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SECOND ORDER NONLINEAR EVOLUTIONARY SYSTEMS DRIVEN BY GENERALIZED MIXED VARIATIONAL INEQUALITIES

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Abstract. In this paper, we deal with the system formulated by abstract second order nonlinear evolution differential equations which are subject to a generalized mixed variational inequalities. Firstly, based on Ky Fan inequality theorem, we examine that the solution set of variational inequalities is bounded, closed and convex by getting rid of the rigid restriction of monotonicity. Afterwards, the existence of solutions for a class of nonlinear differential equation is discussed.

1. Introduction

Let X and E be two Banach spaces, $V \subset E$ and E^* be the dual space of E. In this paper, we concentrate on the following problem:

$$\begin{cases} x''(t) - Ax(t) = f(t, x(t), u(t)), \quad t \in [0,T], \\ u(t) \in S(V, Q(t, x(t), \cdot), \varphi), \quad t \in [0,T], \\ x(0) = x_0, \quad x'(0) = y_0, \end{cases} \tag{1.1}$$

where A is a closed, linear and densely defined operator that generates a family of cosines; f (resp. Q) is a function defined from $[0,+\infty) \times X \times E$ to X (resp. $\mathscr{P}(E^*)$); $S(V, Q(t, x(t), \cdot), \varphi)$ describes the solution set of variational inequality, which is consisted of $u(t) \in V$ such that $u^*(t) \in Q(t, x(t), u(t))$ for some $u^*(t) \in E^*$, and

$$\langle u^*(t), v - u(t) \rangle + \varphi(v) - \varphi(u(t)) \geqslant 0, \quad \forall v \in V. \tag{1.2}$$

Here, $\varphi : E \to (-\infty, +\infty]$ is a proper convex and lower semicontinuous function.

Recently, the nonlinear evolution differential inclusions have attracted the attention of many scholars, see [1, 4, 7, 14, 18, 19]. Also, some researchers [11, 8, 6] have been dedicated to the survey of the solution's existence for the second (higher) order differential equations. For instance, the existence of semilinear evolution differential inclusions was studied by Cardinali and Rubbioni in [3]. Liu and Migórski et al. addressed successfully some existence issues of nonlinear evolutionary systems under compactness of semigroups, see [13, 7] and references therein.

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Differential variational inequalities (for short, DVIs) have earned enthusiastic concern during the recent decade. In [13, 12], the authors studied a class of differential equations subject to a generalized mixed variational inequalities formulated by

$$\begin{cases} x'(t) - Ax(t) = f(t,x(t),u(t)), \quad t \in [0,T], \\ u(t) \in S(V, Q(t,x(t),\cdot), \varphi), \\ x(0) = x_0, \end{cases} \tag{1.3}$$

where A is an infinitesimal generator of semigroup. They verified that the solutions of system (1.3) is bounded, closed and convex under the hypothesis that the mapping Q is monotone or ϕ-pseudomonotone. In the field of non-stationary variational inequalities and optimization, there are substantial practical problems for which the specific concordance rules and the monotonic assumptions are not necessary indeed, see e.g. [2, 9, 10, 16, 17, 20].

The motivation of this paper is to discuss the existence of solutions for generalized mixed variational inequalities without monotonicity. Secondly, we are devoted to establish the relationship between variational inequalities and differential equations. We shall reveal the solvability of second order nonlinear evolutionary dynamical system (1.1) by exploiting the Hausdorff measure of noncompactness.

This paper is organized in the following way. In Section 2, a brief overview of some elementary notions involving with nonlinear analysis is showcased. In section 3, the existence result for the mild solutions of a variational inequality is verified. In section 4, the existence of second order nonlinear evolutionary dynamical system (1.1) is exhibited in the case of Banach space.

2. Preliminaries

Let V be a nonempty convex subset of Banach space E. In the sequel, the symbol $\mathbb{R}$ (resp. $\mathbb{R}_+$) stands for the set of all real (resp. positive real) numbers. The strong and weak convergence of $\{x_k\}$ to u are denoted by $x_k \to x$ and $x_k \rightharpoonup x$, respectively.

DEFINITION 2.1. A function $\mu : V \to \mathbb{R}$ is said to be

(1) convex, if for each $u, v \in V$ and $\alpha \in [0,1]$, it is true that

$$\mu(\alpha u + (1-\alpha)v) \leqslant \alpha\mu(u) + (1-\alpha)\mu(v).$$

(2) quasiconvex, if for each $u, v \in V$ and $\alpha \in [0,1]$, it is true that

$$\mu(\alpha u + (1-\alpha)v) \leqslant \max\{\mu(u), \mu(v)\}.$$

(3) coercive, if $\|u\| \to \infty$, we have $\mu(u) \to +\infty$.

(4) weakly coercive, if the set $\{u \in V | \mu(u) \leqslant \rho\}$ is nonempty and bounded for some constant $\rho \in \mathbb{R}$.

The function μ is referred to as concave (quasiconcave), provided that $-\mu$ is convex (quasiconvex).

It is straightforward that $(1) \Rightarrow (2)$ and $(3) \Rightarrow (4)$, but the reverse implications are not true in general.

Let $I = [0,T]$, the set of functions with $\|\eta\| = (\int_0^T \|\eta(t)\|^2 dt)^{\frac{1}{2}}$ as the norm is denoted by $L^2(I,E)$. $\Omega \subset L^2(I,E)$ is said to be integrably bounded, if for all $\eta \in \Omega$, there exists $g \in L^2(I,\mathbb{R}_+)$ with $\|\eta(t)\| \leqslant g(t)$ for almost $t \in I$. The sequence $\{x_n\} \subset L^2(I,E)$ is termed as semicompact, if $\{x_n(t)\}_{n=1}^{\infty}$ is relatively compact for almost $t \in I$ and $\{x_n\}$ is integrably bounded.

We say that $\Phi : V \times V \to \mathbb{R}$ is an equilibrium bi-function, if $\Phi(u,u) = 0$ for all $u \in V$. Variational inequalities can be regarded as special equilibrium problems (EP, for short), which find $u^* \in V$ satisfying

$$\Phi(u^*,v) \geqslant 0 \qquad \forall v \in V, \tag{2.1}$$

where Φ is an equilibrium bi-function. We now give an existence result related to the equilibrium problem, see [5].

LEMMA 2.1. *Suppose that $\Phi : V \times V \to \mathbb{R}$ is an equilibrium bi-function satisfying:*

(i) $\Phi(\cdot,v)$ is a weakly upper semicontinuous for each $v \in V$;

(ii) $\Phi(u,\cdot)$ is a quasiconvex for each $u \in V$.

Then equilibrium problem (2.1) admits a solution.

For simplicity, we introduce the following notations:

$$\mathscr{P}_{f(c)}(E) := \{B \subset E : nonempty, closed, (convex)\}$$

$$\mathscr{P}_{(\omega)k(c)}(E) := \{B \subset E : nonempty, (weakly)\ compact, (convex)\}$$

DEFINITION 2.2. A function $G : X \to \mathscr{P}(E)$ is said to be

(1) upper semicontinuous (u.s.c), if $G^{-1}(C)$ is closed for each closed subset $C \subset E$, where $G^{-1}(C) = \{u \in X : G(u) \bigcap C \neq \emptyset\}$.

(2) closed, if the set $\Gamma_G = \{(u,v) : v \in G(u)\}$ is closed in $X \times E$.

(3) compact, if $G(B)$ is relatively compact in E for every bounded subset $B \subset X$.

PROPOSITION 2.1. ([15, Proposition 3.12]) *Let X and E be Banach spaces. Suppose that $G : X \to \mathscr{P}(E)$ has compact and convex values, then G is u.s.c iff $\{x_n\} \subset X$ with $x_n \to x_0 \in X$ and $y_n \in G(x_n)$, there exist a subsequence, also denoted by y_n, such that $y_n \to y_0 \in G(x_0)$.*

PROPOSITION 2.2. ([21, Proposition 1.3.4]) *Given a function* $g : X \to \mathbb{R}\bigcup\{+\infty\}$ *with* X *being a topological space. The following statements are equivalent:*

(i) g *is (weakly) u.s.c.;*

(ii) for every $\lambda \in \mathbb{R}$, $G_\lambda := \{x \in X : g(x) \geqslant \lambda\}$ *is (weakly) closed in* X*;*

In general, considering the mild solution of DVI (1.1), we need to deal with the following differential inclusion:

$$\begin{cases} x''(t) - Ax(t) \in F(t,x(t)), & t \in I, \\ x(0) = x_0, \quad x'(0) = y_0. \end{cases} \tag{2.2}$$

The space of all bounded linear operators from X to X^* is denoted by $\mathscr{L}(X)$. We call $C : \mathbb{R} \to \mathscr{L}(X)$ a strongly continuous cosine family, if $C(0) = I$ and $C(t_1+t_2)+C(t_1-t_2) = 2C(t_1)C(t_2)$ for all $t_1,t_2 \in \mathbb{R}$.

For the generator of the cosine operator $C(t)$, let $A : X \to X$ be expressed by

$$Ax = \frac{d^2}{dt^2}C(t)x \,|_{t=0}, \quad \forall x \in D(A),$$

where $D(A) = \{x \in X : C(t)x \in C^2(\mathbb{R},X)\}$ is domain. It is well known that A is a linear, closed and densely defined operator on X, see [6]. The sine operator $S : \mathbb{R} \to \mathscr{L}(E)$ is defined as follows:

$$S(t)x = \int_0^t C(s)x\, ds, \quad \forall\, t \in \mathbb{R},\ x \in X.$$

DEFINITION 2.3. We say that the system (2.2) has a mild solution $x \in C(I,X)$, if there exists $\eta \in L^2(I,X)$ such that $\eta(t) \in F(t,x(t))$ for $t \in I$ and

$$x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)\eta(s)\, ds.$$

For every bounded subset $B \subset X$, the Hausdorff measure of noncompactness (MNC, for short) is defined by

$$\chi(B) = \inf\{\varepsilon > 0 : B\ has\ a\ finite\ \varepsilon - net\}.$$

DEFINITION 2.4. Let χ be a Hausdorff measure of noncompactness and $0 \leqslant \kappa < 1$. A multi-valued mapping $\Gamma : X \to \mathscr{P}_k(X)$ is said to be κ-condensing, provided that for each bounded $B \subset X$, it holds that

$$\chi(\Gamma(B)) \leqslant \kappa \cdot \chi(B).$$

The open and the closed balls with the origin as the center and $r > 0$ as the radius are denoted by $B_r(0)$ and $\overline{B_r(0)}$, respectively. As an important tool, the following fixed point theorem used in our main results of section 3.

LEMMA 2.2. ([4, Corollary 3.1]) *Suppose that* $\mathscr{A} : \overline{B_r(0)} \to X$ *and* $\mathscr{B} : \overline{B_r(0)} \to \mathscr{P}_{kc}(X)$ *are two functions satisfying*

(1) $\mathscr{A}$ *is a contraction function with contraction coefficient* $\lambda < \frac{1}{2}$*;*

(2) $\mathscr{B}$ *is compact and u.s.c.*

Then one of the following conclusions holds

(i) there exist an element $\omega \in \overline{B_r(0)} \setminus B_r(0)$ *such that* $\rho\omega \in \mathscr{A}\omega + \mathscr{B}\omega$ *for some* $\rho > 1$*;*

(ii) the inclusion $x \in \mathscr{A}x + \mathscr{B}x$ *admits a solution in* $\overline{B_r(0)}$*.*

3. Solutions of variational inequalities

In this section, we consider the following generalized mixed variational inequality problem (GMVI, for short): find $u \in V, u^* \in G(u)$ satisfying

$$\langle u^*, v-u\rangle + \varphi(v) - \varphi(u) \geqslant 0, \quad \forall v \in V. \tag{3.1}$$

$(\mathscr{H}_V)$ $V \subset E$ is a closed and convex subset;

$(\mathscr{H}_\varphi)$ $\varphi : V \to \mathbb{R}\bigcup\{+\infty\}$ is proper convex and l.s.c.;

$(\mathscr{H}_G)$ $G : V \to \mathscr{P}_{kc}(E^*)$ is u.s.c. such that $\Psi(\cdot, v)$ is a quasiconcave for each $v \in V$, provide that $\Psi(u,v) = \sup\limits_{u^* \in G(u)} \langle u^*, v-u\rangle$.

We will present an existence result of GMVI (3.1) in the following bounded case.

PROPOSITION 3.1. *Let* V *be a bounded subset of* E *and* $(\mathscr{H}_V)$, $(\mathscr{H}_\varphi)$, $(\mathscr{H}_G)$ *hold, then the system GMVI* (3.1) *has at least a solution.*

Proof. Obviously, $\Psi(\cdot, v)$ is convex for every $u \in V$. Let $u_n \to \overline{u}$. In order to show that $\Psi(\cdot, v)$ is u.s.c. for each $v \in V$, we need to prove

$$\limsup_{n\to\infty} \sup_{u_n^* \in G(u_n)} \langle u_n^*, v-u_n\rangle \leqslant \sup_{u^* \in G(\overline{u})} \langle u^*, v-\overline{u}\rangle.$$

Without loss of generality, we let $\{u_n\} \subset V$ be such that

$$\limsup_{n\to\infty} \sup_{u_n^* \in G(u_n)} \langle u_n^*, v-u_n\rangle = \lim_{n\to\infty} \sup_{u_n^* \in G(u_n)} \langle u_n^*, v-u_n\rangle.$$

Since G has compact values, for every n there exist $w_n^* \in G(u_n)$ such that

$$\sup_{u_n^* \in G(u_n)} \langle u_n^*, v-u_n\rangle = \langle w_n^*, v-u_n\rangle.$$

By Proposition 2.1, there exist subsequence $\{w_{n_k}^*\}$ satisfying

$$w_{n_k}^* \to \overline{u}^* \in G(\overline{u}) \quad \text{as } n_k \to \infty.$$

Hence,

$$\limsup_{n\to\infty} \sup_{u_n^*\in G(u_n)} \langle u_n^*, v-u_n\rangle = \langle \overline{u}^*, v-\overline{u}\rangle \leqslant \sup_{u^*\in G(\overline{u})} \langle u^*, v-\overline{u}\rangle.$$

For convenience, we write

$$\Phi(u,v) = \Psi(u,v) + \varphi(v) - \varphi(u). \tag{3.2}$$

Note that Φ is an equilibrium bi-function and $\limsup_{n\to\infty} \Psi(u_n, v) \leqslant \Psi(\overline{u}, v)$. Therefore, $\Psi(\cdot, v)$ is u.s.c. for each $v \in V$.

On the other hand, from Proposition 2.2, we get the set

$$F_\lambda = \{u\in V | \sup_{u^*\in G(u)} \langle u^*, v-u\rangle \geqslant \lambda\}$$

is closed for any $\lambda \in \mathbb{R}$. Note that F_λ is a convex subset in E, so it is weakly closed. Proposition 2.2 implies that $\Psi(\cdot, v)$ and $\Phi(\cdot, v)$ are weakly u.s.c for any $v \in V$. Then the problem EP (3.2) has a solution from Lemma 2.1. Furthermore, GMVI (3.1) has a solution on account of the well-known minimax theorem. □

Next we consider the case where V is unbounded. For sake of convenience, we write

$$\triangle(u^*, u, v) = \langle u^*, v-u\rangle + \varphi(v) - \varphi(u),$$

and

$$W_\rho = \{u\in V | \mu(u) \leqslant \rho\}, \quad L_\rho = \{u \in V | \mu(u) < \rho\},$$

where $\mu : E \to \mathbb{R}$ is a function.

PROPOSITION 3.2. *Let $\mu : E \to \mathbb{R}$ be a convex function and $(\mathscr{H}_V)$, $(\mathscr{H}_\varphi)$, $(\mathscr{H}_G)$ hold. Assume that there exist $u_\rho \in W_\rho$, $u_\rho^* \in G(u_\rho)$ and $z \in L_\rho$ for some $\rho \in \mathbb{R}$ satisfying*

$$\triangle(u_\rho^*, u_\rho, z) \leqslant 0 \tag{3.3}$$

and for all $v \in W_\rho$

$$\triangle(u_\rho^*, u_\rho, v) \geqslant 0. \tag{3.4}$$

Then u_ρ is a solution of the system GMVI(3.1).

Proof. We write

$$\psi(v) = \triangle(u_\rho^*, u_\rho, v),$$

then ψ is a convex function and $\psi(z) = 0$, that is, z is the minimum element of function ψ on W_ρ. We will show that

$$\psi(v) = \triangle(u_\rho^*, u_\rho, v) \geqslant 0, \quad \textit{for all } v \in V \setminus W_\rho. \tag{3.5}$$

Conversely, we suppose that there exists a point $v_0 \in V \setminus W_\rho$ such that

$$\psi(v_0) < \psi(z) = 0.$$

Then $v(\varepsilon) \in V$ with $v(\varepsilon) = \varepsilon v_0 + (1-\varepsilon)z$ for any $\varepsilon \in (0,1)$. Furthermore, we obtain that for sufficiently small $\varepsilon > 0$

$$\mu(v(\varepsilon)) \leqslant \varepsilon\mu(v_0) + (1-\varepsilon)\mu(z) = \varepsilon[\mu(v_0) - \mu(z)] + \mu(z) \leqslant \rho,$$

on account of the convexity of μ. This implies $v(\varepsilon) \in W_\rho$ for a sufficiently small $\varepsilon > 0$. However, $\psi(v(\varepsilon)) \leqslant \alpha\psi(v_0) + (1-\alpha)\psi(z) < 0$, which is a contradiction.

Combining (3.4) with (3.5), we deduce

$$\triangle(u_\rho^*, u_\rho, v) \geqslant 0, \quad \forall v \in V,$$

that is, u_ρ is a solution of GMVI (3.1). $\square$

Let us introduce a suitable coercivity condition for problem GMVI (3.1), see [10] and references therein.

(C) $\mu : E \to \mathbb{R}$ is convex and l.s.c. and weakly coercive on V. If for some $r > 0$ and for all $u^* \in G(u)$

$$\inf_{v \in W_r} \triangle(u^*, u, v) \geqslant 0 \quad \forall u \in V \setminus W_r, \tag{3.6}$$

then there exists $z \in V$ satisfying

$$\begin{gathered} min\{\triangle(u^*, u, z), \mu(z) - \mu(u)\} < 0 \\ \text{and} \\ max\{\triangle(u^*, u, z), \mu(z) - \mu(u)\} \leqslant 0. \end{gathered} \tag{3.7}$$

Note that, if the set W_ρ is nonempty for every $\rho \in \mathbb{R}$, then it is bounded in V on account of the weakly coercivity of μ.

PROPOSITION 3.3. *Suppose that* $(\mathscr{H}_V)$, $(\mathscr{H}_\varphi)$, $(\mathscr{H}_G)$ *and* (C) *are fulfilled, then* $W_r \neq \emptyset$.

Proof. Since μ is convex, l.s.c. and weakly coercive with respect to the set V, then there exists $z_0 \in V$ satisfying

$$\mu(z_0) = r_0 \triangleq \inf_{u \in V} \mu(u).$$

Therefore, W_{r_0} is a nonempty, closed, convex and bounded subset. Put $V = W_{r_0}$ in Proposition 3.1. Then there exist $\bar{u} \in W_{r_0}$ and $\overline{u}^* \in G(\bar{u})$ such that

$$\triangle(\overline{u}^*, \bar{u}, v) \geqslant 0 \quad \forall v \in W_{r_0}. \tag{3.8}$$

Conversely, we suppose that $W_r = \varnothing$, then $r < r_0$ and $\bar{u} \notin W_r$. From the hypothesis (C), there exists $z \in V$ satisfing $\mu(z) \leqslant \mu(\overline{u}) = r_0$, that is, $z \in W_{r_0}$.

On the other hand, by the definition of r_0, we get $\mu(\bar{u}) = r_0 \leqslant \mu(z)$ for any $z \in V$. Therefore, we have $\mu(\bar{u}) = \mu(z)$ and $\triangle(\overline{u}^*, \bar{u}, z) < 0$ on account of (3.7), which is contradictory to (3.8). $\square$

THEOREM 3.1. *Suppose that* $(\mathscr{H}_V)$, $(\mathscr{H}_\varphi)$, $(\mathscr{H}_G)$ *and* (C) *hold, then problem GMVI* (3.1) *admits a solution.*

Proof. Using the weakly coercivity of μ and the assumption (C), W_ρ is nonempty, closed, convex and bounded in V for every $\rho > r$, By applying Proposition 3.1, there exist $\bar{u} \in W_\rho$ and $\bar{u}^* \in G(\bar{u})$ satisfying for all $v \in W_\rho$

$$\triangle(\bar{u}^*, \bar{u}, v) \geqslant 0.$$

This implies the relations (3.4) with $u_\rho = \bar{u}$ and (3.6) hold.

Next, we will show that $\bar{u}$ is a solution of GMVI (3.1). Indeed, if $\bar{u} \in L_\rho$, by choosing $z = \bar{u}$, then $\bar{u}$ is a solution of GMVI (3.1) from Proposition 3.2. Otherwise, we get $\mu(\bar{u}) = \rho$ and $\bar{u} \notin L_\rho$. From hypothesis(C), there exist $z \in V$ satisfying

$$min\{\triangle(\bar{u}^*, \bar{u}, z), \mu(z) - \mu(\bar{u})\} < 0 \quad and \quad max\{\triangle(\bar{u}^*, \bar{u}, z), \mu(z) - \mu(\bar{u})\} \leqslant 0.$$

This implies $\mu(z) \leqslant \mu(\bar{x}) = \rho$, that is, $z \in W_\rho$. Therefore, we obtain $\mu(z) < \mu(\bar{u}) = \rho$ and $\triangle(\bar{g}, \bar{x}, z) = 0$ due to (3.7). By applying Proposition 3.2 again, we get that $\bar{u}$ solves GMVI (3.1). The proof is complete. $\square$

As V is a compact convex set, we consider the following assumption $(\mathscr{H}'_G)$ taking place of $(\mathscr{H}_G)$.

$(\mathscr{H}'_G)$ $G : V \to \mathscr{P}_{kc}(E^*)$ is u.s.c. set-valued mapping.

COROLLARY 3.2. *Suppose that* $V \subset E$ *is compact and convex set and* (H_φ), $(\mathscr{H}'_G)$ *hold, then GMVI* (3.1) *admits a solution.*

REMARK 3.3. Our conclusion of Theorem 3.1 generalizes the classical result [9, Theorem 1] when E is a finite dimensional space.

Below we give the relationship between the solution of the variational inequality and evolutionary system.

THEOREM 3.4. *Let* X *be a separable Banach space and* E *be a reflexive Banach space,* $V \subset E$ *be a compact and convex subset,* $Q : I \times X \times V \to \mathscr{P}(E^*)$ *be u.s.c. mapping. In addition, we suppose that* $G(\cdot) = Q(t,x,\cdot) : V \to \mathscr{P}(E^*)$ *and* $\varphi : E \to \mathbb{R}\bigcup\{+\infty\}$ *satisfy assumptions* $(H_\varphi) - (\mathscr{H}'_G)$ *for any* $(t,x) \in I \times X$. *Then* $U : I \times X \to \mathscr{P}(V)$ *is defined as follow*

$$U(t,x) := \{u \in V : \exists u^* \in Q(t,x,u) \; s.t. \; \langle u^*, v-u\rangle + \varphi(v) - \varphi(u) \geqslant 0, \quad \forall v \in V\}, \quad (3.9)$$

such that

(U_1) $U(t,x)$ *is nonempty closed convex and bounded in* V *for any* $(t,x) \in I \times X$*;*

(U_2) U *is u.s.c.;*

(U_3) $U(\cdot,x): I \to \mathscr{P}(V)$ *is measurable for any* $x \in X$.

Proof. Corollary 3.2 guarantees that the mapping U is well defined. From hypothesis $(\mathscr{H}'_G)$, we get that $U(t,x)$ is convex set in V. We will show that $U(t,x)$ is closed in V. Let $u_n \in U(t,x)$ and $u_n \to u$, there exist $u_n^* \in Q(t,x,u_n)$ such that for all $v \in V$

$$\langle u_n^*, v-u_n\rangle + \varphi(v) - \varphi(u_n) \geqslant 0.$$

From $(\mathscr{H}'_G)$ and Proposition 2.1, there exists subsequence $u_{n_k}^* \in Q(t,x,u_{n_k})$ such that $u_{n_k}^* \to u^* \in Q(t,x,u)$. Furthermore, we obtain for all $v \in V$

$$\langle u^*, v-u\rangle + \varphi(v) - \varphi(u) \geqslant 0$$

on account of the lower semicontinuity of φ. This implies $u \in U(t,x)$ and $U(t,x)$ is closed.

To verify (U_2), we proceed to check that

$$U^{-1}(C) := \{(t,x) \in I \times X : U(t,x) \bigcap C \neq \emptyset\} \tag{3.10}$$

is closed in $\mathbb{R} \times X$ for arbitrary closed subset C of V. Let $(t_n,x_n) \in U^{-1}(C)$ and $(t_n,x_n) \to (t,x)$ in $\mathbb{R} \times X$. Thus, by the definition of U, there exist $u_n \in U(t_n,x_n) \bigcap C$ and $u_n^* \in Q(t_n,x_n,u_n)$ satisfying

$$\langle u_n^*, v-u_n\rangle + \varphi(v) - \varphi(u_n) \geqslant 0, \quad \forall v \in V. \tag{3.11}$$

From the compactness of V, we may assume that $u_n \to u \in V \bigcap C$. By taking the limit of the above inequality, we obtain that there exist subsequence $u_{n_k}^* \in Q(t_{n_k},x_{n_k},u_{n_k})$ such that $u_{n_k}^* \to u^* \in Q(t,x,u)$ and

$$\langle u^*, v-u\rangle + \varphi(v) - \psi(u) \geqslant 0, \quad \forall v \in V$$

on account of Proposition 2.1. Thus $(t,x) \in U^{-1}(C)$.

Finally, by analogizing to the proof of Theorem 3.4 of [12], one can examine that (U_3) is fulfilled. This completes the proof. □

REMARK 3.5. Compared with the results of [13, Theorem 3.4] and [12, Theorem 3.4], Theorem 3.4 in this paper is of more novelty and interest due to the removal of the rigid restriction of monotonicity.

4. Solutions of differential equation

In this section, we turn to consider the system (1.1) with the constraints set $V \subset E$ and give the following hypotheses of $f: I \times X \times V \to X$.

(f_1) For each convex subset C of V, $f(t,x,C)$ is a convex subset of X for any $(t,x) \in I \times X$;

(f_2) There exists $h \in L^2(I,\mathbb{R}_+)$ satisfying

$$\|f(t,x,u)\|_X \leqslant h(t) + b\|x\|_X$$

for all $(t,x,u) \in I \times X \times V$.

(f_3) $f(\cdot,x,u) : I \to X$ is measurable for any $(x,u) \in X \times V$;

(f_4) $f(t,\cdot,\cdot) : X \times E \to X$ is continuous for all $t \in I$;

(f_5) For all $x_1, x_2 \in X$ and almost $t \in I$, it holds for some $q \in L^2(I,\mathbb{R}_+)$ that

$$\|f(t,x_1,u) - f(t,x_2,u)\|_X \leqslant q(t)\|x_1 - x_2\|_X, \ \ \forall u \in C.$$

To study system (1.1), we define a set-valued mapping $F : [0,T] \times X \to \mathscr{P}(X)$ as follows:

$$F(t,x) = f(t,x,U(t,x)), \tag{4.1}$$

where U is specified in (3.9).

LEMMA 4.1. *Suppose that the assumptions of Theorem 3.4 are fulfilled and the above conditions (f_1)–(f_5) are satisfied. Then the following statements are true:*

(F_1) For each $(t,x) \in I \times X$, $F(t,x)$ admits convex and compact values;

(F_2) For each $x \in X$, $F(\cdot,x)$ admits a strongly measurable selection;

(F_3) For each $t \in I$, $F(t,\cdot)$ is u.s.c.;

(F_4) For each bounded subset $B \subset X$, there exists $q \in L^2(I,\mathbb{R}_+)$ which guarantees that for almost $t \in I$,

$$\chi(F(t,B)) \leqslant q(t)\chi(B).$$

Proof. (F_1). From (U_1) of Theorem 3.4, we obtain that $U(t,x) \in \mathscr{P}_{kc}(E)$. Therefore, $F(t,x)$ has convex and compact values for every $(t,x) \in I \times X$ due to (f_1) and (f_4).

(F_2). From (f_3) and (f_4), we know that $f(\cdot,x,\cdot) : I \times E \to X$ is a Carathéodory function. By virtue of (U_3), it can be checked that $F(\cdot,x) = f(\cdot,x,U(\cdot,x))$ is measurable for every $x \in X$, see [8, Proposition 1.3.1]. Furthermore, the separability of X implies $F(\cdot,x)$ is strongly measurable, see [8, Theorem 1.3.1]. Since $F(t,x)$ has convex and compact values, $F(\cdot,x)$ admits a strongly measurable selection (see [8]).

(F_3). From the upper semicontinuity of U and the continuity of $f(t,\cdot,\cdot)$, we obtain easily that $F(t,\cdot)$ is upper semicontinuous for each $t \in I$.

(F_4). For each $t \in I$, $y \in X$, we write $\Gamma(x,y) = f(t,y,U(t,x))$. From the continuity of $f(t,\cdot,\cdot)$, it is easy to see that $\Gamma(B,y) = f(t,y,U(t,B))$ is a relatively compact subset of X for each bounded subset $B \subset X$.

Next, we will show that $\Gamma(x,\cdot)$ is $q(t)$-Lipschitz with respect to the Hausdorff metric. Indeed, let $y_1, y_2 \in X$ and $z_1 \in \Gamma(x,y_1)$, then there exists $u \in U(t,x)$ such that

$z_1 = f(t,y_1,u)$. Furthermore, we have $z_2 = f(t,y_2,u) \in \Gamma(x,y_2)$. By Assumption (f_5), we get

$$\|z_1 - z_2\|_X = \|f(t,y_1,u) - f(t,y_2,u)\|_X \leqslant q(t)\|y_1 - y_2\|_X.$$

This implies that $\Gamma(x,\cdot)$ is $q(t)$-Lipschitz. By applying [8, Proposition 2.2.2], we have $\Gamma(x,x)$ is $q(t)$-condensing, that is, $F(t,\cdot)$ is $q(t)$-condensing. Therefore, for every bounded subset B of X

$$\chi(F(t,B)) \leqslant q(t)\chi(B), \quad a.e.\, t \in I.$$

The proof is complete. □

Subsequently, we define a set-valued operator $\mathscr{P}_F : C(I,X) \to \mathscr{P}(L^2(I,X))$ in the next form:

$$\mathscr{P}_F(x) = \{\eta \in L^2(I,X) : \eta(t) \in F(t,x(t)) \; for \; a.e.\, t \in I\}.$$

Note that by using (F_1)-(F_3) and (f_2), it is known that $\mathscr{P}_F$ is well-defined, see [8, section 1.3.3]. This section aims to solve the existence issue of mild solutions of differential inclusion. To realize this, we invoke the next assumptions for the problem (2.2):

$\mathscr{H}(S)$ there exist constants $M_A \geqslant 1$, $w \geqslant 0$ with $\|C(t)\| \leqslant M_A e^{w|t|}$ and $\|S(t)\| \leqslant M_A e^{w|t|}$ fulfilled, and the operator $S(t)$ is compact for all $t \in I$.

Note that the uniform boundedness principle implies that $C(t)$ and $S(t)$ are uniformly bounded on I. We set $M = \sup_{t\in I}\{\|C(t)\|, \|S(t)\|\}$.

In the sequel, we define two set-valued operators $\mathscr{B} : C(I,X) \to C(I,X)$ and $\mathscr{W} : L^2(I,X) \to C(I,X)$ as below:

$$\mathscr{B}(x) = \{\phi \in C(I,X) : \phi(t) = \int_0^t S(t-s)\eta(s)\, ds, \quad \eta \in \mathscr{P}_F(x)\}, \tag{4.2}$$

$$\mathscr{W}(\eta)(t) = \int_0^t S(t-s)\eta(s)\, ds.$$

Note that every semicompact sequence $\{\eta_n\}_{n=1}^{\infty} \subset L^1(I,E)$ is weakly compact. In addition, based on [8, Theorem 5.1.1], we will demonstrate a very applicable property in terms of semicompactness.

LEMMA 4.2. *Suppose that all conditions of Lemma 4.1 and $\mathscr{H}(S)$ are satisfied. Then the sequence $\{\mathscr{W}\eta_n\}_{n=1}^{\infty}$ is relatively compact for each semicompact sequence $\{\eta_n\}_{n=1}^{\infty} \subset L^1(I,E)$. In particular, if $\eta_n \rightharpoonup \eta_0$, then $\mathscr{W}\eta_n \to \mathscr{W}\eta_0$.*

LEMMA 4.3. *Suppose that the conditions of Lemma 4.1 and $\mathscr{H}(S)$ are satisfied, the set-valued mapping $\mathscr{B} : C(I,X) \to C(I,X)$ is u.s.c. and has convex, compact values.*

Proof. Note that the operator $\mathscr{B}$ has convex values for any $x \in C(I,X)$ from the convexity of $\mathscr{P}_F$. The proof is divided into the following four steps to complete.

Step 1: $\mathscr{B}$ is bounded operator on $C(I,X)$. For arbitrary $x \in B_r(0)$ and $\phi \in \mathscr{B}(x)$, we deduce from (f_2) and Hölder inequality that

$$\begin{aligned}\|\phi(t)\|_X &\leqslant \int_0^t \|S(t-s)\eta(s)\|_X\, ds \leqslant M\int_0^t h(s)+b\|x(s)\|_X\, ds\\ &\leqslant M(\|h\|_{L^2(I,\mathbb{R}_+)}\sqrt{T}+brT).\end{aligned}$$

Hence, $\mathscr{B}(B_r(0))$ is bounded subset of $C(I,X)$, i.e. the operator $\mathscr{B}$ is bounded operator.

Step 2: The equicontinuity of $\{\mathscr{B}(x)|x \in B_r(0)\}$. Let σ be a sufficiently small positive number and $0 < t_1 < t_2 \leqslant T$, we get

$$\begin{aligned}\|\phi(t_2)-\phi(t_1)\|_X &= \|\int_0^{t_2} S(t_2-\tau)\eta(\tau)\, d\tau - \int_0^{t_1} S(t_1-\tau)\eta(\tau)\, d\tau\|_X\\ &\leqslant \int_0^{t_1} \|[S(t_2-\tau)-S(t_1-\tau)]\eta(\tau)\|\, d\tau + \int_{t_1}^{t_2} \|S(t_2-\tau)\eta(\tau)\|\, d\tau\\ &\leqslant \int_0^{t_1} \|[S(t_2-\tau)-S(t_1-\tau)]\|[h(\tau)+br]\, d\tau + M\int_{t_1}^{t_2}[h(\tau)+br]\, d\tau\\ &\leqslant \sup_{\tau\in[0,t_1-\sigma]} \|[S(t_2-\tau)-S(t_1-\tau)]\|(\|h\|_{L^2(I,\mathbb{R}_+)}\sqrt{T}+brT)\\ &\quad +M[\|h\|_{L^2(I,\mathbb{R}_+)}(2\sqrt{\sigma}+\sqrt{t_2-t_1})+br(2\sigma+t_2-t_1)].\end{aligned}$$

From the continuity of sine operator $S(t)$, when $t_2 \to t_1$, the right side of the above inequality is independent of x and tends to zero. This proves our conclusion.

Step 3: For almost $t \in I$, the set $\Pi(t) = \{\phi(t) : \phi \in \mathscr{B}(B_r(0))\}$ is relatively compact in X. Clearly, $\Pi(0) = \{0\}$ is compact. For any $x \in B_r(0)$ and $\phi \in \mathscr{B}(B_r(0))$, there exists $\eta \in \mathscr{P}_F(x)$ satisfying fixed $t \in (0,T]$,

$$\phi(t) = \int_0^t S(t-\tau)\eta(\tau)\, d\tau.$$

For every $\varepsilon \in (0,t)$, defined $\phi^\varepsilon : I \to X$ by

$$\phi^\varepsilon(t) = \int_0^{t-\varepsilon} S(t-\tau)\eta(\tau)\, d\tau = S(\varepsilon)\int_0^{t-\varepsilon} S(t-\tau-\varepsilon)\eta(\tau)\, d\tau.$$

From the compactness of $S(\cdot)$ and the boundedness of $\int_0^{t-\varepsilon} S(t-\tau-\varepsilon)\eta(\tau)\, d\tau$, we get that the set $\Pi_\varepsilon(t) = \{\phi^\varepsilon(t) : \phi \in \mathscr{B}(B_r(0))\}$ is relatively compact in X. Furthermore, we have

$$\begin{aligned}\|\phi(t)-\phi^\varepsilon(t)\|_X &\leqslant M\int_{t-\varepsilon}^t h(\tau)+b\|x(\tau)\|_X\, d\tau\\ &\leqslant M(\|h\|_{L^2(I,\mathbb{R}_+)}\sqrt{\varepsilon}+br\varepsilon).\end{aligned}$$

This implies that the set $\Pi(t)$ $(t > 0)$ is also relatively compact.

Finally, by synthesizing the results of Steps 1 and 2 together with applying Ascoli-Arzelà Theorem, we obtain that $\mathscr{B}$ is a compact mapping.

Step 4: The graph of $\mathscr{B}$ is closed. Actually, let $\{x_n\},\{\phi_n\}\subset C(I,X), x_n\to \bar{x}, \phi_n\in \mathscr{W}\circ\mathscr{P}_F(x_n)$ and $\phi_n\to\bar{\phi}$. Choosing any sequence $\eta_n\subset L^2(I,X)$ such that $\eta_n\in\mathscr{P}_F(x_n), \phi_n=\mathscr{W}(\eta_n)$. From (f_2), we get the sequence η_n is integrable bounded. Hence, it is weakly compact in $L^2(I,X)$. For convenience, we may assume that $\eta_n\rightharpoonup\bar{\eta}$. From (F_4), it is easily seen that for almost $t\in I$,

$$\chi(\{\eta_n(t)\})\leqslant q(t)\chi(\{x_n(t)\}).$$

Therefore, $\{\eta_n(t)\}$ is relatively compact in X. Due to Lemma 4.2, we get

$$\phi_n=\mathscr{W}(\eta_n)\to\mathscr{W}(\bar{\eta})=\bar{\phi}.$$

On the other hand, taking account of Proposition 2.1 yields $\bar{\eta}\in\mathscr{P}_F(\bar{x})$. Therefore, $\mathscr{B}$ has a closed graph. The proof is completed. $\square$

THEOREM 4.1. *Suppose that the conditions of Lemma 4.1 and $\mathscr{H}(S)$ are satisfied, the system (2.2) admits one mild solution for each pair of initial data $x_0,y_0\in X$.*

Proof. The mapping $\mathscr{F}:C(I,X)\to C(I,X)$ is defined as follows:

$$\mathscr{F}(u)=\{\varphi\in C(I,X):\varphi(t)=C(t)x_0+S(t)y_0+\int_0^t S(t-\tau)\eta(\tau)\,d\tau,\quad \eta\in\mathscr{P}_F(x)\}.$$

For convenience, we write $\mathscr{F}=\mathscr{A}+\mathscr{B}$, where $\mathscr{A}(x)=C(t)x_0+S(t)y_0$ with $t\in I$ and $\mathscr{B}$ is defined as (4.2). Obviously, system (1.1) admits a mild solution iff $\mathscr{F}$ has a fixed point. Owing to Lemma 2.2, we need to prove that (i) of Lemma 2.2 is not true.

In fact, suppose that $\rho x\in\mathscr{A}x+\mathscr{B}x$ with $\rho>1$ and there exists $\eta\in\mathscr{P}_F(x)$ with

$$\rho x(t)=C(t)x_0+S(t)y_0+\int_0^t S(t-\tau)\eta(\tau)\,d\tau.$$

Then we obtain

$$\begin{aligned}\|x(t)\|_X&\leqslant\|C(t)x_0\|_X+\|S(t)y_0\|_X+\|\int_0^t S(t-\tau)\eta(\tau)\,d\tau\|_X\\&\leqslant M\|x_0\|_X+M\|y_0\|_X+M\int_0^t h(\tau)+b\|x(\tau)\|_X\,d\tau\\&\leqslant d+Mb\int_0^t\|x(\tau)\|_X\,d\tau,\end{aligned}$$

where $d=M\|x_0\|_X+M\|y_0\|_X+M\|h\|_{L^2(I,\mathbb{R}_+)}\sqrt{T}$.

Utilizing the Gronwall inequality gives

$$\|x(t)\|_X\leqslant de^{Mbt}.$$

This implies

$$\|x\|_{C(I,X)}\leqslant de^{MbT}=:r.$$

Furthermore, let

$$Q_r = \{x \in C(I,X) : \|x\|_{C(I,X)} < r+1\}.$$

It yields that $\mathscr{B} : \overline{Q_r} \to \mathscr{P}_{kc}(X)$ is compact and u.s.c. by employing Lemma 4.3. Note that $\mathscr{A} : \overline{Q_r} \to X$ is a single value mapping with a contraction factor of less than $\frac{1}{2}$ Therefore, according to the choice of $\overline{Q_r}$, on the boundary of $\overline{Q_r}$ there is no $x \in X$ with $\|x\| = r$ satisfying $\rho x \in \mathscr{A}x + \mathscr{B}x$ for some $\rho > 1$.

Applying Lemma 2.2, it is evident that all the conditions are fulfilled and hence $\mathscr{F}$ has a fixed point. Hence, system (1.1) admits a mild solution. □

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