NONLINEAR INEQUALITIES AND RELATED FIXED POINT PROBLEMS

MUHAMMAD NAZAM, ESKANDAR AMEER,
MOHAMMAD MURSALEEN* AND ÖZLEM ACAR

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Abstract. In this paper, we introduce a nonlinear inequality based on four self-mappings. We give necessary conditions which ensure the existence of a common fixed point of four self-mappings satisfying said inequality defined in Ś-metric spaces. A common fixed point problem is discussed. We set up an example to elucidate our main result. Moreover, the existence of a common solution to a system of four integral equations is shown by application of main result.

1. Introduction

The study of metric fixed point theory was initiated by Banach (1922) [4]. The metric fixed point theory has been enriching by introducing several structures, generalizing underlying metric space and contractive condition in it. The Banach contraction principle [4] has been proved in several different abstract metric spaces (see [6, 7, 8, 10, 11, 13, 18]). Before the publication of the paper by Mustafa *et al.* [13] (2006), the metric mappings were defined on X^2 (X is any non-empty set), Mustafa *et al.* [13] introduced a metric mapping defined on X^3 and it is known as G-metric. Mustafa *et al.* [13] proved Banach contraction principle in G-metric space and gave several examples to establish its superiority over Banach contraction principle in metric space. Sedghi *et al.* [17] (2012), following the abstract metric introduced by Mustafa *et al.* [13], introduced another abstract metric defined on X^3 called S-metric and proved analogue of Banach contraction principle (see also [3, 12, 16]).

The concept of F-contraction given by Wardowski [21] proved to be another milestone in fixed point theory and numerous research papers on F-contraction have been published (see [2, 9, 1, 14, 20, 22] and references therein).

In this paper, we follow Mustafa *et al.* [13] and Sedghi *et al.* [17] to introduce another nonlinear inequality known as (F, Ω)-contraction and discuss some new common fixed point problems in Ś-complete metric spaces. Examples are also given for explanation. An application to existence of the solution to a system of integral equations is presented.

2. Preliminaries

Let Ψ represents the collection of mappings $\Omega : [0,\infty) \to [0,\infty)$ satisfying:

($\Omega 1$) Ω is strictly increasing;

($\Omega 2$) $\Omega^{-1}(t) \leqslant t \leqslant \Omega(t)$ and $\Omega^{-1}(0) = 0 = \Omega(0)$;

($\Omega 3$) Ω is continuous.

Following are some examples of members of this family.

EXAMPLE 1. Let $\Omega_i : [0,\infty) \to [0,\infty)$, $i \in \{1,2,3\}$ defined by
$\Omega_1(t) = e^t - 1$, $t \geqslant 0$,
$\Omega_2(t) = te^t$, $t \geqslant 0$,
$\Omega_3(t) = t^2 + 2t$, $t \geqslant 0$.

DEFINITION 1. [17] Let $\mathscr{A}$ be a nonempty set. A mapping $\widetilde{d} : \mathscr{A} \times \mathscr{A} \to [0,\infty)$ is said to be a p-metric if if there exists $\Omega \in \Psi$ such that for all $\zeta, \eta, z \in \mathscr{A}$, the following axioms hold:

($\widetilde{d}_1$) $\widetilde{d}(\zeta,\eta) = 0$ if and only if $\zeta = \eta$;

($\widetilde{d}_2$) $\widetilde{d}(\zeta,\zeta) \leqslant \widetilde{d}(\zeta,\eta)$;

($\widetilde{d}_3$) $\widetilde{d}(\zeta,\eta) = \widetilde{d}(\eta,\zeta)$;

($\widetilde{d}_4$) $\widetilde{d}(\zeta,z) \leqslant \Omega\left[\widetilde{d}(\zeta,\eta) + \widetilde{d}(\eta,z)\right]$.

The pair $(\mathscr{A}, \widetilde{d})$ is called a p-metric space.

A b-metric [6] is a p-metric with $\Omega(t) = bt$ for some $b \geqslant 1$. If $\Omega \in \Psi$, then every metric is a p-metric while a p-metric space becomes a metric space with $\Omega(t) = t$.

DEFINITION 2. [17] Let $\mathscr{A}$ be a nonempty set. A mapping $S : \mathscr{A} \times \mathscr{A} \times \mathscr{A} \to [0,\infty)$ satisfying following conditions, for all $\zeta, \eta, \upsilon, \sigma \in \mathscr{A}$,

($\acute{S}1$) $S(\zeta,\eta,\upsilon) = 0$ if and only if $\zeta = \eta = \upsilon$;

($\acute{S}2$) $S(\zeta,\eta,\upsilon) \leqslant S(\zeta,\zeta,\sigma) + S(\eta,\eta,\sigma) + S(\upsilon,\upsilon,\sigma)$.

is called an S-metric on $\mathscr{A}$ and the pair $(\mathscr{A}, S)$ is called an S-metric space.

The S-metric space is a generalization of a G-metric space [13], that is, every G-metric space is an S-metric space but, in general, the converse is not true. Following example explains this fact.

EXAMPLE 2. Let $\mathscr{A} = \mathbb{R}^n$ and $\|.\|$ be a norm on $\mathscr{A}$, then $S(\zeta, \eta, \upsilon) = \|\eta + \upsilon - 2\zeta\| + \|\eta - \upsilon\|$ is an S-metric on $\mathscr{A}$. However, due to lack of symmetry, it is not G-metric on $\mathscr{A}$.

Every S-metric space is an S_b-metric space with $b = 1$ but, in general, the converse is not true, see the following example.

EXAMPLE 3. *Let $\mathscr{A} = \mathbb{R}$, then $S_*(\zeta, \eta, \upsilon) = [|\eta + \upsilon - 2\zeta| + |\eta - \upsilon|]^2$ is an S_b-metric on $\mathscr{A}$ with $b = 4$, for all $\zeta, \eta, \upsilon \in \mathscr{A}$. Nevertheless, it is not an S-metric on $\mathscr{A}$. Indeed, let $\zeta = 3, \eta = 5, \upsilon = 7$, and $a = \frac{7}{2}$. Hence,*

$$S_*(3,5,7) = [|5+7-6| + |5-7|]^2 = 8^2 = 64,$$

$$S_*\left(3,3,\frac{7}{2}\right) = \left[\left|3 + \frac{7}{2} - 6\right| + \left|3 - \frac{7}{2}\right|\right]^2 = 1^2 = 1,$$

$$S_*\left(5,5,\frac{7}{2}\right) = \left[\left|5 + \frac{7}{2} - 10\right| + \left|5 - \frac{7}{2}\right|\right]^2 = 3^2 = 9,$$

$$S_*\left(7,7,\frac{7}{2}\right) = \left[\left|7 + \frac{7}{2} - 14\right| + \left|7 - \frac{7}{2}\right|\right]^2 = 7^2 = 49.$$

Thus,

$$S_*(3,5,7) = 64$$
$$> 59 = S_*\left(3,3,\frac{7}{2}\right) + S_*\left(5,5,\frac{7}{2}\right) + S_*\left(7,7,\frac{7}{2}\right).$$

DEFINITION 3. Let $\mathscr{A}$ be a nonempty set. A mapping $\acute{S} : \mathscr{A} \times \mathscr{A} \times \mathscr{A} \to [0,\infty)$ is said to be an $\acute{S}$-metric if there exists $\Omega \in \Psi$ such that for all $\zeta, \eta, \upsilon, \sigma \in \mathscr{A}$, we have

($\acute{S}$1) $\acute{S}(\zeta, \eta, \upsilon) = 0$ if and only if $\zeta = \eta = \upsilon$;

($\acute{S}$2) $\acute{S}(\zeta, \eta, \upsilon) \leqslant \Omega\left[\acute{S}(\zeta, \zeta, \sigma) + \acute{S}(\eta, \eta, \sigma) + \acute{S}(\upsilon, \upsilon, \sigma)\right].$

The pair $(\mathscr{A}, \acute{S})$ is called an $\acute{S}$-metric space.

Every metric space is an $\acute{S}$-metric space but in general the converse is not true (see Example 4).

EXAMPLE 4. Let $(\mathscr{A}, S)$ be an S-metric space, then $\acute{S}(\zeta, \eta, \upsilon) = e^{S(\zeta, \eta, \upsilon)} - 1$ is an $\acute{S}$-metric with $\Omega(t) = e^t - 1$ but it is not metric on $\mathscr{A}$ since, it is not symmetric.

The $\acute{S}$-metric space is a generalization of S-metric space and S_b-metric space [19]. The S-metric space is an $\acute{S}$-metric space for each $\Omega \in \Psi$. Every $\acute{S}$-metric space is an S_b-metric space [19] with $b \geqslant 1$ and $\Omega(t) = bt$.

DEFINITION 4. The $\acute{S}$-metric is called symmetric if $\acute{S}(\zeta, \zeta, \eta) = \acute{S}(\eta, \eta, \zeta)$, for all $\zeta, \eta \in \mathscr{A}$.

EXAMPLE 5. *Let* $(\mathscr{A}, S)$ *be an S-metric space. Then*

(i) $\acute{S}(\zeta, \eta, \upsilon) = e^{S(\zeta, \eta, \upsilon)} - 1$ *is an* $\acute{S}$*-metric with* $\Omega(t) = e^t - 1$.

(ii) $\acute{S}(\zeta, \eta, \upsilon) = S(\zeta, \eta, \upsilon) e^{S(\zeta, \eta, \upsilon)}$ *is an* $\acute{S}$*-metric with* $\Omega(t) = te^t$.

(iii) $\acute{S}(\zeta, \eta, \upsilon) = \Omega(S(\zeta, \eta, \upsilon))$ *is an* $\acute{S}$*-metric and it is symmetric for each* $\Omega \in \Psi$.

DEFINITION 5. Let $(\mathscr{A}, \acute{S})$ be an $\acute{S}$-metric space. A sequence $\{\zeta_n\} \subseteq \mathscr{A}$ is said to be:

(i) $\acute{S}$-Cauchy if for every $\varepsilon > 0$, there exists a positive integer n_0 such that for all $m, n \geqslant n_0$, $\acute{S}(\zeta_n, \zeta_n, \zeta_m) < \varepsilon$,

(ii) $\acute{S}$-convergent to a point $\zeta \in \mathscr{A}$ if for every $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n \geqslant n_0$, $\acute{S}(\zeta, \zeta, \zeta_n) < \varepsilon$. Similarly, for every $\varepsilon > 0$, there exists a positive integer n_0 such that for all $n \geqslant n_0$, $\acute{S}(\zeta_n, \zeta_n, \zeta) < \varepsilon$ and $\acute{S}(\zeta_n, \zeta, \zeta_n) < \varepsilon$.

(iii) An $\acute{S}$-metric space $\mathscr{A}$ is called $\acute{S}$-complete, if each $\acute{S}$-Cauchy is $\acute{S}$-convergent in $\mathscr{A}$.

LEMMA 1. Let $(\mathscr{A}, \acute{S})$ be an $\acute{S}$-metric space.
(i) Assume that $\{\zeta_n\}$ and $\{\eta_n\}$ are $\acute{S}$-convergent to ζ and η, respectively. Then we have

$$\Omega^{-2}\left(\acute{S}(\zeta, \zeta, \eta)\right) \leqslant \liminf_{n \to \infty} \acute{S}(\zeta_n, \zeta_n, \eta_n)$$
$$\leqslant \limsup_{n \to \infty} \acute{S}(\zeta_n, \zeta_n, \eta_n) \leqslant \Omega^2\left(\acute{S}(\zeta, \zeta, \eta)\right).$$

In particular, if $\zeta = \eta$, then we have $\lim_{n \to \infty} \acute{S}(\zeta_n, \zeta_n, \eta_n) = 0$.
(ii) Assume that $\acute{S}$-metric is symmetric and $\{\zeta_n\}$ is $\acute{S}$-convergent to ζ and $\upsilon \in \mathscr{A}$ is arbitrary. Then we have

$$\Omega^{-1}\left(\acute{S}(\zeta, \zeta, \upsilon)\right) \leqslant \liminf_{n \to \infty} \acute{S}(\zeta_n, \zeta_n, \upsilon)$$
$$\leqslant \limsup_{n \to \infty} \acute{S}(\zeta_n, \zeta_n, \upsilon) \leqslant \Omega\left(\acute{S}(\zeta, \zeta, \eta)\right).$$

DEFINITION 6. [21] Let $(\mathscr{A}, d)$ be a metric space. A map $T : \mathscr{A} \to \mathscr{A}$ is said to be an F_w-contraction, if there exist $\tau > 0$ and $F \in \mathscr{F}_1$ such that

$$\forall \zeta, \eta \in \mathscr{A}, \ d(T(\zeta), T(\eta)) > 0 \Rightarrow \tau + F(d(T(\zeta), T(\eta))) \leqslant F(d(\zeta, \eta)),$$

where $\mathscr{F}_1$ is the family of mappings $F_w : (0, \infty) \to (-\infty, \infty)$ satisfying: $(WF1)$ F is strictly increasing; $(WF2)$ for each sequence $\{t_n\}_{n=1}^{\infty} \subset (0, \infty)$,

$$\lim_{n \to \infty} F(t_n) = -\infty \Leftrightarrow \lim_{n \to \infty} t_n = 0;$$

and $(WF3)$ there exists $l \in (0, 1)$ such that $\lim_{t \to 0^+} t^l F(t) = 0$.

THEOREM 1. [21] *Let $(\mathscr{A},d)$ be a complete metric space and let $T : \mathscr{A} \to \mathscr{A}$ be an F_w- contraction. Then T has a unique fixed point in $\mathscr{A}$.*

Piri and Kumam [15] modified the notion of F_w-contraction as follows:

DEFINITION 7. [15] *Let $(\mathscr{A},d)$ be a metric space. A mapping $T : \mathscr{A} \to \mathscr{A}$ is said to be an F-contraction if there exist $F \in \mathscr{F}$ and $\tau > 0$ such that*

$$\forall \zeta, \eta \in \mathscr{A}, d(T(\zeta), T(\eta)) > 0 \Rightarrow \tau + F\left(d\left(T(\zeta), T(\eta)\right)\right) \leqslant F\left(d(\zeta, \eta)\right),$$

where $\mathscr{F}$ is the set of mappings $F : (0,\infty) \to (-\infty,\infty)$ satisfying the following conditions:

(F1) *For all $\zeta, \eta \in \mathbb{R}^+ = (0,\infty)$ such that $\zeta < \eta \Leftrightarrow F(\zeta) < F(\eta)$;*
(F2) *For each sequence $\{t_n\}_{n=1}^{\infty}$ of positive numbers,*

$$\lim_{n \to \infty} F(t_n) = -\infty \text{ if and only if } \lim_{n \to \infty} t_n = 0;$$

(F3) *F is continuous.*

3. Existence Theorems

Let $\mathbb{R}_0^+$ represents non-negative real numbers and Δ_D be the set of all continuous mappings $D : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}^+$ with the property (P).

(P): if the numbers $t_1, t_2, t_3, , t_4 \in \mathbb{R}^+$ have zero product, then there exists $\tau > 0$ such that $D(t_1, t_2, t_3, t_4) = \tau$.

Let the mapping $E : \mathscr{A} \times \mathscr{A} \times \mathscr{A} \to \mathbb{R}$ be defined by

$$E(\zeta, \eta, \sigma) = \max \left\{ \begin{array}{l} \Omega^{-1}\left(\acute{S}(S(\zeta), S(\eta), R(\sigma))\right), \\[2mm] \Omega^{-1}\left(\frac{1}{3}\left(\begin{array}{l} \acute{S}(S(\zeta), S(\zeta), f(\zeta)) \\ +\acute{S}(S(\eta), S(\eta), f(\eta)) \\ +\acute{S}(R(\sigma), R(\sigma), g(\sigma)) \end{array}\right)\right), \\[6mm] \frac{1}{3}\Omega^{-1}\left(\frac{1}{2}\left(\begin{array}{l} \acute{S}(S(\zeta), S(\zeta), g(\sigma)) \\ +\acute{S}(S(\eta), S(\eta), g(\sigma)) \\ +\acute{S}(R(\sigma), R(\sigma), f(\zeta)) \\ +\acute{S}(R(\sigma), R(\sigma), f(\eta)) \end{array}\right)\right) \end{array} \right\}.$$

DEFINITION 8. *Let $(\mathscr{A}, \acute{S})$ be an $\acute{S}$-metric space. The self-mappings $f, g, R, S : \mathscr{A} \to \mathscr{A}$ are said to form (F, Ω)-contraction, if there exist $F \in \mathscr{F}$ and $D \in \Delta_D$ such that*

$$\begin{aligned} & \Omega\left(\acute{S}(f(\zeta), f(\eta), g(\sigma))\right) > 0 \\ & \Rightarrow F\left(\Omega\left(\acute{S}(f(\zeta), f(\eta), g(\sigma))\right)\right) \leqslant F(E(\zeta, \eta, \sigma)) \\ & \qquad -D\left(\begin{array}{l} \acute{S}(S(\zeta), S(\zeta), g(\sigma)), \acute{S}(S(\eta), S(\eta), g(\sigma)), \\ \acute{S}(R(\sigma), R(\sigma), f(\zeta)), \acute{S}(R(\sigma), R(\sigma), f(\eta)) \end{array}\right). \end{aligned} \qquad (1)$$

EXAMPLE 6. Let $\mathscr{A} = [0,1]$. Define $\acute{S} : \mathscr{A} \times \mathscr{A} \times \mathscr{A} \to [0,+\infty)$ by,

$$\acute{S}(\zeta,\eta,\sigma) = e^{[|\zeta-\sigma|+|\eta-\sigma|]} - 1,$$

for all $\zeta,\eta,\sigma \in \mathscr{A}$ and $\Omega(t) = e^t - 1$ (so, $\Omega^{-1}(t) = \ln(t+1)$). Then, $(\mathscr{A},\acute{S})$ is a complete $\acute{S}$-metric space. Define, $f,S,g,R : \mathscr{A} \to \mathscr{A}$ by

$$f(\beta) = \ln\left(1 + \frac{\beta}{8}\right),$$
$$S(\beta) = e^{7\beta} - 1,$$
$$g(\beta) = \ln\left(1 + \frac{\beta}{7}\right),$$
$$R(\beta) = e^{6\beta} - 1,$$

for all $\beta \in \mathscr{A}$. Define the mappings $F : (0,\infty) \to (-\infty,\infty)$ and $D : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}^+$ by,

$$F(\alpha) = \ln(\alpha), \text{ for } \alpha > 0, \text{ and } D(t_1,t_2,t_3,t_4) = 1.$$

Now, for all $\zeta,\eta,\sigma \in \mathscr{A}$, with $\Omega\big(\acute{S}(f(\zeta),f(\eta),g(\sigma))\big) > 0$, we have,

$$\begin{aligned}
&F\big(\Omega\big(\acute{S}(f(\zeta),f(\eta),g(\sigma))\big)\big)\\
&\leqslant F\left(E\left(\zeta,\eta,\sigma\right)\right)\\
&-D\left(\begin{array}{l}\acute{S}(S(\zeta),S(\zeta),g(\sigma)),\acute{S}(S(\eta),S(\eta),g(\sigma)),\\ \acute{S}(R(\sigma),R(\sigma),f(\zeta)),\acute{S}(R(\sigma),R(\sigma),f(\eta))\end{array}\right).
\end{aligned}$$

Hence, the maps $f,g,R,S : \mathscr{A} \to \mathscr{A}$ form (F,Ω)-contraction.

THEOREM 2. *Let $(\mathscr{A},\acute{S})$ be a complete symmetric $\acute{S}$-metric space and $f,g,R,S : \mathscr{A} \to \mathscr{A}$ be four self-mappings forming (F,Ω)-contraction. Let $\Omega(a+b) \geqslant \Omega(a) + \Omega(b)$, for each $a,b \in [0,\infty)$. Moreover, if*

(i) *$f(\mathscr{A}) \subseteq R(\mathscr{A})$, $g(\mathscr{A}) \subseteq S(\mathscr{A})$ and either $R(\mathscr{A})$ or $S(\mathscr{A})$ is a closed subset of $\mathscr{A}$;*

(ii) *the pair (f,S) and (g,R) are weak compatible.*

Then f,g,R and S admit a unique common fixed point in $\mathscr{A}$.

Proof. Let $\zeta_0 \in \mathscr{A}$ be an arbitrary point. By (i), there exist $\zeta_1,\zeta_2 \in \mathscr{A}$ such that

$$f(\zeta_0) = R(\zeta_1) = \eta_0 \text{ and } g(\zeta_1) = S(\zeta_2) = \eta_0.$$

Repeating above steps, we construct a sequence $\{\eta_n\}$ in $\mathscr{A}$ such that

$$\eta_{2n} = f(\zeta_{2n}) = R(\zeta_{2n+1}) \text{ and } \eta_{2n+1} = g(\zeta_{2n+1}) = S(\zeta_{2n+2}); \, n = 0,1,2,\ldots.$$

We show that $\{\eta_n\}$ is a $\acute{S}$-Cauchy sequence. If $\Omega\left(\acute{S}(f(\zeta_{2n}),f(\zeta_{2n}),g(\zeta_{2n+1}))\right) > 0$, then by contractive condition (1), we get,

$$
\begin{aligned}
F\left(\Omega\left(\acute{S}(\eta_{2n},\eta_{2n},\eta_{2n+1}))\right)\right) &= F\left(\Omega\left(\acute{S}(f(\zeta_{2n}),f(\zeta_{2n}),g(\zeta_{2n+1})))\right)\right) \\
&\leqslant F\left(E\left(\zeta_n,\eta_n,\sigma_n\right)\right) \\
&\quad -D\left(\begin{array}{c}
\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n+1}), \\
\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n+1}), \\
\acute{S}(\eta_{2n},\eta_{2n},\eta_{2n}), \\
\acute{S}(\eta_{2n},\eta_{2n},\eta_{2n})
\end{array}\right),
\end{aligned}
\tag{2}
$$

where,

$$
\begin{aligned}
&E\left(\zeta_n,\eta_n,\sigma_n\right) \\
&= \max\left\{
\begin{array}{l}
\Omega^{-1}\left(\acute{S}(S(\zeta_{2n}),S(\zeta_{2n}),R(\zeta_{2n+1}))\right), \\[2mm]
\Omega^{-1}\left(\frac{1}{3}\left(\begin{array}{l}
\acute{S}(S(\zeta_{2n}),S(\zeta_{2n}),f(\zeta_{2n})) \\
+\acute{S}(S(\zeta_{2n}),S(\zeta_{2n}),f(\zeta_{2n}))+ \\
\acute{S}(R(\zeta_{2n+1}),R(\zeta_{2n+1}),g(\zeta_{2n+1}))
\end{array}\right)\right), \\[5mm]
\frac{1}{3}\Omega^{-1}\left(\frac{1}{2}\left(\begin{array}{l}
2\acute{S}(S(\zeta_{2n}),S(\zeta_{2n}),g(\zeta_{2n+1})) \\
+2\acute{S}(R(\zeta_{2n+1}),R(\zeta_{2n+1}),f(\zeta_{2n}))
\end{array}\right)\right)
\end{array}
\right\} \\[3mm]
&= \max\left\{
\begin{array}{l}
\Omega^{-1}\left(\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n})\right), \\[2mm]
\Omega^{-1}\left(\frac{1}{3}\left(\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n})+\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n})+\acute{S}(\eta_{2n},\eta_{2n},\eta_{2n+1}))\right)\right), \\[2mm]
\frac{1}{3}\Omega^{-1}\left(\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n+1})+\acute{S}(\eta_{2n},\eta_{2n},\eta_{2n}))\right)
\end{array}
\right\} \\[3mm]
&= \max\left\{
\begin{array}{l}
\Omega^{-1}\left(\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n})\right), \\[2mm]
\Omega^{-1}\left(\frac{1}{3}\left(2\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n})+\acute{S}(\eta_{2n},\eta_{2n},\eta_{2n+1}))\right)\right), \\[2mm]
\frac{1}{3}\Omega^{-1}\left(\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n+1})\right)
\end{array}
\right\}.
\end{aligned}
$$

Thus, by the definition of mapping D, inequality (2) turns into the following:

$$
\begin{aligned}
&F\left(\Omega\left(\acute{S}(\eta_{2n},\eta_{2n},\eta_{2n+1}))\right)\right) \\
&= F\left(\Omega\left(\acute{S}(f(\zeta_{2n}),f(\zeta_{2n}),g(\zeta_{2n+1})))\right)\right) \\
&\leqslant F\left(\max\left\{
\begin{array}{l}
\Omega^{-1}\left(\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n})\right), \\[2mm]
\Omega^{-1}\left(\frac{1}{3}\left(2\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n})+\acute{S}(\eta_{2n},\eta_{2n},\eta_{2n+1}))\right)\right), \\[2mm]
\frac{1}{3}\Omega^{-1}\left(\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n+1})\right)
\end{array}
\right\}\right) - \tau.
\end{aligned}
\tag{3}
$$

As

$$
\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n+1}) \leqslant \Omega\left[2\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n})+\acute{S}(\eta_{2n},\eta_{2n},\eta_{2n+1})\right],
$$

so,

$$\frac{1}{3}\Omega^{-1}\left[\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n+1})\right] \leqslant \frac{1}{3}\left[2\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n})+\acute{S}(\eta_{2n},\eta_{2n},\eta_{2n+1})\right].$$

We show that,

$$F\left(\Omega\left(\acute{S}(\eta_{2n},\eta_{2n},\eta_{2n+1})\right)\right) \leqslant F\left(\Omega^{-1}\left(\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n})\right)\right)-\tau.$$

By (F1), we have $\Omega\left(\acute{S}(\eta_{2n},\eta_{2n},\eta_{2n+1})\right) \leqslant \Omega^{-1}\left(\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n})\right)$, that is, $\acute{S}(\eta_{2n},\eta_{2n},\eta_{2n+1}) \leqslant \acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n})$, for all $n\in\mathbb{N}$. If

$$\acute{S}(\eta_{2n},\eta_{2n},\eta_{2n+1}) > \acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n}),$$

for some $n\in\mathbb{N}$, we have

$$\frac{1}{3}\Omega^{-1}\left[\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n+1})\right] \leqslant \acute{S}(\eta_{2n},\eta_{2n},\eta_{2n+1}). \tag{4}$$

Hence by (4), we get

$$\begin{aligned}
F\left(\acute{S}(\eta_{2n},\eta_{2n},\eta_{2n+1})\right) &\leqslant F\left(\Omega\left(\acute{S}(\eta_{2n},\eta_{2n},\eta_{2n+1})\right)\right)\\
&\leqslant F\left(\Omega^{-1}\left(\acute{S}(\eta_{2n},\eta_{2n},\eta_{2n+1})\right)\right)-\tau\\
&\leqslant F\left(\acute{S}(\eta_{2n},\eta_{2n},\eta_{2n+1})\right)-\tau,
\end{aligned}$$

which is a contradiction. Now for $n+1$

$$\begin{aligned}
F\left(\Omega\left(\acute{S}(\eta_{2n+1},\eta_{2n+1},\eta_{2n+2})\right)\right) &= F\left(\Omega\left(\acute{S}(\eta_{2n+2},\eta_{2n+2},\eta_{2n+1})\right)\right)\\
&= F\left(\Omega\left(\acute{S}(f\left(\zeta_{2n+2}\right),f\left(\zeta_{2n+2}\right),g\left(\zeta_{2n+1}\right))\right)\right)\\
&\leqslant F\left(E\left(\zeta_{n+1},\eta_{n+1},\sigma_{n+1}\right)\right)\\
&\quad -D\begin{pmatrix}\acute{S}(\eta_{2n+1},\eta_{2n+1},\eta_{2n+1}),\\ \acute{S}(\eta_{2n+1},\eta_{2n+1},\eta_{2n+1}),\\ \acute{S}(\eta_{2n},\eta_{2n},\eta_{2n+2}),\\ \acute{S}(\eta_{2n},\eta_{2n},\eta_{2n+2})\end{pmatrix},
\end{aligned} \tag{5}$$

where,

$$E\left(\zeta_{n+1},\eta_{n+1},\sigma_{n+1}\right)$$
$$= \max\begin{cases}\Omega^{-1}\left(\acute{S}(S\left(\zeta_{2n+2}\right),S\left(\zeta_{2n+2}\right),R\left(\zeta_{2n+1}\right))\right),\\[2mm]
\Omega^{-1}\left(\frac{1}{3}\begin{pmatrix}\acute{S}(S\left(\zeta_{2n+2}\right),S\left(\zeta_{2n+2}\right),f\left(\zeta_{2n+2}\right))\\ +\acute{S}(S\left(\zeta_{2n+2}\right),S\left(\zeta_{2n+2}\right),f\left(\zeta_{2n+2}\right))\\ +\acute{S}(R\left(\zeta_{2n+1}\right),R\left(\zeta_{2n+1}\right),g\left(\zeta_{2n+1}\right))\end{pmatrix}\right),\\[4mm]
\frac{1}{3}\Omega^{-1}\left(\frac{1}{2}\begin{pmatrix}2\acute{S}(S\left(\zeta_{2n+2}\right),S\left(\zeta_{2n+2}\right),g\left(\zeta_{2n+1}\right))\\ +2\acute{S}(R\left(\zeta_{2n+1}\right),R\left(\zeta_{2n+1}\right),f\left(\zeta_{2n+2}\right))\end{pmatrix}\right)\end{cases}$$

$$= \max \begin{cases} \Omega^{-1}\left(\acute{S}(\eta_{2n+1}, \eta_{2n+1}, \eta_{2n})\right), \\ \Omega^{-1}\left(\frac{1}{3}\begin{pmatrix} \acute{S}(\eta_{2n+1}, \eta_{2n+1}, \eta_{2n+2}) + \acute{S}(\eta_{2n+1}, \eta_{2n+1}, \eta_{2n+2}) \\ +\acute{S}(\eta_{2n}, \eta_{2n}, \eta_{2n+1}) \end{pmatrix}\right), \\ \frac{1}{3}\Omega^{-1}\left(\acute{S}(\eta_{2n+1}, \eta_{2n+1}, \eta_{2n+1}) + \acute{S}(\eta_{2n}, \eta_{2n}, \eta_{2n+2})\right) \end{cases}$$

$$= \max \begin{cases} \Omega^{-1}\left(\acute{S}(\eta_{2n+1}, \eta_{2n+1}, \eta_{2n})\right), \\ \Omega^{-1}\left(\frac{1}{3}\left(2\acute{S}(\eta_{2n-1}, \eta_{2n-1}, \eta_{2n}) + \acute{S}(\eta_{2n}, \eta_{2n}, \eta_{2n+1}))\right)\right), \\ \frac{1}{3}\Omega^{-1}\left(\acute{S}(\eta_{2n-1}, \eta_{2n-1}, \eta_{2n+1})\right) \end{cases}.$$

Thus, by the definition of mapping D, inequality (5) becomes

$$F\left(\Omega\left(\acute{S}(\eta_{2n+1}, \eta_{2n+1}, \eta_{2n+2})\right)\right)$$

$$= F\left(\Omega\left(\acute{S}(f\left(\zeta_{2n+2}\right), f\left(\zeta_{2n+2}\right), g\left(\zeta_{2n+1}\right))\right)\right)$$

$$\leqslant F\left(\max \begin{cases} \Omega^{-1}\left(\acute{S}(\eta_{2n+1}, \eta_{2n+1}, \eta_{2n})\right), \\ \Omega^{-1}\left(\frac{1}{3}\left(2\acute{S}(\eta_{2n-1}, \eta_{2n-1}, \eta_{2n}) + \acute{S}(\eta_{2n}, \eta_{2n}, \eta_{2n+1}))\right)\right), \\ \frac{1}{3}\Omega^{-1}\left(\acute{S}(\eta_{2n-1}, \eta_{2n-1}, \eta_{2n+1})\right) \end{cases}\right) - \tau.$$

From

$$\acute{S}(\eta_{2n}, \eta_{2n}, \eta_{2n+2}) \leqslant \Omega\left[2\acute{S}(\eta_{2n}, \eta_{2n}, \eta_{2n+1}) + \acute{S}(\eta_{2n+1}, \eta_{2n+1}, \eta_{2n+2})\right],$$

we have,

$$\frac{1}{3}\Omega^{-1}\left[\acute{S}(\eta_{2n}, \eta_{2n}, \eta_{2n+2})\right] \leqslant \frac{1}{3}\left[2\acute{S}(\eta_{2n}, \eta_{2n}, \eta_{2n+1}) + \acute{S}(\eta_{2n+1}, \eta_{2n+1}, \eta_{2n+2})\right].$$

Similarly, if

$$\acute{S}(\eta_{2n+1}, \eta_{2n+1}, \eta_{2n+2}) > \acute{S}(\eta\acute{S}(\eta_{2n}, \eta_{2n}, \eta_{2n+1}),$$

for some $n \in \mathbb{N}$, we have

$$\frac{1}{3}\Omega^{-1}\left[\acute{S}(\eta_{2n}, \eta_{2n}, \eta_{2n+2})\right] \leqslant \acute{S}(\eta_{2n+1}, \eta_{2n+1}, \eta_{2n+2}). \tag{6}$$

Hence, by (6), we get

$$F\left(\acute{S}(\eta_{2n+1}, \eta_{2n+1}, \eta_{2n+2})\right) \leqslant F\left(\Omega\left(\acute{S}(\eta_{2n+1}, \eta_{2n+1}, \eta_{2n+2})\right)\right)$$
$$\leqslant F\left(\Omega^{-1}\acute{S}(\eta_{2n+1}, \eta_{2n+1}, \eta_{2n+2})\right) - \tau$$
$$\leqslant F\left(\acute{S}(\eta_{2n+1}, \eta_{2n+1}, \eta_{2n+2})\right) - \tau,$$

it is a contradiction. Therefore,

$$
\begin{aligned}
F\left(\acute{S}(\eta_{2n}, \eta_{2n}, \eta_{2n+1})\right) &\leqslant F\left(\Omega\left(\acute{S}(\eta_{2n}, \eta_{2n}, \eta_{2n+1})\right)\right) \\
&\leqslant F\left(\Omega^{-1}\left(\acute{S}(\eta_{2n-1}, \eta_{2n-1}, \eta_{2n})\right)\right) - \tau \\
&\leqslant F\left(\acute{S}(\eta_{2n-1}, \eta_{2n-1}, \eta_{2n})\right) - \tau.
\end{aligned}
$$

Thus, for each $n \in \mathbb{N}$, we have,

$$
F\left(\acute{S}(\eta_n, \eta_n, \eta_{n+1})\right) \leqslant F\left(\acute{S}(\eta_{n-1}, \eta_{n-1}, \eta_n)\right) - \tau,
$$

which further implies,

$$
\begin{aligned}
F\left(\acute{S}(\eta_n, \eta_n, \eta_{2n+1})\right) &\leqslant F\left(\acute{S}(\eta_{n-1}, \eta_{n-1}, \eta_n)\right) - \tau \\
&\leqslant F\left(\acute{S}(\eta_{n-2}, \eta_{n-2}, \eta_{n-1})\right) - 2\tau \\
&\quad \cdots \\
&\leqslant F\left(\acute{S}(\eta_0, \eta_0, \eta_1)\right) - n\tau.
\end{aligned}
$$

Letting $n \longrightarrow \infty$ in the above inequality, we get

$$
\lim_{n \longrightarrow \infty} F\left(\acute{S}(\eta_n, \eta_n, \eta_{n+1})\right) = -\infty.
$$

By $(F2)$, we get

$$
\lim_{n \longrightarrow \infty} \acute{S}(\eta_n, \eta_n, \eta_{n+1}) = 0, \tag{7}
$$

Now, we will prove that the sequence $\{\eta_n\}$ is $\acute{S}$-Cauchy. Assume the contrary, there exist $\varepsilon > 0$ and a subsequences $\left\{\eta_{2\hat{h}_n}\right\}_{n=1}^{\infty}$ and $\{\eta_{2\hat{j}_n}\}_{n=1}^{\infty}$ of $\{\eta_n\}$ such for all $n \in \mathbb{N}$,

$$
2\hat{j}_n > 2\hat{h}_n > n \text{ and } \acute{S}\left(\eta_{2\hat{h}_n}, \eta_{2\hat{h}_n}, \eta_{2\hat{j}_n}\right) \geqslant \varepsilon, \tag{8}
$$

then,

$$
\acute{S}\left(\eta_{2\hat{h}_n}, \eta_{2\hat{h}_n}, \eta_{2\hat{j}_n-1}\right) < \varepsilon. \tag{9}
$$

By contractive condition (1), we get,

$$
\begin{aligned}
F\left(\Omega\left(\acute{S}(\eta_{2\hat{j}_n}, \eta_{2\hat{j}_n}, \eta_{2\hat{h}_n+1})\right)\right) &= F\left(\Omega\left(\acute{S}(f\left(\zeta_{2\hat{j}_n}\right), f\left(\zeta_{2\hat{j}_n}\right), g\left(\zeta_{2\hat{h}_n+1}\right))\right)\right) \\
&\leqslant F\left(E\left(\eta_{2\hat{h}_n}, \eta_{2\hat{h}_n}, \eta_{2\hat{j}_n}\right)\right) \\
&\quad -D\begin{pmatrix} \acute{S}(S\left(\zeta_{2\hat{j}_n}\right), S\left(\zeta_{2\hat{j}_n}\right), g\left(\zeta_{2\hat{h}_n+1}\right)), \\ \acute{S}(S\left(\zeta_{2\hat{j}_n}\right), S\left(\zeta_{2\hat{j}_n}\right), g\left(\zeta_{2\hat{h}_n+1}\right)), \\ \acute{S}(R\left(\zeta_{2\hat{h}_n+1}\right), R\left(\zeta_{2\hat{h}_n+1}\right), f\left(\zeta_{2\hat{j}_n}\right), \\ \acute{S}(R\left(\zeta_{2\hat{h}_n+1}\right), R\left(\zeta_{2\hat{h}_n+1}\right), f\left(\zeta_{2\hat{j}_n}\right) \end{pmatrix}, \tag{10}
\end{aligned}
$$

where,

$$E\left(\eta_{2\hat{h}_n}, \eta_{2\hat{h}_n}, \eta_{2\hat{j}_n}\right)$$

$$= \max \left\{ \begin{array}{l} \Omega^{-1}\left(\acute{S}(S(\zeta_{2\hat{j}_n}), S(\zeta_{2\hat{j}_n}), R\left(\zeta_{2\hat{h}_n+1}\right))\right), \\ \Omega^{-1}\left(\frac{1}{3}\left(\begin{array}{l}\acute{S}(S(\zeta_{2\hat{j}_n}), S(\zeta_{2\hat{j}_n}), f(\zeta_{2\hat{j}_n})) \\ +\acute{S}(S(\zeta_{2\hat{j}_n}), S(\zeta_{2\hat{j}_n}), f(\zeta_{2\hat{j}_n})) \\ +\acute{S}(R\left(\zeta_{2\hat{h}_n+1}\right), R\left(\zeta_{2\hat{h}_n+1}\right), g\left(\zeta_{2\hat{h}_n+1}\right))\end{array}\right)\right), \\ \frac{1}{3}\Omega^{-1}\left(\frac{1}{2}\left(\begin{array}{l}2\acute{S}(S(\zeta_{2\hat{j}_n}), S(\zeta_{2\hat{j}_n}), g\left(\zeta_{2\hat{h}_n+1}\right)) \\ +2\acute{S}(R\left(\zeta_{2\hat{h}_n+1}\right), R\left(\zeta_{2\hat{h}_n+1}\right), f(\zeta_{2\hat{j}_n}))\end{array}\right)\right) \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} \Omega^{-1}\left(\acute{S}(\eta_{2\hat{j}_n-1}, \eta_{2\hat{j}_n-1}, \eta_{2\hat{h}_n})\right), \\ \Omega^{-1}\left(\frac{1}{3}\left(\begin{array}{l}\acute{S}(\eta_{2\hat{j}_n-1}, \eta_{2\hat{j}_n-1}, \eta_{2\hat{j}_n}) + \acute{S}(\eta_{2\hat{j}_n-1}, \eta_{2\hat{j}_n-1}, \eta_{2\hat{j}_n}) \\ +\acute{S}(\eta_{2\hat{h}_n}, \eta_{2\hat{h}_n}, \eta_{2\hat{h}_n+1})\end{array}\right)\right), \\ \frac{1}{3}\Omega^{-1}\left(\acute{S}(\eta_{2\hat{j}_n-1}, \eta_{2\hat{j}_n-1}, \eta_{2\hat{h}_n+1}) + \acute{S}(\eta_{2\hat{h}_n}, \eta_{2\hat{h}_n}, \eta_{2\hat{j}_n})\right) \end{array} \right\}.$$

Thus, by the definition of mapping D, inequality (10) becomes,

$$F\left(\Omega\left(\acute{S}(\eta_{2\hat{j}_n}, \eta_{2\hat{j}_n}, \eta_{2\hat{h}_n+1})\right)\right) \tag{11}$$

$$\leqslant F\left(\max \left\{ \begin{array}{l} \Omega^{-1}\left(\acute{S}(\eta_{2\hat{j}_n-1}, \eta_{2\hat{j}_n-1}, \eta_{2\hat{h}_n})\right), \\ \Omega^{-1}\left(\frac{1}{3}\left(\begin{array}{l}\acute{S}(\eta_{2\hat{j}_n-1}, \eta_{2\hat{j}_n-1}, \eta_{2\hat{j}_n}) + \acute{S}(\eta_{2\hat{j}_n-1}, \eta_{2\hat{j}_n-1}, \eta_{2\hat{j}_n}) \\ +\acute{S}(\eta_{2\hat{h}_n}, \eta_{2\hat{h}_n}, \eta_{2\hat{h}_n+1})\end{array}\right)\right), \\ \frac{1}{3}\Omega^{-1}\left(\acute{S}(\eta_{2\hat{j}_n-1}, \eta_{2\hat{j}_n-1}, \eta_{2\hat{h}_n+1}) + \acute{S}(\eta_{2\hat{h}_n}, \eta_{2\hat{h}_n}, \eta_{2\hat{j}_n})\right) \end{array} \right\}\right) - \tau.$$

As,

$$\acute{S}(\eta_{2\hat{j}_n}, \eta_{2\hat{j}_n}, \eta_{2\hat{h}_n}) \leqslant \Omega\left(2\acute{S}(\eta_{2\hat{j}_n}, \eta_{2\hat{j}_n}, \eta_{2\hat{j}_n-1}) + \acute{S}(\eta_{2\hat{h}_n}, \eta_{2\hat{h}_n}, \eta_{2\hat{j}_n-1})\right) \tag{12}$$
$$\leqslant \Omega\left(2\acute{S}(\eta_{2\hat{j}_n}, \eta_{2\hat{j}_n}, \eta_{2\hat{j}_n-1}) + \varepsilon\right).$$

Letting the upper limit as $n \to \infty$ in (12), we get

$$\varepsilon \leqslant \limsup_{n\to\infty} \acute{S}(\eta_{2\hat{j}_n}, \eta_{2\hat{j}_n}, \eta_{2\hat{h}_n})$$
$$\leqslant \Omega\left(2\limsup_{n\to\infty} \acute{S}(\eta_{2\hat{j}_n}, \eta_{2\hat{j}_n}, \eta_{2\hat{j}_n-1}) + \varepsilon\right)$$
$$\leqslant \Omega(\varepsilon).$$

Since,

$$\begin{aligned}
\acute{S}(\eta_{2\hat{\jmath}_n-1}, \eta_{2\hat{\jmath}_n-1}, \eta_{2\hat{h}_n+1}) &= \acute{S}(\eta_{2\hat{h}_n+1}, \eta_{2\hat{h}_n+1}, \eta_{2\hat{\jmath}_n-1}) \hspace{2cm} (13) \\
&\leqslant \Omega\left(2\acute{S}(\eta_{2\hat{h}_n+1}, \eta_{2\hat{h}_n+1}, \eta_{2\hat{h}_n}) + \acute{S}(\eta_{2\hat{\jmath}_n-1}, \eta_{2\hat{\jmath}_n-1}, \eta_{2\hat{h}_n})\right) \\
&\leqslant \Omega\left(2\acute{S}(\eta_{2\hat{h}_n+1}, \eta_{2\hat{h}_n+1}, \eta_{2\hat{h}_n}) + \varepsilon\right),
\end{aligned}$$

taking the upper limit as $n \to \infty$ in (13), we get

$$\begin{aligned}
\varepsilon &\leqslant \lim_{n\to\infty}\sup \acute{S}(\eta_{2\hat{\jmath}_n-1}, \eta_{2\hat{\jmath}_n-1}, \eta_{2\hat{h}_n+1}) \\
&\leqslant \Omega\left(2\lim_{n\to\infty}\sup \acute{S}(\eta_{2\hat{h}_n+1}, \eta_{2\hat{h}_n+1}, \eta_{2\hat{h}_n}) + \varepsilon\right) \\
&\leqslant \Omega(\varepsilon).
\end{aligned}$$

Now, since,

$$\begin{aligned}
\acute{S}(\eta_{2\hat{\jmath}_n}, \eta_{2\hat{\jmath}_n}, \eta_{2\hat{h}_n}) &= \acute{S}(\eta_{2\hat{h}_n}, \eta_{2\hat{h}_n}, \eta_{2\hat{\jmath}_n}) \hspace{3cm} (14) \\
&\leqslant \Omega\left(2\acute{S}(\eta_{2\hat{h}_n}, \eta_{2\hat{h}_n}, \eta_{2\hat{h}_n+1}) + \acute{S}(\eta_{2\hat{\jmath}_n}, \eta_{2\hat{\jmath}_n}, \eta_{2\hat{h}_n+1})\right),
\end{aligned}$$

taking the upper limit as $n \to \infty$ in (14), we get

$$\begin{aligned}
\varepsilon &\leqslant \lim_{n\to\infty}\sup \acute{S}(\eta_{2\hat{\jmath}_n}, \eta_{2\hat{\jmath}_n}, \eta_{2\hat{h}_n}) \\
&\leqslant \Omega\left(2\lim_{n\to\infty}\sup \acute{S}(\acute{S}(\eta_{2\hat{h}_n}, \eta_{2\hat{h}_n}, \eta_{2\hat{h}_n+1})) + \lim_{n\to\infty}\sup \acute{S}(\acute{S}(\eta_{2\hat{\jmath}_n}, \eta_{2\hat{\jmath}_n}, \eta_{2\hat{h}_n+1}))\right) \\
&\leqslant \Omega\left(0 + \lim_{n\to\infty}\sup \acute{S}(\acute{S}(\eta_{2\hat{\jmath}_n}, \eta_{2\hat{\jmath}_n}, \eta_{2\hat{h}_n+1}))\right) \\
&= \Omega\left(\lim_{n\to\infty}\sup \acute{S}(\acute{S}(\eta_{2\hat{\jmath}_n}, \eta_{2\hat{\jmath}_n}, \eta_{2\hat{h}_n+1}))\right),
\end{aligned}$$

which then implies,

$$\varepsilon \leqslant \lim_{n\to\infty}\sup \acute{S}(\eta_{2\hat{\jmath}_n}, \eta_{2\hat{\jmath}_n}, \eta_{2\hat{h}_n}) \leqslant \Omega\left(\lim_{n\to\infty}\sup \acute{S}(\acute{S}(\eta_{2\hat{\jmath}_n}, \eta_{2\hat{\jmath}_n}, \eta_{2\hat{h}_n+1}))\right). \hspace{0.5cm} (15)$$

From (11), (15), and since F and Ω are continuous, we get,

$$\begin{aligned}
&F(\varepsilon) \leqslant F\left(\lim_{n\to\infty}\sup \acute{S}(\eta_{2\hat{\jmath}_n}, \eta_{2\hat{\jmath}_n}, \eta_{2\hat{h}_n})\right) \\
&\leqslant F\left(\Omega\left(\lim_{n\to\infty}\sup \acute{S}(\eta_{2\hat{\jmath}_n}, \eta_{2\hat{\jmath}_n}, \eta_{2\hat{h}_n+1})\right)\right) \\
&\leqslant F\left(\max\left\{
\begin{array}{l}
\Omega^{-1}\left(\lim_{n\to\infty}\sup \acute{S}(\eta_{2\hat{\jmath}_n-1}, \eta_{2\hat{\jmath}_n-1}, \eta_{2\hat{h}_n})\right), \\[2mm]
\Omega^{-1}\left(\lim_{n\to\infty}\sup\left(\frac{1}{3}\begin{pmatrix}\acute{S}(\eta_{2\hat{\jmath}_n-1}, \eta_{2\hat{\jmath}_n-1}, \eta_{2\hat{\jmath}_n})+ \\ \acute{S}(\eta_{2\hat{\jmath}_n-1}, \eta_{2\hat{\jmath}_n-1}, \eta_{2\hat{\jmath}_n}) \\ +\acute{S}(\eta_{2\hat{h}_n}, \eta_{2\hat{h}_n}, \eta_{2\hat{h}_n+1})\end{pmatrix}\right)\right), \\[4mm]
\frac{1}{3}\Omega^{-1}\left(\begin{array}{l}\lim_{n\to\infty}\sup \acute{S}(\eta_{2\hat{\jmath}_n-1}, \eta_{2\hat{\jmath}_n-1}, \eta_{2\hat{h}_n+1}) \\ +\lim_{n\to\infty}\sup \acute{S}(\eta_{2\hat{h}_n}, \eta_{2\hat{h}_n}, \eta_{2\hat{\jmath}_n})\end{array}\right)
\end{array}\right\}\right) - \tau
\end{aligned}$$

$$\leqslant F\left(\max\left\{\Omega^{-1}\left(\Omega(\varepsilon)\right),\Omega^{-1}(0),\frac{1}{3}\Omega^{-1}\left(2\Omega(\varepsilon)\right)\right\}\right)-\tau$$

$$\leqslant F\left(\frac{1}{3}\Omega^{-1}\left(2\Omega(\varepsilon)\right)\right)-\tau.$$

This implies,

$$F\left(\varepsilon\right)\leqslant F\left(\lim_{n\to\infty}\sup \acute{S}(\eta_{2\hat{\jmath}_n},\eta_{2\hat{\jmath}_n},\eta_{2\hat{h}_n})\right)\leqslant F\left(\frac{1}{3}\Omega^{-1}\left(2\Omega(\varepsilon)\right)\right)-\tau$$

$$< F\left(\frac{1}{3}\Omega^{-1}\left(2\Omega(\varepsilon)\right)\right).$$

By (F1), we get,

$$\varepsilon<\frac{1}{3}\Omega^{-1}\left(2\Omega(\varepsilon)\right).$$

Thus,

$$\Omega(3\varepsilon)<2\Omega(\varepsilon).$$

As $\Omega(a+b)\geqslant\Omega(a)+\Omega(b)$, so $3\Omega(\varepsilon)\leqslant\Omega(3\varepsilon)<2\Omega(\varepsilon)$, this is a contradiction. Hence, $\{\eta_n\}$ is an $\acute{S}$-Cauchy sequence. Since $\mathscr{A}$ is an $\acute{S}$-complete, so $\{\eta_n\}$ $\acute{S}$-converges to a point $\eta^*\in\omega$, i.e., $\lim_{n\to\infty}\eta_n=\eta^*$,

$$\lim_{n\to\infty}\eta_{2n}=\lim_{n\to\infty}f\left(\zeta_{2n}\right)=\lim_{n\to\infty}R\left(\zeta_{2n+1}\right)$$
$$=\lim_{n\to\infty}\eta_{2n+1}=\lim_{n\to\infty}g\left(\zeta_{2n+1}\right)$$
$$=\lim_{n\to\infty}S\left(\zeta_{2n+2}\right)=\eta^*.$$

Let $R\left(\mathscr{A}\right)$ be a closed subset of $\mathscr{A}$, then, there exists $u\in\mathscr{A}$ such that $R\left(u\right)=\eta^*$. we show that $g\left(u\right)=\eta^*$. Now,

$$F\left(\Omega\left(\acute{S}(f\left(\zeta_{2n}\right),f\left(\zeta_{2n}\right),g\left(u\right))\right)\right)$$
$$\leqslant F\left(E\left(\zeta_n,\zeta_n,u\right)\right)$$
$$-D\begin{pmatrix}\acute{S}(S\left(\zeta_{2n}\right),S\left(\zeta_{2n}\right),g\left(u\right)),\\ \acute{S}(S\left(\zeta_{2n}\right),S\left(\zeta_{2n}\right),g\left(u\right)),\\ \acute{S}(R\left(u\right),R\left(u\right),f\left(\zeta_{2n}\right)),\\ \acute{S}(R\left(u\right),R\left(u\right),f\left(\zeta_{2n}\right))\end{pmatrix},\tag{16}$$

where,

$$E\left(\zeta_n,\zeta_n,u\right)=\max\left\{\begin{array}{l}\Omega^{-1}\left(\acute{S}(S\left(\zeta_{2n}\right),S\left(\zeta_{2n}\right),R\left(u\right))\right),\\[2mm]\Omega^{-1}\left(\frac{1}{3}\begin{pmatrix}\acute{S}(S\left(\zeta_{2n}\right),S\left(\zeta_{2n}\right),f\left(\zeta_{2n}\right))\\+\acute{S}(S\left(\zeta_{2n}\right),S\left(\zeta_{2n}\right),f\left(\zeta_{2n}\right))\\+\acute{S}(R\left(u\right),R\left(u\right),g\left(u\right))\end{pmatrix}\right),\\[2mm]\frac{1}{3}\Omega^{-1}\left(\frac{1}{2}\left(2\acute{S}(S\left(\zeta_{2n}\right),S\left(\zeta_{2n}\right),g\left(u\right))+2\acute{S}(R\left(u\right),R\left(u\right),f\left(\zeta_{2n}\right))\right)\right)\end{array}\right\}.$$

Thus, by the definition of mapping D, inequality (16) turns into the following:

$$F\left(\Omega\left(\acute{S}(f\left(\zeta_{2n}\right),f\left(\zeta_{2n}\right),g\left(u\right)\right)\right)\right) \tag{17}$$

$$\leqslant F\left(\max\left\{\begin{array}{l}\Omega^{-1}\left(\acute{S}(S\left(\zeta_{2n}\right),S\left(\zeta_{2n}\right),R\left(u\right))\right), \\ \Omega^{-1}\left(\frac{1}{3}\left(\begin{array}{l}\acute{S}(S\left(\zeta_{2n}\right),S\left(\zeta_{2n}\right),f\left(\zeta_{2n}\right)) \\ +\acute{S}(S\left(\zeta_{2n}\right),S\left(\zeta_{2n}\right),f\left(\zeta_{2n}\right)) \\ +\acute{S}(R\left(u\right),R\left(u\right),g\left(u\right))\end{array}\right)\right), \\ \frac{1}{3}\Omega^{-1}\left(\frac{1}{2}\left(2\acute{S}(S\left(\zeta_{2n}\right),S\left(\zeta_{2n}\right),g\left(u\right))+2\acute{S}(R\left(u\right),R\left(u\right),f\left(\zeta_{2n}\right))\right)\right)\end{array}\right\}\right)-\tau.$$

Since Ω and F are continuous, so by taking upper limit as $n\to\infty$ in (17), and applying Lemma 1, we have,

$$F\left(\acute{S}(\eta^*,\eta^*,g\left(u\right))\right)=F\left(\Omega\left(\Omega^{-1}\left(\acute{S}(\eta^*,\eta^*,g\left(u\right))\right)\right)\right) \tag{18}$$

$$\leqslant F\left(\Omega\left(\limsup_{n\to\infty}\acute{S}(\eta^*,\eta^*,g\left(u\right))\right)\right)$$

$$\leqslant F\left(\max\left\{\begin{array}{l}0, \\ \Omega^{-1}\left(\frac{1}{3}\left(0+0+\acute{S}(\eta^*,\eta^*,g\left(u\right))\right)\right), \\ \frac{1}{3}\Omega^{-1}\left(\frac{1}{2}\left(2\limsup_{n\to\infty}\acute{S}(S\left(\zeta_{2n}\right),S\left(\zeta_{2n}\right),g\left(u\right))+0\right)\right)\end{array}\right\}\right)-\tau$$

$$\leqslant F\left(\max\left\{\begin{array}{l}0,\Omega^{-1}\left(\frac{\acute{S}(\eta^*,\eta^*,g(u))}{3}\right), \\ \frac{1}{3}\Omega^{-1}\left(\Omega\left(\acute{S}(\eta^*,\eta^*,g\left(u\right))\right)\right)\end{array}\right\}\right)-\tau$$

$$\leqslant F\left(\max\left\{\begin{array}{l}0,\frac{\acute{S}(\eta^*,\eta^*,g(u))}{3}, \\ \frac{1}{3}\acute{S}(\eta^*,\eta^*,g\left(u\right))\end{array}\right\}\right)-\tau$$

$$=F\left(\frac{1}{3}\acute{S}(\eta^*,\eta^*,g\left(u\right))\right)-\tau.$$

Together with (F1), we have,

$$\acute{S}(\eta^*,\eta^*,g\left(u\right))<\frac{1}{3}\acute{S}(\eta^*,\eta^*,g\left(u\right)).$$

If $\acute{S}(\eta^*,\eta^*,g\left(u\right))>0$, then it is a contradiction. Hence, $\eta^*=g\left(u\right)$. As the pair (R,g) is weak compatible, we have $g\left(R\left(u\right)\right)=R\left(g\left(u\right)\right)$, then, $g\left(\eta^*\right)=R\left(\eta^*\right)$. Now we prove that $g\left(\eta^*\right)=\eta^*$, if $g\left(\eta^*\right)\neq\eta^*$, then,

$$F\left(\Omega\left(\acute{S}(f\left(\zeta_{2n}\right),f\left(\zeta_{2n}\right),g\left(\eta^*\right)\right)\right)\right)$$

$$\leqslant F\left(E\left(\zeta_n,\zeta_n,\eta^*\right)\right)$$

$$-D\left(\begin{array}{l}\acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n+1}), \\ \acute{S}(\eta_{2n-1},\eta_{2n-1},\eta_{2n+1}), \\ \acute{S}(\eta_{2n},\eta_{2n},\eta_{2n}), \\ \acute{S}(\eta_{2n},\eta_{2n},\eta_{2n})\end{array}\right), \tag{19}$$

where,

$$E\left(\zeta_n,\zeta_n,\eta^*\right)=\max\left\{\begin{array}{l}\Omega^{-1}\left(\acute{S}(S\left(\zeta_{2n}\right),S\left(\zeta_{2n}\right),R\left(\eta^*\right))\right),\\[2ex]\Omega^{-1}\left(\frac{1}{3}\left(\begin{array}{l}\acute{S}(S\left(\zeta_{2n}\right),S\left(\zeta_{2n}\right),f\left(\zeta_{2n}\right))\\+\acute{S}(S\left(\zeta_{2n}\right),S\left(\zeta_{2n}\right),f\left(\zeta_{2n}\right))\\+\acute{S}(R\left(\eta^*\right),R\left(\eta^*\right),g\left(\eta^*\right))\end{array}\right)\right),\\[4ex]\frac{1}{3}\Omega^{-1}\left(\frac{1}{2}\left(2\acute{S}(S\left(\zeta_{2n}\right),S\left(\zeta_{2n}\right),g\left(\eta^*\right))\\+2\acute{S}(R\left(\eta^*\right),R\left(\eta^*\right),f\left(\zeta_{2n}\right))\right)\right)\end{array}\right\}.$$

Thus, by the definition of mapping D, inequality (19) turns into the following:

$$F\left(\Omega\left(\acute{S}(f\left(\zeta_{2n}\right),f\left(\zeta_{2n}\right),g\left(\eta^*\right))\right)\right) \tag{20}$$

$$\leqslant F\left(\max\left\{\begin{array}{l}\Omega^{-1}\left(\acute{S}(S\left(\zeta_{2n}\right),S\left(\zeta_{2n}\right),R\left(\eta^*\right))\right),\\[2ex]\Omega^{-1}\left(\frac{1}{3}\left(\begin{array}{l}\acute{S}(S\left(\zeta_{2n}\right),S\left(\zeta_{2n}\right),f\left(\zeta_{2n}\right))\\+\acute{S}(S\left(\zeta_{2n}\right),S\left(\zeta_{2n}\right),f\left(\zeta_{2n}\right))\\+\acute{S}(R\left(\eta^*\right),R\left(\eta^*\right),g\left(\eta^*\right))\end{array}\right)\right),\\[4ex]\frac{1}{3}\Omega^{-1}\left(\frac{1}{2}\left(2\acute{S}(S\left(\zeta_{2n}\right),S\left(\zeta_{2n}\right),g\left(\eta^*\right))\\+2\acute{S}(R\left(\eta^*\right),R\left(\eta^*\right),f\left(\zeta_{2n}\right))\right)\right)\end{array}\right\}\right)-\tau.$$

Again, as, Ω and F are continuous, so by taking upper limit as $n\to\infty$ in (20), and applying Lemma 1, we have,

$$F\left(\acute{S}(\eta^*,\eta^*,g\left(\eta^*\right))\right) \tag{21}$$

$$\leqslant F\left(\max\left\{\begin{array}{l}\Omega^{-1}\left(\Omega\left(\acute{S}(\eta^*,\eta^*,g\left(\eta^*\right))\right)\right),\\[2ex]\Omega^{-1}\left(\frac{0+0+0}{3}\right),\\[2ex]\frac{1}{3}\Omega^{-1}\left(\frac{1}{2}\left(\begin{array}{l}\Omega\left(\acute{S}(\eta^*,\eta^*,g\left(\eta^*\right))\right)\\+\Omega\left(\acute{S}(\eta^*,\eta^*,g\left(\eta^*\right))\right)\\+\Omega\left(\acute{S}(g\left(\eta^*\right),g\left(\eta^*\right),\eta^*)\right)\\+\Omega\left(\acute{S}(g\left(\eta^*\right),g\left(\eta^*\right),\eta^*)\right)\end{array}\right)\right)\end{array}\right\}\right)-\tau$$

$$\leqslant F\left(\max\left\{\begin{array}{l}\acute{S}(\eta^*,\eta^*,g\left(\eta^*\right)),\\\frac{1}{3}\Omega^{-1}\left(\Omega\left(2\acute{S}(\eta^*,\eta^*,g\left(\eta^*\right))\right)\right)\end{array}\right\}\right)-\tau$$

$$=\leqslant F\left(\max\left\{\begin{array}{l}\acute{S}(\eta^*,\eta^*,g\left(\eta^*\right)),\\\frac{2}{3}\acute{S}(\eta^*,\eta^*,g\left(\eta^*\right))\end{array}\right\}\right)-\tau$$

$$=F\left(\acute{S}(\eta^*,\eta^*,g\left(\eta^*\right))\right)-\tau,$$

a contradiction. Hence, $\eta^*=R\left(\eta^*\right)=g\left(\eta^*\right)$, that is, η^* is a common fixed point of R and g. Since, $\eta^*=g\left(\eta^*\right)\in g\left(\mathscr{A}\right)\subseteq S\left(\mathscr{A}\right)$, then there exists $\upsilon\in\mathscr{A}$ such that

$S(\upsilon) = \eta^*$. we show that $f(\upsilon) = \eta^*$,

$$F\left(\Omega\left(\acute{S}(f(\upsilon), f(\upsilon), g(\zeta_{2n+1}))\right)\right)$$
$$\leqslant F\left(E\left(\zeta_n, \eta_n, \upsilon\right)\right)$$
$$-D\begin{pmatrix} \acute{S}(S(\upsilon), S(\upsilon), g(\zeta_{2n+1})), \\ \acute{S}(S(\upsilon), S(\upsilon), g(\zeta_{2n+1})), \\ \acute{S}(R(\zeta_{2n+1}), R(\zeta_{2n+1}), f(\upsilon)), \\ \acute{S}(R(\zeta_{2n+1}), R(\zeta_{2n+1}), f(\upsilon)) \end{pmatrix}, \tag{22}$$

where,

$$E(\zeta_n, \eta_n, \upsilon) = \max\left\{\begin{array}{l} \Omega^{-1}\left(\acute{S}(S(\upsilon), S(\upsilon), R(\zeta_{2n+1}))\right), \\ \Omega^{-1}\left(\frac{1}{3}\begin{pmatrix} \acute{S}(S(\upsilon), S(\upsilon), f(\upsilon)) \\ +\acute{S}(S(\upsilon), S(\upsilon), f(\upsilon)) \\ +\acute{S}(R(\zeta_{2n+1}), R(\zeta_{2n+1}), g(\zeta_{2n+1})) \end{pmatrix}\right), \\ \frac{1}{3}\Omega^{-1}\left(\frac{1}{2}\begin{pmatrix} 2\acute{S}(S(\upsilon), S(\upsilon), g(\zeta_{2n+1})) \\ +2\acute{S}(R(\zeta_{2n+1}), R(\zeta_{2n+1}), f(\upsilon)) \end{pmatrix}\right) \end{array}\right\}.$$

Thus, by the definition of mapping D, inequality (22) turns into the following:

$$F\left(\Omega\left(\acute{S}(f(\upsilon), f(\upsilon), g(\zeta_{2n+1}))\right)\right) \tag{23}$$
$$\leqslant F\left(\max\left\{\begin{array}{l} \Omega^{-1}\left(\acute{S}(S(\upsilon), S(\upsilon), R(\zeta_{2n+1}))\right), \\ \Omega^{-1}\left(\frac{1}{3}\begin{pmatrix} \acute{S}(S(\upsilon), S(\upsilon), f(\upsilon)) \\ +\acute{S}(S(\upsilon), S(\upsilon), f(\upsilon)) \\ +\acute{S}(R(\zeta_{2n+1}), R(\zeta_{2n+1}), g(\zeta_{2n+1})) \end{pmatrix}\right), \\ \frac{1}{3}\Omega^{-1}\left(\frac{1}{2}\begin{pmatrix} 2\acute{S}(S(\upsilon), S(\upsilon), g(\zeta_{2n+1})) \\ +2\acute{S}(R(\zeta_{2n+1}), R(\zeta_{2n+1}), f(\upsilon)) \end{pmatrix}\right) \end{array}\right\}\right) - \tau.$$

Similarly, as, Ω and F are continuous, so by taking upper limit as $n \to \infty$ in (23), and applying Lemma 1, we have

$$F\left(\acute{S}(f(\upsilon), f(\upsilon), \eta^*)\right)$$
$$\leqslant F\left(\max\left\{\begin{array}{l} 0, \\ \frac{1}{3}\left(2\acute{S}(\eta^*, \eta^*, f(\upsilon))\right), \\ \frac{1}{3}\Omega^{-1}\left(\Omega\left(\acute{S}(\eta^*, \eta^*, f(\upsilon))\right)\right) \end{array}\right\}\right) - \tau$$
$$\leqslant F\left(\frac{1}{3}\left(2\acute{S}(f(\upsilon), f(\upsilon), \eta^*)\right)\right).$$

Hence, $f(\upsilon) = \eta^*$. By the weak compatibility of the pair (f, g), we have $S(f(\upsilon)) = f(S(\upsilon))$. Therefore, $f(\eta^*) = S(\eta^*)$. We prove that $f(\eta^*) = \eta^*$. If $f(\eta^*) \neq \eta^*$, we

get,

$$
\begin{aligned}
& F\left(\Omega\left(\acute{S}(f(\eta^*),f(\eta^*),g(\eta^*))\right)\right) \\
& \leqslant F\left(E\left(\zeta_n,\eta_n,\eta^*\right)\right) \\
& \quad -D\begin{pmatrix}
\acute{S}(\acute{S}(S(\eta^*),S(\eta^*),g(\eta^*)), \\
\acute{S}(\acute{S}(S(\eta^*),S(\eta^*),g(\eta^*)), \\
\acute{S}(R(\eta^*),R(\eta^*),f(\eta^*)), \\
\acute{S}(R(\eta^*),R(\eta^*),f(\eta^*))
\end{pmatrix},
\end{aligned}
\tag{24}
$$

where,

$$
E\left(\zeta_n,\eta_n,\eta^*\right)=\max\left\{
\begin{array}{l}
\Omega^{-1}\left(\acute{S}(S(\eta^*),S(\eta^*),R(\eta^*))\right), \\[4pt]
\Omega^{-1}\left(\dfrac{1}{3}\begin{pmatrix}
\acute{S}(S(\eta^*),S(\eta^*),f(\eta^*)) \\
+\acute{S}(S(\eta^*),S(\eta^*),f(\eta^*)) \\
+\acute{S}(R(\eta^*),R(\eta^*),g(\eta^*))
\end{pmatrix}\right), \\[14pt]
\dfrac{1}{3}\Omega^{-1}\left(\dfrac{1}{2}\begin{pmatrix}
2\acute{S}(S(\eta^*),S(\eta^*),g(\eta^*)) \\
+2\acute{S}(R(\eta^*),R(\eta^*),f(\eta^*))
\end{pmatrix}\right)
\end{array}
\right\}.
$$

Then,

$$
F\left(\acute{S}(f(\eta^*),f(\eta^*),\eta^*)\right)
\tag{25}
$$
$$
\leqslant F\left(\max\left\{
\begin{array}{l}
\Omega^{-1}\left(\acute{S}(f(\eta^*),f(\eta^*),\eta^*)\right), \\
0, \\
\frac{1}{3}\Omega^{-1}\left(2\acute{S}(f(\eta^*),f(\eta^*),\eta^*)\right)
\end{array}
\right\}\right)-\tau
$$
$$
\leqslant F\left(\acute{S}(f(\eta^*),f(\eta^*),\eta^*)\right)-\tau,
$$

which is a contradiction. Hence, $f(\eta^*)=S(\eta^*)=\eta^*$, that is, η^* is a common fixed point of f,S. Thus,

$$
f(\eta^*)=S(\eta^*)=g(\eta^*)=R(\eta^*)=\eta^*.
$$

Therefore, η^* is a common fixed point of f,g,R and S. Next, assume that υ^* is another common fixed point of f,g,R and S. If $\acute{S}(\eta^*,\eta^*,\upsilon^*)>0$, then,

$$
\begin{aligned}
F\left(\acute{S}(\eta^*,\eta^*,\upsilon^*)\right)&=F\left(\Omega\left(\acute{S}(f(\eta^*),f(\eta^*),g(\upsilon^*))\right)\right) \\
&\leqslant F\left(E\left(\zeta_n,\eta_n,\upsilon^*\right)\right) \\
&\quad -D\begin{pmatrix}
\acute{S}\acute{S}(S(\eta^*),S(\eta^*),g(\upsilon^*)), \\
\acute{S}(S(\eta^*),S(\eta^*),g(\upsilon^*)), \\
\acute{S}(R(\upsilon^*),R(\upsilon^*),f(\eta^*)), \\
\acute{S}(R(\upsilon^*),R(\upsilon^*),f(\eta^*))
\end{pmatrix},
\end{aligned}
$$

where,

$$E\left(\zeta,\eta,\sigma\right)=\max\left\{\begin{array}{l}\Omega^{-1}\left(\acute{S}(S(\eta^*),S(\eta^*),R(\upsilon^*))\right),\\[2mm]\Omega^{-1}\left(\frac{1}{3}\left(\begin{array}{l}\acute{S}(S(\eta^*),S(\eta^*),f(\eta^*))\\+\acute{S}(S(\eta^*),S(\eta^*),f(\eta^*))\\+\acute{S}(R(\upsilon^*),R(\upsilon^*),g(\upsilon^*))\end{array}\right)\right),\\[4mm]\frac{1}{3}\Omega^{-1}\left(\frac{1}{2}\left(\begin{array}{l}2\acute{S}(S(\eta^*),S(\eta^*),g(\upsilon^*))\\+2\acute{S}(R(\upsilon^*),R(\upsilon^*),f(\eta^*))\end{array}\right)\right)\end{array}\right\}.$$

Thus,

$$F\left(\acute{S}(\eta^*,\eta^*,\upsilon^*)\right)$$

$$\leqslant F\left(\max\left\{\begin{array}{l}\Omega^{-1}\left(\acute{S}(S(\eta^*),S(\eta^*),R(\upsilon^*))\right),\\[2mm]\Omega^{-1}\left(\frac{1}{3}\left(\begin{array}{l}\acute{S}(S(\eta^*),S(\eta^*),f(\eta^*))\\+\acute{S}(S(\eta^*),S(\eta^*),f(\eta^*))\\+\acute{S}(R(\upsilon^*),R(\upsilon^*),g(\upsilon^*))\end{array}\right)\right),\\[4mm]\frac{1}{3}\Omega^{-1}\left(\frac{1}{2}\left(\begin{array}{l}2\acute{S}(S(\eta^*),S(\eta^*),g(\upsilon^*))\\+2\acute{S}(R(\upsilon^*),R(\upsilon^*),f(\eta^*))\end{array}\right)\right)\end{array}\right\}\right)-\tau$$

$$\leqslant F\left(\acute{S}(\eta^*,\eta^*,\upsilon^*)\right)-\tau,$$

a contradiction. Therefore, $\eta^* = \upsilon^*$ is a unique common fixed point of f, g, R and S. $\square$

COROLLARY 1. *Let* $(\mathscr{A},\S)$ *be a complete symmetric* $\S_b$- *metric space with* $b \geqslant 1$ *and* $f,g,R,S : \mathscr{A} \to \mathscr{A}$ *be four self-mappings with* $\Omega(x+y) \geqslant \Omega(x) + \Omega(y)$, *for each* $x,y \in [0,\infty)$. *Moreover, if*

(i) $f(\mathscr{A}) \subseteq R(\mathscr{A})$, $g(\mathscr{A}) \subseteq S(\mathscr{A})$ *and either* $R(\mathscr{A})$ *or* $S(\mathscr{A})$ *is a closed subset of* $\mathscr{A}$;

(ii) *for* $\zeta,\eta,\sigma \in \mathscr{A}$, *there exist* $F \in \mathscr{F}$, $b \geqslant 1$ *and* $D \in \Delta_D$ *such that*

$$\S(f(\zeta),f(\eta),g(\sigma)) > 0$$
$$\Rightarrow F\left(b^2\S(f(\zeta),f(\eta),g(\sigma))\right) \leqslant F\left(E(\zeta,\eta,\sigma)\right)$$
$$-D\left(\begin{array}{l}\S(S(\zeta),S(\zeta),g(\sigma)),\S(S(\eta),S(\eta),g(\sigma)),\\\S(R(\sigma),R(\sigma),f(\zeta)),\S(R(\sigma),R(\sigma),f(\eta))\end{array}\right),$$

where,

$$E\left(\zeta,\eta,\sigma\right)=\max\left\{\begin{array}{l}\mathcal{S}(S\left(\zeta\right),S\left(\eta\right),R\left(\sigma\right)),\\[2mm]\dfrac{1}{3}\left(\begin{array}{l}\mathcal{S}(S\left(\zeta\right),S\left(\zeta\right),f\left(\zeta\right))\\+\mathcal{S}(S\left(\eta\right),S\left(\eta\right),f\left(\eta\right))\\+\mathcal{S}(R\left(\sigma\right),R\left(\sigma\right),g\left(\sigma\right))\end{array}\right),\\[6mm]\dfrac{1}{6}\left(\begin{array}{l}\mathcal{S}(S\left(\zeta\right),S\left(\zeta\right),g\left(\sigma\right))\\+\mathcal{S}(S\left(\eta\right),S\left(\eta\right),g\left(\sigma\right))\\+\mathcal{S}(R\left(\sigma\right),R\left(\sigma\right),f\left(\zeta\right))+\mathcal{S}(R\left(\sigma\right),R\left(\sigma\right),f\left(\eta\right))\end{array}\right)\end{array}\right\};$$

(iii) the pair (f,S) and (g,R) are weak compatible.

Then f,g,R and S have a unique common fixed point in $\mathscr{A}$.

Proof. Take, $\Omega(t)=bt$ $(b\geqslant 1)$ in Theorem 2 □

COROLLARY 2. *Let $\left(\mathscr{A},\acute{S}\right)$ be a complete symmetric $\acute{S}$-metric space and $f,g:\mathscr{A}\to\mathscr{A}$ be two self-mappings with $\Omega(x+y)\geqslant\Omega(x)+\Omega(y)$, for each $x,y\in[0,\infty)$. Moreover, if*

(i) for $\zeta,\eta,\sigma\in\mathscr{A}$, there exist $F\in\mathscr{F}$ and $D\in\Delta_D$ such that

$$\Omega\left(\acute{S}(f\left(\zeta\right),f\left(\eta\right),g\left(\sigma\right))\right)>0$$
$$\Rightarrow F\left(\Omega\left(\acute{S}(f\left(\zeta\right),f\left(\eta\right),g\left(\sigma\right))\right)\right)\leqslant F\left(E\left(\zeta,\eta,\sigma\right)\right)$$
$$-D\left(\begin{array}{l}\acute{S}(\zeta,\zeta,g\left(\sigma\right)),\acute{S}(\eta,\eta,g\left(\sigma\right)),\\\acute{S}(\sigma,\sigma,f\left(\zeta\right)),\acute{S}(\sigma,\sigma,f\left(\eta\right))\end{array}\right),$$

where,

$$E\left(\zeta,\eta,\sigma\right)=\max\left\{\begin{array}{l}\Omega^{-1}\left(\acute{S}(\zeta,\eta,\sigma)\right),\\[2mm]\Omega^{-1}\left(\dfrac{1}{3}\left(\begin{array}{l}\acute{S}(\zeta,\zeta,g\left(\zeta\right))+\acute{S}(\eta,\eta,g\left(\eta\right))\\+\acute{S}(\sigma,\sigma,g\left(\sigma\right))\end{array}\right)\right),\\[6mm]\dfrac{1}{3}\Omega^{-1}\left(\dfrac{1}{2}\left(\begin{array}{l}\acute{S}(\zeta,\zeta,g\left(\sigma\right))+\acute{S}(\eta,\eta,g\left(\sigma\right))\\+\acute{S}(\sigma,\sigma,f\left(\zeta\right))+\acute{S}(\sigma,\sigma,f\left(\eta\right))\end{array}\right)\right)\end{array}\right\}.$$

Then f and g admit a unique common fixed point in $\mathscr{A}$.

COROLLARY 3. *Let $\left(\mathscr{A},\acute{S}\right)$ be a complete symmetric $\acute{S}$-metric space and $g,R:\mathscr{A}\to\mathscr{A}$ be two self-mappings with $\Omega(x+y)\geqslant\Omega(x)+\Omega(y)$, for each $x,y\in[0,\infty)$. Moreover, if*

(i) $g\left(\mathscr{A}\right)\subseteq R\left(\mathscr{A}\right)$ and $R\left(\mathscr{A}\right)$ is a closed subset of $\mathscr{A}$;

(ii) for $\zeta, \eta, \sigma \in \mathscr{A}$, there exist $F \in \mathscr{F}$ and $D \in \Delta_D$ such that

$$\Omega\left(\acute{S}(g(\zeta), g(\eta), g(\sigma))\right) > 0$$
$$\Rightarrow F\left(\Omega\left(\acute{S}(g(\zeta), g(\eta), g(\sigma)))\right)\right) \leqslant F\left(E(\zeta, \eta, \sigma)\right)$$
$$-D\left(\begin{matrix} \acute{S}(R(\zeta), R(\zeta), g(\sigma)), \acute{S}(R(\eta), R(\eta), g(\sigma)), \\ \acute{S}(R(\sigma), R(\sigma), g(\zeta)), \acute{S}(R(\sigma), R(\sigma), g(\eta)) \end{matrix}\right),$$

where,

$$E(\zeta, \eta, \sigma) = \max \left\{ \begin{matrix} \Omega^{-1}\left(\acute{S}(R(\zeta), R(\eta), R(\sigma))\right), \\ \\ \Omega^{-1}\left(\frac{1}{3}\left(\begin{matrix} \acute{S}(R(\zeta), R(\zeta), g(\zeta)) \\ +\acute{S}(R(\eta), R(\eta), g(\eta)) \\ +\acute{S}(R(\sigma), R(\sigma), g(\sigma)) \end{matrix}\right)\right), \\ \\ \frac{1}{3}\Omega^{-1}\left(\frac{1}{2}\left(\begin{matrix} \acute{S}(R(\zeta), R(\zeta), g(\sigma)) \\ +\acute{S}(R(\eta), R(\eta), g(\sigma)) \\ +\acute{S}(R(\sigma), R(\sigma), g(\zeta)) \\ +\acute{S}(R(\sigma), R(\sigma), g(\eta)) \end{matrix}\right)\right) \end{matrix} \right\};$$

(iii) the pair (g, R) are weak compatible.

Then g and R admit a unique common fixed point in $\mathscr{A}$.

COROLLARY 4. *Let $\left(\mathscr{A}, \acute{S}\right)$ be a complete symmetric $\acute{S}$-metric space and $S : \mathscr{A} \to \mathscr{A}$ be a self-mapping with $\Omega(x+y) \geqslant \Omega(x) + \Omega(y)$, for each $x, y \in [0, \infty)$. Moreover if*

(i) for $\zeta, \eta, \sigma \in \mathscr{A}$, there exist $F \in \mathscr{F}$ and $D \in \Delta_D$ such that

$$\Omega\left(\acute{S}(S(\zeta), S(\eta), S(\sigma))\right) > 0$$
$$\Rightarrow F\left(\Omega\left(\acute{S}(S(\zeta), S(\eta), S(\sigma)))\right)\right) \leqslant F\left(E(\zeta, \eta, \sigma)\right)$$
$$-D\left(\begin{matrix} \acute{S}(\zeta, \zeta, S(\sigma)), \acute{S}(\eta, \eta, S(\sigma)), \\ \acute{S}(\sigma, \sigma, S(\zeta)), \acute{S}(\sigma, \sigma, S(\eta)) \end{matrix}\right),$$

where,

$$E(\zeta, \eta, \sigma) = \max \left\{ \begin{matrix} \Omega^{-1}\left(\acute{S}(\zeta, \eta, \sigma)\right), \\ \\ \Omega^{-1}\left(\frac{1}{3}\left(\begin{matrix} \acute{S}(\zeta, \zeta, S(\zeta)) + \acute{S}(\eta, \eta, S(\eta)) \\ +\acute{S}(\sigma, \sigma, S(\sigma)) \end{matrix}\right)\right), \\ \\ \frac{1}{3}\Omega^{-1}\left(\frac{1}{2}\left(\begin{matrix} \acute{S}(\zeta, \zeta, S(\sigma)) + \acute{S}(\eta, \eta, S(\sigma)) \\ +\acute{S}(\sigma, \sigma, S(\zeta)) + \acute{S}(\sigma, \sigma, S(\eta)) \end{matrix}\right)\right) \end{matrix} \right\}.$$

Then S admits a unique fixed point in $\mathscr{A}$.

COROLLARY 5. *Let $(\mathscr{A},\acute{S})$ be a complete symmetric $\acute{S}$-metric space and $f,g,R,S:$ $\mathscr{A} \to \mathscr{A}$ be four self-mappings with $\Omega(x+y) \geqslant \Omega(x)+\Omega(y)$, for each $x,y \in [0,\infty)$. Moreover, if*

(i) *$f(\mathscr{A}) \subseteq R(\mathscr{A})$, $g(\mathscr{A}) \subseteq S(\mathscr{A})$ and either $R(\mathscr{A})$ or $S(\mathscr{A})$ is a closed subset of $\mathscr{A}$;*

(ii) *for $\zeta,\eta,\sigma \in \mathscr{A}$, $\exists \, \lambda \in (0,1)$ such that*

$$\Omega\big(\acute{S}(f(\zeta),f(\eta),g(\sigma))\big)$$
$$\leqslant \lambda \max \begin{cases} \Omega^{-1}\big(\acute{S}(S(\zeta),S(\eta),R(\sigma))\big), \\[2mm] \Omega^{-1}\left(\frac{1}{3}\begin{pmatrix} \acute{S}(S(\zeta),S(\zeta),f(\zeta)) \\ +\acute{S}(S(\eta),S(\eta),f(\eta)) \\ +\acute{S}(R(\sigma),R(\sigma),g(\sigma)) \end{pmatrix}\right), \\[5mm] \frac{1}{3}\Omega^{-1}\left(\frac{1}{2}\begin{pmatrix} \acute{S}(S(\zeta),S(\zeta),g(\sigma))+\acute{S}(S(\eta),S(\eta),g(\sigma)) \\ +\acute{S}(R(\sigma),R(\sigma),f(\zeta))+\acute{S}(R(\sigma),R(\sigma),f(\eta)) \end{pmatrix}\right) \end{cases};$$

(iii) *the pair (f,S) and (g,R) are weak compatible.*

Then f,g,R and S admit a unique common fixed point in $\mathscr{A}$.

EXAMPLE 7. Let $\mathscr{A} = l^P(0,1)$, $P > 1$ and

$$\|k\|_P = \left(\sum_{i=0}^{\infty} |k_i(t)|^P\right)^{\frac{1}{P}}, \text{ for } t \in (0,\infty).$$

Define $\acute{S}: \mathscr{A} \times \mathscr{A} \times \mathscr{A} \to [0,+\infty)$ by $\acute{S}(k,q,l) = \|k-l\|_P + \|q-l\|_P$, for all $k,q,l \in \mathscr{A}$ and $\Omega(t) = t$. Then, $(\mathscr{A},\acute{S})$ is a complete $\acute{S}$-metric space and it is symmetric. Define $f,S,g,R: \mathscr{A} \to \mathscr{A}$ by

$$f(k(t)) = \left[\frac{k(t)}{2}\right]^{16},$$

$$S(k(t)) = \left[\frac{k(t)}{2}\right]^{8},$$

$$g(k(\zeta)) = \left[\frac{k(t)}{2}\right]^{8},$$

$$R(k(t)) = \left[\frac{k(t)}{2}\right]^{4},$$

where, $\|k(t)\|_P \leqslant \frac{17}{10}$. Clearly, $f(\mathscr{A}) \subseteq R(\mathscr{A})$ and $g(\mathscr{A}) \subseteq S(\mathscr{A})$. We can easily, see that, $\{f,S\}$ is a weak compatible, if $\{k_n\}$ is a sequence in $\mathscr{A}$ such that,

$$\lim_{n\to\infty} f(k_n) = \lim_{n\to\infty} S(k_n) = t, \text{ for some } t \in \mathscr{A},$$

then,

$$\lim_{n\to\infty} |f(k_n) - t| = \lim_{n\to\infty} |S(k_n) - t| = 0,$$

and equivalently,

$$\lim_{n\to\infty} \left| \left[\frac{\eta_n(t)}{2} \right]^{16} - t \right| = \lim_{n\to\infty} \left| \left[\frac{\eta_n(t)}{2} \right]^{8} - t \right| = 0.$$

Thus,

$$\lim_{n\to\infty} \left| [k_n(t)]^{16} - 2^{16}t \right| = \lim_{n\to\infty} \left| [k_n(t)]^{8} - 2^{8}t \right| = 0.$$

By uniqueness of limit, we have $t^{\frac{1}{16}} = t^{\frac{1}{16}}$, hence, $t = 0, 1$. Using continuity of f and S, one get,

$$\lim_{n\to\infty} \|fS(k_n) - Sf(k_n)\|_P = \lim_{n\to\infty} \left(\sum_{n=0}^{\infty} |fS(k_n(t)) - Sf(k_n(t))|^P \right)^{\frac{1}{P}},$$

$$\left(\sum_{0}^{\infty} |f(t) - S(t)|^P \right)^{\frac{1}{P}} = \left(\sum_{0}^{\infty} |0 - 0|^P \right)^{\frac{1}{P}} = 0, \text{ for } t = 0 \in \mathscr{A}.$$

Hence,

$$\lim_{n\to\infty} \acute{S}(fS(k_n), fS(k_n), Sf(k_n))$$
$$= \lim_{n\to\infty} \|fS(k_n) - Sf(k_n)\|_P + \lim_{n\to\infty} \|fS(k_n) - Sf(k_n)\|_P = 0.$$

Similarly, the pair $\{g, R\}$ is weak compatible. Define the mapping $F : (0, \infty) \to (-\infty, \infty)$ by,

$$F(\alpha) = \ln(\alpha), \text{ for } \alpha > 0, \text{ and } D(t_1, t_2, t_3, t_4) = \ln\left(\frac{1}{r} \right), 0 < r < 1.$$

Now, from

$$\|f(k) - g(k)\|_P = \left\| \left[\frac{k(t)}{2} \right]^{16} - \left[\frac{l(t)}{2} \right]^{8} \right\|_P$$

$$= \left\| \left[\frac{k(t)}{2} \right]^{8} + \left[\frac{l(t)}{2} \right]^{4} \right\|_P \left\| \left[\frac{k(t)}{2} \right]^{8} - \left[\frac{l(t)}{2} \right]^{4} \right\|_P$$

$$= \left\| \left[\frac{k(t)}{2} \right]^{8} + \left[\frac{l(t)}{2} \right]^{4} \right\|_P \|S(k) - R(l)\|_P$$

$$\leqslant r \|S(k) - R(k)\|_P,$$

where, $r = \left\| \left[\frac{k(t)}{2} \right]^{8} + \left[\frac{l(t)}{2} \right]^{4} \right\|_P < 1$, (since $\|k(t)\|_P \leqslant \frac{17}{10}$ and $\|l(t)\|_P \leqslant \frac{17}{10}$). We can get,

$$\|f(k) - g(l)\|_P \leqslant r \|S(k) - R(l)\|_P$$

and
$$\|f(q) - g(l)\|_P \leqslant r\|S(q) - R(l)\|_P,$$

so, for all $k, q, l \in \mathscr{A}$,

$$
\begin{aligned}
\acute{S}(f(k), f(q), g(l)) &= \|f(k) - g(l)\|_P + \|f(q) - g(l)\|_P \\
&\leqslant r\|S(k) - R(l)\|_P + r\|S(q) - R(l)\|_P \\
&\leqslant r[\|S(k) - R(l)\|_P + \|S(q) - R(l)\|_P] \\
&= r\acute{S}(S(k), S(q), R(l)) \\
&\leqslant r\max \left\{
\begin{array}{l}
\acute{S}(S(k), S(q), R(l)), \\[4pt]
\frac{1}{3}\left(\begin{array}{l} \acute{S}(S(k), S(k), g(k)) + \acute{S}(S(q), S(q), f(q)) \\ +\acute{S}(R(l), R(l), g(l)) \end{array}\right), \\[10pt]
\frac{1}{3}\frac{1}{2}\left(\begin{array}{l} \acute{S}(S(k), S(k), g(l)) + \acute{S}(S(q), S(q), g(l)) \\ +\acute{S}(R(l), R(l), f(k)) + \acute{S}(R(l), R(l), S(q)) \end{array}\right)
\end{array}
\right\}.
\end{aligned}
$$

This implies,

$$
\begin{aligned}
&F\left(\acute{S}(f(k), f(q), g(l))\right) \\
&\leqslant F\left(\max \left\{
\begin{array}{l}
\acute{S}(S(k), S(q), R(l)), \\[4pt]
\frac{1}{3}\left(\begin{array}{l} \acute{S}(S(k), S(k), g(k)) + \acute{S}(S(q), S(q), f(q)) \\ +\acute{S}(R(l), R(l), g(l)) \end{array}\right), \\[10pt]
\frac{1}{3}\frac{1}{2}\left(\begin{array}{l} \acute{S}(S(k), S(k), g(l)) \\ +\acute{S}(S(q), S(q), g(l)) + \acute{S}(R(l), R(l), f(k)) \\ +\acute{S}(R(l), R(l), S(q)) \end{array}\right)
\end{array}
\right\}\right) \\
&\quad -\ln\left(\frac{1}{r}\right).
\end{aligned}
$$

Hence all the hypotheses of Theorem 2 are satisfied. Thus, f, S, g and R have a unique common fixed point.

COROLLARY 6. *Let $(\mathscr{A}, \acute{S})$ be a complete symmetric $\acute{S}$-metric space and $f, g, R, S:$ $\mathscr{A} \to \mathscr{A}$ be four self-mappings with $\Omega(x + y) \geqslant \Omega(x) + \Omega(y)$, for each $x, y \in [0, \infty)$. Moreover, if*

(i) *$f(\mathscr{A}) \subseteq R(\mathscr{A})$, $g(\mathscr{A}) \subseteq S(\mathscr{A})$ and either $R(\mathscr{A})$ or $S(\mathscr{A})$ is a closed subset of $\mathscr{A}$;*

(ii) *for $\zeta, \eta, \sigma \in \mathscr{A}$, there exist $F \in \mathscr{F}$ and $\tau > 0$ such that*

$$
\begin{aligned}
&\Omega\left(\acute{S}(f(\zeta), f(\eta), g(\sigma))\right) > 0 \\
&\Rightarrow \tau + F\left(\Omega\left(\acute{S}(f(\zeta), f(\eta), g(\sigma))\right)\right) \leqslant F\left(\Omega^{-1}\left(\acute{S}(S(\zeta), S(\eta), R(\sigma))\right)\right);
\end{aligned}
$$

(iii) the pair (f,S) and (g,R) are weak compatible.

Then f,g,R and S admit a unique common fixed point in $\mathscr{A}$.

4. Application to the system of Volterra type integral equations

Let $I = [0,K] \subset \mathbb{R}$ be a closed and bounded interval and $K > 0$. We consider the system of Volterra type integral equations:

$$y(t) = q(t) + \int_0^t K_1(t,s,y(s))ds, \tag{26}$$

$$y(t) = q(t) + \int_0^t K_2(t,s,y(s))ds, \tag{27}$$

$$y(t) = q(t) + \int_0^t K_3(t,s,y(s))ds, \tag{28}$$

$$y(t) = q(t) + \int_0^t K_4(t,s,y(s))ds, \tag{29}$$

where, $t \in I$, and $K_i : I \times I \times \mathbb{R} \to \mathbb{R}$, $i \in \{1,2,3,4\}$ and $q : \mathbb{R} \to \mathbb{R}$ are continuous mappings. Let $\mathscr{A} = C(I,\mathbb{R})$ be the set of real continuous mappings defined on I and

$$Q_i y(t) = q(t) + \int_0^t K_i(t,s,y(s))ds,$$

for all $t \in I$, $y \in C(I,\mathbb{R})$ and $i \in \{1,2,3,4\}$. Obviously, y is a solution of system (26)–(29) if and only if it is a common fixed point of Q_i for $i \in \{1,2,3,4\}$.

THEOREM 3. *Assume that the following conditions hold:*

(a) $|K(t,s,y(s)) - K(t,s,w(s))| \leqslant |Q_4(y(s)) - Q_3(w(s))|$, *for all $s \in [0,K]$ and $y,w \in C(I,\mathbb{R})$;*

(b) *there exists a sequence $\{\eta_n\}$ in $\mathscr{A}$ such that,*

$$\lim_{n\to\infty} f(S(\eta_n)) = \lim_{n\to\infty} S(f(\eta_n)) \text{ and } \lim_{n\to\infty} g(R(\eta_n)) = \lim_{n\to\infty} R(g(\eta_n)),$$

whenever,

$$\lim_{n\to\infty} f(\eta_n) = \lim_{n\to\infty} S(\eta_n) = t$$
$$\text{and } \lim_{n\to\infty} g(\eta_n) = \lim_{n\to\infty} R(\eta_n) = t,$$

for some $t \in \mathscr{A}$;

(c) *for every $t \in I$ and $\tau \geqslant 1$ is taken arbitrary,*

$$\sup_{t\in I} \int_0^t ds < e^{-\tau}.$$

Then system (26)–(29) of integral equations has a solution $y^* \in C(I,\mathbb{R})$.

Proof. We note that $\zeta \in \mathscr{A} = C(I, \mathbb{R})$. Define $\|\zeta\| = \sup_{t \in I} \{|\zeta(t)|\}$. $\|.\|_{\tau}$ is a norm equivalent to the superimum norm and $(\mathscr{A}, \|.\|)$ is a Banach space. The metric induced by this norm is given by

$$d(\zeta, \eta) = \|\zeta - \eta\| = \sup_{t \in I} \{|\zeta(t) - \eta(t)|\}, \text{ for all } \zeta, \eta \in \mathscr{A}.$$

Define the Ś-metric on $\mathscr{A}$ by $\acute{S}(\zeta, \eta, \delta) = \|\zeta - \delta\| + \|\eta - \delta\|$, for all $\zeta, \eta, \delta \in \mathscr{A}$. Clearly, $(\mathscr{A}, \acute{S})$ is a complete Ś-metric space. Now, let $y(t), v(t) \in \mathscr{A}$. Then, we have,

$$
\begin{aligned}
|S_1 y(t) - S_2 v(t)| &= \left| \int_0^t K_1(t, s, y(s)) ds - \int_0^t K_2(t, s, v(s)) ds \right| \\
&\leqslant \int_0^t |K_1(t, s, y(s)) - K_2(t, s, v(s))| \, ds \\
&\leqslant \int_0^t |S_4 y(t)) - S_3 v(t))| \, ds \\
&\leqslant \int_0^t \sup_{s \in I} |S_4 y(s)) - S_3 v(s))| \, ds \\
&= \sup_{t \in I} |S_4 y(t) - S_3 v(t)| \int_0^t ds \\
&= \sup_{t \in I} |S_4 y(t) - S_3 v(t)| e^{-2\tau}.
\end{aligned}
$$

Hence,

$$\max_{t \in I} |S_1 y(t) - S_2 v(t)| \leqslant \max_{t \in I} |S_4 y(t) - S_3 v(t)| e^{-\tau}. \tag{30}$$

Similarly, we have,

$$\max_{t \in I} |S_1 w(t) - S_2 v(t)| \leqslant \max_{t \in I} |S_4 w(t) - S_3 v(t)| e^{-\tau}. \tag{31}$$

Therefore, from (30) and (31), we get,

$$\max_{t \in I} |S_1 y(t) - S_2 v(t)| + \max_{t \in I} |S_1 w(t) - S_2 v(t)|$$
$$\leqslant e^{-\tau} \left[\max_{t \in I} |S_4 y(t)) - S_3 v(t))| + \max_{t \in I} |S_4 w(t)) - S_3 v(t))| \right].$$

This implies,

$$\acute{S}(S_1 y(t), S_1 w(t), S_2 v(t))$$
$$\leqslant e^{-\tau} \acute{S}(S_4 y(t), S_4 w(t), S_3 v(t)).$$

A slight modification gives

$$e^{\acute{S}(S_1 y(t), S_1 w(t), S_2 v(t))} - 1 \leqslant e^{-\tau} \ln \left(\acute{S}(S_4 y(t), S_4 w(t), S_3 v(t)) + 1 \right)$$

and

$$\Omega\left(\acute{S}(S_1 y(t), S_1 w(t), S_2 v(t))\right) \leqslant e^{-\tau}\Omega^{-1}\left(\acute{S}(S_4 y(t), S_4 w(t), S_3 v(t))\right).$$

Thus,

$$\tau + \ln\left[\Omega\left(\acute{S}(S_1 y(t), S_1 w(t), S_2 v(t))\right)\right]$$
$$\leqslant \ln\left[\Omega^{-1}\left(\acute{S}(S_4 y(t), S_4 w(t), S_3 v(t))\right)\right].$$

Putting, $S_1 = f$, $S_2 = g$, $S_3 = R$ and $S_4 = S$. Then, all the hypotheses of Corollary 6 are satisfied for $F(r) = \ln(r)$, $\tau > 0$ and $\Omega(t) = e^t - 1$ (so, $\Omega^{-1}(t) = \ln(t+1)$). Therefore, f, g, R and S have a common fixed point $y^* \in C(I, \mathbb{R})$; that is, $y^* \in C(I, \mathbb{R})$ is a solution of system (26–29). □

5. Conclusion

This paper suggests a solution for a new common fixed point problem addressing generalized (F, Ω)-contraction in $\acute{S}$-complete metric spaces. This paper produces a new fixed point method for the existence of a solution for a system of integral equations. The new problem can lead to further investigations and applications.

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Muhammad Nazam
Department of Mathematics
Allama Iqbal Open University
H-8, Islamabad, Pakistan
e-mail: muhammad.nazam@aiou.edu.pk

Eskandar Ameer
Department of Mathematics
International Islamic University
H-10, Islamabad – 44000, Pakistan
and
Department of Mathematics
Taiz University
Taiz, Yemen
e-mail: eskandarameer@gmail.com

Mohammad Mursaleen
China Medical University Hospital
China Medical University
Taichung, Taiwan
and
Department of Mathematics
Aligarh Muslim University
Aligarh – 202002, India
e-mail: mursaleenm@gmail.com

Özlem Acar
Department of Mathematics
Faculty of Science, Selcuk University
Konya, Turkey
e-mail: ozlem.acar@selcuk.edu.tr

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