\alpha)|}$$

and

$$|a_2| \leqslant \sqrt{\frac{2(|h_0|+|l_0|)+(|h_1|+|l_1|)}{|1+\lambda([3]_{s,t}-\alpha)+\frac{\lambda(1-\alpha)}{2}(2[2]_{s,t}-\alpha)|}}$$

which gives us the desired estimate on $|a_2|$ as asserted in (3.10).

Next, in order to find the bound on $|a_3|$, by subtracting (3.15) from (3.13), we obtain

$$a_3 = a_2^2 + \frac{\xi_1(h_0c_2 + h_1c_1 - l_0d_2 - l_1d_1) + \xi_2(h_0c_1^2 - l_0d_1^2)}{2[1+\lambda([3]_{s,t}-\alpha)]}. \tag{3.18}$$

Thus, upon substituting the value of $a_2^2$ from (3.16) into (3.18), it follows that

$$a_3 = \frac{\xi_1^2(h_0^2c_1^2 + l_0^2d_1^2)}{2[1+\lambda([2]_{s,t}-\alpha)]^2} + \frac{\xi_1(h_0c_2 + h_1c_1 - l_0d_2 - l_1d_1) + \xi_2(h_0c_1^2 - l_0d_1^2)}{2[1+\lambda([3]_{s,t}-\alpha)]}$$

which yields

$$|a_3| \leqslant \frac{2(h_0^2+l_0^2)}{[1+\lambda([2]_{s,t}-\alpha)]^2} + \frac{2(|h_0|+|l_0|)+(|h_1|+|l_1|)}{|1+\lambda([3]_{s,t}-\alpha)|}. \tag{3.19}$$

On the other hand, upon substituting the value of $a_2^2$ from (3.17) into (3.18), we obtain

$$a_3 = \frac{\xi_1(h_0c_2 + h_1c_1 + l_0d_2 + l_1d_1) + \xi_2(h_0c_1^2 + l_0d_1^2)}{2[1+\lambda([3]_{s,t}-\alpha)] + \lambda(1-\alpha)(2[2]_{s,t}-\alpha)} + \frac{\xi_1(h_0c_2 + h_1c_1 - l_0d_2 - l_1d_1) + \xi_2(h_0c_1^2 - l_0d_1^2)}{2[1+\lambda([3]_{s,t}-\alpha)]}.$$

It follows that

$$|a_3| \leqslant \frac{|4+\lambda[4([3]_{s,t}-\alpha)+(1-\alpha)(2[2]_{s,t}-\alpha)]|(2|h_0|+|h_1|)+|\lambda(1-\alpha)(2[2]_{s,t}-\alpha)|(2|l_0|+|l_1|)}{|[1+\lambda([3]_{s,t}-\alpha)](2+\lambda[2([3]_{s,t}-\alpha)+(1-\alpha)(2[2]_{s,t}-\alpha)])|}. \tag{3.20}$$

Combining (3.19) and (3.20), we get the desired estimate on the coefficient $|a_3|$ as asserted in (3.11). This evidently completes the proof of Theorem 3.7. □

REMARK 3.8. (1) For $\alpha = 1$, $s = 1$ and $t \to 1^-$ in Theorem 3.7, we obtain the bounds on $|a_2|$ and $|a_3|$ which are the improved results ([6], Theorem 2.2). (2) For $\lambda = 1$, $s = 1$ and $t \to 1^-$ in Theorem 3.7, we obtain the bounds on $|a_2|$ and $|a_3|$ which are the improved results ([7], Theorem 2.1). (3) For $\lambda = 1$, $s = 1$, $\psi(z) = \phi(\omega) = 1$ and $t \to 1^-$ in Theorem 3.7, we obtain the bounds on $|a_2|$ and $|a_3|$ which are the improved results ([33], Theorem 2).

### 3.2. Fekete-Szegö problem

In this section, we obtain Fekete-Szeg$\ddot{o}$ inequality for the function class $\mathscr{B}^q_\Sigma(\lambda,\alpha, s,t,\varphi)$.

THEOREM 3.9. *Let the function* $f(z)\in\mathscr{B}^q_\Sigma(\lambda,\alpha,s,t,\varphi)$ *be given by* (1.1)*, then for any number* $\mu\in\mathbb{C}$ *and* $\lambda([3]_{s,t}-\alpha)>-1$

$$|a_3-\mu a_2^2|\leqslant\begin{cases}\frac{2\xi_1(|h_0|+|h_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)}{2[1+\lambda([3]_{s,t}-\alpha)]}, & 0\leqslant|M(\mu)|\leqslant\frac{1}{2[1+\lambda([3]_{s,t}-\alpha)]},\\ [2\xi_1(|l_0|+|l_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)]|M(\mu)|, & |M(\mu)|\geqslant\frac{1}{2[1+\lambda([3]_{s,t}-\alpha)]}.\end{cases}\tag{3.21}$$

*For any number* $\mu\in\mathbb{C}$ *and* $\lambda([3]_{s,t}-\alpha)<-1$

$$|a_3-\mu a_2^2|\leqslant\begin{cases}[2\xi_1(|l_0|+|l_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)]|M(\mu)|, & 0\leqslant|M(\mu)|\leqslant-\frac{1}{2[1+\lambda([3]_{s,t}-\alpha)]},\\ -\frac{2\xi_1(|h_0|+|h_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)}{2[1+\lambda([3]_{s,t}-\alpha)]}, & |M(\mu)|\geqslant-\frac{1}{2[1+\lambda([3]_{s,t}-\alpha)]},\end{cases}\tag{3.22}$$

*where*

$$M(\mu)=\frac{h_0l_0\xi_1^2(1-\mu)}{[2(1+\lambda([3]_{s,t}-\alpha))+\lambda(1-\alpha)(2[2]_{s,t}-\alpha)]h_0l_0\xi_1^2-[1+\lambda([2]_{s,t}-\alpha)]^2(h_0+l_0)(\xi_2-\xi_1)}.$$

*Proof.* Since $f(z)\in\mathscr{B}^q_\Sigma(\lambda,\alpha,s,t,\varphi)$, then there exist analytic functions $u,v:\mathbb{D}\to\mathbb{D}$, with $u(0)=0=v(0)$, $|u(z)|<1$, $|v(\omega)|<1$ and analytic functions $\psi,\phi$ defined by (3.2) such that

$$(1-\lambda)\frac{f(z)}{z}+\lambda(D_{s,t}f)(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}-1=\psi(z)[\varphi(u(z))-1]\tag{3.23}$$

and

$$(1-\lambda)\frac{g(\omega)}{\omega}+\lambda(D_{s,t}g)(\omega)\left(\frac{g(\omega)}{\omega}\right)^{\alpha-1}-1=\phi(\omega)[\varphi(v(\omega))-1].\tag{3.24}$$

Define the functions $p_1$ and $p_2$ in $\mathscr{P}$ given by

$$p_1(z)=\frac{1+u(z)}{1-u(z)}=1+p_1z+p_2z^2+\cdots$$

and

$$p_2(\omega) = \frac{1+v(\omega)}{1-v(\omega)} = 1 + q_1\omega + q_2\omega^2 + \cdots.$$

It follows

$$u(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}p_1 z + \frac{1}{2}\left(p_2 - \frac{p_1^2}{2}\right)z^2 + \cdots \tag{3.25}$$

and

$$v(\omega) = \frac{p_2(\omega)-1}{p_2(\omega)+1} = \frac{1}{2}q_1\omega + \frac{1}{2}\left(q_2 - \frac{q_1^2}{2}\right)\omega^2 + \cdots. \tag{3.26}$$

Using (3.1), (3.2), (3.25) and (3.26), it is evident that

$$\psi(z)[\varphi(u(z))-1] = \frac{1}{2}h_0\xi_1 p_1 z + \left[\frac{1}{2}h_1\xi_1 p_1 + \frac{1}{2}h_0\xi_1 p_2 + \frac{1}{4}h_0(\xi_2-\xi_1)p_1^2\right]z^2 + \cdots \tag{3.27}$$

and

$$\phi(\omega)[\varphi(v(\omega))-1] = \frac{1}{2}l_0\xi_1 q_1\omega + \left[\frac{1}{2}l_1\xi_1 q_1 + \frac{1}{2}l_0\xi_1 q_2 + \frac{1}{4}l_0(\xi_2-\xi_1)q_1^2\right]\omega^2 + \cdots. \tag{3.28}$$

Using (2.3) and (3.27) in (3.23) and comparing the coefficient of $z$ and $z^2$, we get

$$[1+\lambda([2]_{s,t}-\alpha)]a_2 = \frac{1}{2}h_0\xi_1 p_1, \tag{3.29}$$

$$\begin{aligned}&[1+\lambda([3]_{s,t}-\alpha)]a_3 + \lambda(1-\alpha)\left([2]_{s,t}-\frac{\alpha}{2}\right)a_2^2\\ &= \frac{1}{2}h_1\xi_1 p_1 + \frac{1}{2}h_0\xi_1 p_2 + \frac{1}{4}h_0(\xi_2-\xi_1)p_1^2.\end{aligned} \tag{3.30}$$

Similarly using (2.7) and (3.28) in (3.24) and comparing the coefficient of $\omega$ and $\omega^2$, we get

$$-[1+\lambda([2]_{s,t}-\alpha)]a_2 = \frac{1}{2}l_0\xi_1 q_1, \tag{3.31}$$

$$\begin{aligned}&[1+\lambda([3]_{s,t}-\alpha)](2a_2^2-a_3) + \lambda(1-\alpha)\left([2]_{s,t}-\frac{\alpha}{2}\right)a_2^2\\ &= \frac{1}{2}l_1\xi_1 q_1 + \frac{1}{2}l_0\xi_1 q_2 + \frac{1}{4}l_0(\xi_2-\xi_1)q_1^2.\end{aligned} \tag{3.32}$$

Subtracting (3.32) from (3.30), we get

$$a_3 = a_2^2 + \frac{\frac{1}{2}(h_1p_1 - l_1q_1)\xi_1 + \frac{1}{2}(h_0p_2 - l_0q_2)\xi_1 + \frac{1}{4}(\xi_2-\xi_1)(h_0p_1^2 - l_0q_1^2)}{2[1+\lambda([3]_{s,t}-\alpha)]}. \tag{3.33}$$

By adding (3.30) and (3.32), we have

$$a_2^2 = \frac{\frac{1}{2}(h_1p_1 + l_1q_1)\xi_1 + \frac{1}{2}(h_0p_2 + l_0q_2)\xi_1 + \frac{1}{4}(\xi_2-\xi_1)(h_0p_1^2 + l_0q_1^2)}{2[1+\lambda([3]_{s,t}-\alpha) + \lambda(1-\alpha)([2]_{s,t}-\frac{\alpha}{2})]}. \tag{3.34}$$

Using (3.29) and (3.31), we obtain

$$h_0p_1^2+l_0q_1^2=\frac{4[1+\lambda([2]_{s,t}-\alpha)]^2(h_0+l_0)}{h_0l_0\xi^2}a_2^2. \tag{3.35}$$

From (3.32)–(3.34), we get

$$\begin{aligned} a_3-\mu a_2^2=\frac{\xi_1}{2}\Bigg[\left(M(\mu)+\frac{1}{2[1+\lambda([3]_{s,t}-\alpha)]}\right)(h_1p_1+h_0p_2)\\ +\left(M(\mu)-\frac{1}{2[1+\lambda([3]_{s,t}-\alpha)]}\right)(l_1q_1+l_0q_2)\Bigg]\\ +\frac{(\xi_2-\xi_1)(h_0p_1^2-l_0q_1^2)}{8[1+\lambda([3]_{s,t}-\alpha)]} \end{aligned} \tag{3.36}$$

where

$$M(\mu)=\frac{h_0l_0\xi_1^2(1-\mu)}{[2(1+\lambda([3]_{s,t}-\alpha))+\lambda(1-\alpha)(2[2]_{s,t}-\alpha)]h_0l_0\xi_1^2-[1+\lambda([2]_{s,t}-\alpha)]^2(h_0+l_0)(\xi_2-\xi_1)}.$$

By taking the moduli on both sides of (3.36) and applying Lemma 2.6, we finally obtain (3.21) and (3.22). This evidently completes the proof of Theorem 3.9. □

Setting $\alpha=1$, $s=1$ and $t\to1^-$ in Theorem 3.9, we get the following result.

COROLLARY 3.10. *Let the function $f(z)\in\mathscr{R}_\Sigma^q(\lambda,\varphi)$ be given by (1.1), then for any number $\mu\in\mathbb{C}$*

$$|a_3-\mu a_2^2|\leqslant\begin{cases}\frac{2\xi_1(|h_0|+|h_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)}{2(1+2\lambda)}, & 0\leqslant|M(\mu)|\leqslant\frac{1}{2(1+2\lambda)},\\ [2\xi_1(|l_0|+|l_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)]|M(\mu)|, & |M(\mu)|\geqslant\frac{1}{2(1+2\lambda)}.\end{cases}$$

*where*

$$M(\mu)=\frac{h_0l_0\xi_1^2(1-\mu)}{2(1+2\lambda)h_0l_0\xi_1^2-(1+\lambda)^2(h_0+l_0)(\xi_2-\xi_1)}.$$

Setting $\lambda=1$, $s=1$ and $t\to1^-$ in Theorem 3.8, we get the following result.

COROLLARY 3.11. *Let the function $f(z)\in\mathscr{J}_\Sigma^q(\alpha,\varphi)$ be given by (1.1), then for any number $\mu\in\mathbb{C}$ and $0\leqslant\alpha<4$*

$$|a_3-\mu a_2^2|\leqslant\begin{cases}\frac{2\xi_1(|h_0|+|h_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)}{2(4-\alpha)}, & 0\leqslant|M(\mu)|\leqslant\frac{1}{2(4-\alpha)},\\ [2\xi_1(|l_0|+|l_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)]|M(\mu)|, & |M(\mu)|\geqslant\frac{1}{2(4-\alpha)}.\end{cases}$$

*For any number $\mu\in\mathbb{C}$ and $\alpha>4$*

$$|a_3-\mu a_2^2|\leqslant\begin{cases}[2\xi_1(|l_0|+|l_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)]|M(\mu)|, & 0\leqslant|M(\mu)|\leqslant\frac{1}{2(\alpha-4)},\\ \frac{2\xi_1(|h_0|+|h_1|)+|\xi_2-\xi_1|(|h_0|+|l_0|)}{2(\alpha-4)}, & |M(\mu)|\geqslant\frac{1}{2(\alpha-4)},\end{cases}$$

*where*

$$M(\mu)=\frac{h_0l_0\xi_1^2(1-\mu)}{(4-\alpha)(3-\alpha)h_0l_0\xi_1^2-(3-\alpha)^2(h_0+l_0)(\xi_2-\xi_1)}.$$

*Acknowledgements.* The present investigation is supported by the Natural Science Foundation of China (Grant No. 11561001), the Natural Science Foundation of Inner Mongolia of China (Grant No. 2020MS01010), and the Higher School Foundation of Inner Mongolia of China (Grant No. NJZY19211).

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(Received March 24, 2020)

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