HIGHER DIMENSIONS OPIAL DIAMOND–ALPHA INEQUALITIES ON TIME SCALES

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Abstract. In this paper, we generalize Opial inequality to higher dimensions on time scales. We prove the Opial Delta-nabla inequality of \( n \) variables, and then give two diamond-alpha dynamic inequalities of Opial type of \( n \) variables. As well, we introduce some special cases.

1. Introduction

To unify the representation of discrete and continuous inequalities, Hilger established the theory of time scales in 1998 [1]. Since then, a number of interesting results on time scales have been presented such as Jensen’s equalities, Hardy’s equalities, Hölder’s inequality, et al [2, 3]. Among them, Bohner and Kaymakçalan gave inequality which cover continuous and discrete opial inequalities [4, 5].

THEOREM 1.1. (see [4]) Suppose \( 0, h \in \mathbb{T} \) and \( f : [0, h]_{\mathbb{T}} \to \mathbb{R} \) with \( f(0) = 0 \) is delta-differentiable, then

\[
\int_0^h |(f(t) + f^{\sigma}(t)) f^{\Delta}(t)| \Delta t \leq h \int_0^h |f^{\Delta}(t)|^2 \Delta t,
\]

with equality when \( f(x) = ct \).

THEOREM 1.2. (see [5]) Suppose \( 0, h \in \mathbb{T} \) and \( f : [0, h]_{\mathbb{T}} \to \mathbb{R} \) with \( f(0) = 0 \) is nabla-differentiable, then

\[
\int_0^h |(f(t) + f^{\rho}(t)) f^{\nabla}(t)| \nabla t \leq h \int_0^h |f^{\nabla}(t)|^2 \nabla t,
\]

with equality when \( f(x) = ct \).


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Theorem 1.3. (see [4]) Suppose $f$ is $\Delta$-differentiable defined on $[0, h]_T$, then
\[
\int_0^h |(f(t) + f^\sigma(t)) f^\Delta(t) \Delta t | \leq \alpha \int_0^h |f^\Delta(t)|^2 \Delta t + 2\beta \int_0^h |f^\Delta(t)| \Delta t, \tag{1.3}
\]
where
\[
\alpha = \min_{t \in [0, h]_T} \max\{t, h - t\},
\]
\[
\beta = \max\{|f(0)|, |f(h)|\}.
\]

Theorem 1.4. (see [4]) Suppose $f$ is $\nabla$-differentiable defined on $[0, h]_T$, then
\[
\int_0^h |(f(t) + f^\rho(t)) f^\nabla(t) \nabla t | \leq \alpha \int_0^h |f^\nabla(t)|^2 \nabla t + 2\beta \int_0^h |f^\nabla(t)| \nabla t, \tag{1.4}
\]
where
\[
\alpha = \min_{t \in [0, h]_T} \max\{t, h - t\},
\]
and
\[
\beta = \max\{|f(0)|, |f(h)|\}.
\]

And many generalized Opial inequalities involving more functions or higher order derivatives on time scales were provided in [4, 14, 15, 16, 17, 18, 20].

Theorem 1.5. (see [4]) Suppose $p, q \in C([0, h])$ are positive functions with $\int_0^h \frac{1}{p(t)} \Delta t < \infty$, $q$ nonincreasing. If $f$ is $\Delta$-differentiable defined on $[0, h]_T$ with $f(0) = 0$, then
\[
\int_0^h \left(q^\sigma(t)(f(t) + f^\rho(t)) f^\nabla(t) \nabla t \right) \Delta t \leq \left( \int_0^h \frac{1}{p(t)} \Delta t \right) \left( \int_0^h p(t) q(t) |f^\Delta(t)|^2 \Delta t \right).
\]

Theorem 1.6. (see [4]) Suppose $m, n \in \mathbb{N}$, if $\Delta^n$-differentiable function $f : [0, h]_T \to \mathbb{R}$ with $f(0) = f^\Delta(0) = \cdots = f^{\Delta^{n-1}}(0) = 0$, then
\[
\int_0^h \left( \sum_{k=0}^m f^k(t)(f^\sigma)^{m-k}(t) f^\Delta^n(t) \right) \Delta t \leq h^{nm} \int_0^h \left| f^{\Delta^n}(t) \right|^{m+1} \Delta t.
\]

In [14], it was presented that

Theorem 1.7. Suppose $m, n \in \mathbb{N}$ and $g, p : [0, h]_T \to \mathbb{R}$ are positive $\Delta$-differentiable with $g$ nonincreasing, and $\int_0^h \frac{1}{p(t)} \Delta t < \infty$. If $f : [0, h]_T \to \mathbb{R}$ is $\Delta^n$-differentiable with $f(0) = f^\Delta(0) = \cdots = f^{\Delta^{n-1}}(0)$, then
\[
\int_0^h g^\sigma(t) \left| \left( \sum_{k=1}^m f^k(t)(f^\sigma)^{m-k}(t) \right) f^\Delta^n(t) \right| \Delta t \leq h^{(n-1)m} \left( \int_0^h \frac{1}{p(t)} \Delta t \right)^m \left( \int_0^h p^m(t) g^\sigma(t) \right| f^{\Delta^n}(t) \right|^{m+1} \Delta t \right). \tag{1.5}
\]
Recently, researchers have provided $\diamond_{\alpha}$ as a weighting between $\Delta$ and $\nabla$ dynamic derivatives. Many diamond-alpha inequalities as the generalization of delta inequalities and nabla inequalities were provided, such as [6, 7, 8, 9, 10].

According to Theorems 1.1 and 1.2, Opial diamond-alpha inequality we need to prove is

$$
\int_0^h \left| \frac{\partial f^2(v)}{\diamond_{\alpha} v} \right| \diamond_{\alpha} v \leq h \int_0^h \left| \frac{\partial f(v)}{\diamond_{\alpha} v} \right|^2 \diamond_{\alpha} v.
$$

(1.6)

However, it doesn’t hold when $h = 1, \alpha = \frac{1}{2}, T = \{-1, 0, 1, 2\}, f(0) = f(-1) = 0, f(1) = f(2) = 1$.

In [11], the following inequality was discovered in 2010. For convenience, we call it Right-Opial Diamond-Alpha inequality of one variable.

**THEOREM 1.8.** (Right-Opial Diamond-Alpha inequality of one variable) Suppose $0, h \in T$ and $f : [0, h]_T \rightarrow \mathbb{R}$ with $f(0) = 0$ is $\diamond_{\alpha}$-differentiable, if $f^\Delta f^\nabla$ is non-negative, then

$$
\alpha^3 \int_0^h |(f(t) + f^\sigma(t)) f^\Delta(t)| \Delta t + (1 - \alpha)^3 \int_0^h |(f(t) + f^p(t)) f^\nabla(t)| \nabla t \leq h \int_0^h (f^{\diamond_{\alpha}}(t))^2 \diamond_{\alpha} t
$$

(1.7)

In [15], a generalization about Theorem 1.8 was given that

**THEOREM 1.9.** Suppose $p > 1, q = \frac{p}{p-1}, h > 0, h \in T, w : [0, h]_T \rightarrow (0, \infty)$ is continuous function, $f \in C_{\diamond_{\alpha}}([0, h]_T, \mathbb{R})$ with $f(0) = 0$. If both $f^\Delta$ and $f^\nabla$ are non-negative, then

$$
\alpha^4 \int_0^h |(f^2)^\Delta(t)| \Delta t + (1 - \alpha^4) \int_0^h |(f^2)^\nabla(t)| \nabla t \\
\leq \left( \int_0^h w^{1-q}(t) \diamond_{\alpha} t \right)^\frac{2}{q} \left( \int_0^h w(t) |f^{\diamond_{\alpha}}(t)|^p \diamond_{\alpha} t \right)^\frac{2}{p}.
$$

(1.8)

In this paper, we will firstly give the following inequality called Left-Opial Diamond-Alpha inequality of one variable.

**THEOREM 1.10.** (Left-Opial Diamond-Alpha inequality of one variable) Suppose $0, h \in T, f : T \rightarrow \mathbb{R}$ with $f(\rho(0)) = f(\sigma(0)) = f(0) = 0$, and $f^\Delta, f^\nabla$ are continuous functions, then

$$
\int_0^h \left| \frac{\partial f^2(t)}{\diamond_{\alpha} t} \right| \diamond_{\alpha} t \leq \alpha^2 h \int_0^h \left| \frac{\partial f(t)}{\Delta t} \right|^2 \Delta t + \alpha(1 - \alpha) h \int_0^h \left| \frac{\partial f(t)}{\nabla t} \right|^2 \nabla t \\
+ \alpha(1 - \alpha) h \int_0^h \left| \frac{\partial f(t)}{\Delta t} \right|^2 \Delta t + (1 - \alpha)^2 h \int_0^h \left| \frac{\partial f(t)}{\nabla t} \right|^2 \nabla t,
$$

(1.9)

where $\sigma(t) = \inf\{s \in T : s > t\}$ and $\rho(t) = \sup\{s \in T : s < t\}$.

Secondly, the following inequality called Opial Delta-Nabla Inequality of $n$ variables will be given.
Theorem 1.11. (Opial Delta-Nabla Inequality of $n$ variables) Suppose $0, h_i \in \mathbb{T}_i$ $(i = 1, 2, \ldots, n)$, diamond-$\alpha_i$ $(i = 1, 2, \ldots, n)$ integrable and differential function $f : [0, h_1]_{\mathbb{T}_1} \times [0, h_2]_{\mathbb{T}_2} \times \cdots \times [0, h_n]_{\mathbb{T}_n} \to \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ satisfies $f(v_1, v_2, \ldots, v_n) = 0$ when one of $v_i$ $(i = 1, 2, \ldots, n) = 0$, then

$$
\int_0^{h_n} \int_0^{h_{n-1}} \cdots \int_0^{h_1} \left| \frac{\partial^n f(v_1, v_2, \ldots, v_n)}{\partial v_1 \partial v_2 \cdots \partial v_n} \right| \cdot \omega_1 v_1 \omega_2 v_2 \cdots \omega_n v_n 
\leq \prod_{i=1}^n h_i \int_0^{h_n} \int_0^{h_{n-1}} \cdots \int_0^{h_1} \left| \frac{\partial^n f(v_1, v_2, \ldots, v_n)}{\partial v_1 \partial v_2 \cdots \partial v_n} \right| \cdot \omega_1 v_1 \omega_2 v_2 \cdots \omega_n v_n,
$$

(1.10)

where $\omega_i(i = 1, 2, \ldots, n) = 0$ or 1.

Theorem 1.11 may be helpful when generalizing Theorems 1.3, 1.5, 1.6 and 1.7 to higher dimension.

And then we will prove the following Theorem via Theorem 1.11.

Theorem 1.12. (Right-Opial Diamond-Alpha inequality of $n$ variables) Suppose $0, h_i \in \mathbb{T}_i$ $(i = 1, \ldots, n)$, diamond-$\alpha_i$ $(i = 1, \ldots, n)$ integrable and differential function $f : [0, h_1]_{\mathbb{T}_1} \times [0, h_2]_{\mathbb{T}_2} \times \cdots \times [0, h_n]_{\mathbb{T}_n} \to \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ satisfies

(i) $f(v_1, v_2, \ldots, v_n) = 0$ when one of $v_i$ $(i = 1, \ldots, n) = 0$;
(ii) $f(v_1, v_2, \ldots, v_n) \geq 0$ for all $\omega_i$ $(i = 1, \ldots, n) = 0$ or 1;

then

$$
\sum_{\omega_1, \ldots, \omega_n = 0 \text{ or } 1} \psi_i^3(\omega_1) \cdots \psi_n^3(\omega_n) \int_0^{h_n} \cdots \int_0^{h_1} \left| \frac{\partial^n f(v_1, v_2, \ldots, v_n)}{\partial v_1 \partial v_2 \cdots \partial v_n} \right| \cdot \omega_1 v_1 \cdots \omega_n v_n 
\leq \prod_{i=1}^n h_i \int_0^{h_n} \int_0^{h_{n-1}} \cdots \int_0^{h_1} \left| \frac{\partial^n f(v_1, v_2, \ldots, v_n)}{\partial v_1 \partial v_2 \cdots \partial v_n} \right| \cdot \omega_1 v_1 \omega_2 v_2 \cdots \omega_n v_n,
$$

(1.11)

where $\omega_i(i = 1, \ldots, n) = 0$ or 1, functions $\psi_i : [0, 1] \to [0, 1]$ $(i = 1, \ldots, n)$ are defined as follows,

$$
\psi_i(x) = \begin{cases} 
\alpha_i & \text{if } x = 1, \\
1 - \alpha_i & \text{if } x = 0.
\end{cases}
$$

In the end, we will obtain

Theorem 1.13. (Left-Opial Diamond-Alpha inequality of $n$ variables) Suppose

(i) $f(v_1, v_2, \ldots, v_n) = 0$ when one of $v_i$ $(i = 1, \ldots, n) = 0$;
(ii) $f(v_1, v_2, \ldots, v_n) = 0$ when one of $v_i$ $(i = 1, \ldots, n) = \rho(0)$;
(iii) $f(v_1, v_2, \ldots, v_n) = 0$ when one of $v_i$ $(i = 1, \ldots, n) = \sigma(0)$;
(iv) $f_i^{\Delta}(v_1, v_2, \ldots, v_i, \ldots, v_n) = f_i^{\nabla}(v_1, v_2, \ldots, \sigma(v_i), \ldots, v_n)$ for all $i = 1, 2, \ldots, n$;
(v) $f_i^{\nabla}(v_1, v_2, \ldots, v_i, \ldots, v_n) = f_i^{\Delta}(v_1, v_2, \ldots, \rho(v_i), \ldots, v_n)$ for all $i = 1, 2, \ldots, n$. 


then
\[
\int_0^{h_n} \int_0^{h_{n-1}} \cdots \int_0^{h_1} \left| \frac{\partial^n f^2(v_1, v_2, \cdots, v_n)}{\alpha_1 v_1 \alpha_2 v_2 \cdots \alpha_n v_n} \right| \alpha_1 v_1 \alpha_2 v_2 \cdots \alpha_n v_n \\
\leq \sum_{\omega_1, \omega_2, \ldots, \omega_n = 0 \text{ or } 1} \psi_1(\omega_1) \psi_2(\omega_2) \cdots \psi_n(\omega_n) \\
\times \sum_{\omega_1', \omega_2', \ldots, \omega_n' = 0 \text{ or } 1} \psi_1'(\omega_1') \psi_2'(\omega_2') \cdots \psi_n'(\omega_n') \\
\times \prod_{i=1}^{n} h_i \int_0^{h_n} \int_0^{h_{n-1}} \cdots \int_0^{h_1} \left| \frac{\partial^n f(v_1, v_2, \cdots, v_n)}{\alpha_1 v_1 \alpha_2 v_2 \cdots \alpha_n v_n} \right|^2 \alpha_1 v_1 \alpha_2 v_2 \cdots \alpha_n v_n,
\]

(1.12)

where \(\omega_i(i = 1, \cdots, n) = 0 \text{ or } 1\), functions \(\psi_i : \{0, 1\} \to [0, 1](i = 1, \cdots, n)\) are defined as follows,

\[
\psi_i(x) = \begin{cases} 
\alpha_i & \text{if } x = 1, \\
1 - \alpha_i & \text{if } x = 0.
\end{cases}
\]

The structure of this paper is: In section 2, some preliminaries will be mentioned. And we will prove Theorem 1.10 in section 3. In section 4, we prove Theorems from 1.11 to 1.13 via some lemmas, and introduce some special cases.

2. Preliminaries

For more basic knowledge about time scales, readers can consult [13]. See [6], one-order partial Diamond-Alpha dynamic derivative is denoted by

\[
\frac{\partial f(v_1, v_2, \cdots, v_n)}{\alpha_i v_i} \quad (i = 1, 2, \cdots, n).
\]

Further,

\[
\frac{\partial}{\alpha_j v_j} \frac{\partial f(v_1, v_2, \cdots, v_n)}{\alpha_i v_i} = \frac{\partial^2 f(v_1, v_2, \cdots, v_n)}{\alpha_i v_i \alpha_j v_j}, \quad (1 \leq i, j \leq n),
\]

and higher order partial Diamond-Alpha dynamic derivatives can be defined in the same way.

However in [15], the authors give another denotations: one-order partial Delta dynamic derivative is denoted by

\[
f_i^\Delta(v_1, v_2, \cdots, v_n) \quad (i = 1, 2, \cdots, n),
\]

similarly, one-order partial Diamond-Alpha dynamic derivatives can be denoted by

\[
f_i^{\alpha_i}(v_1, v_2, \cdots, v_n) \quad (i = 1, 2, \cdots, n).
\]
Further, for all $1 \leq i, j \leq n$,

$$\frac{\partial}{\partial \alpha_j} f_i^{\alpha_j} (v_1, v_2, \cdots, v_n) = f_{i,j}^{\alpha_i \alpha_j} (v_1, v_2, \cdots, v_n),$$

and higher order partial Diamond-Alpha dynamic derivatives also can be defined in the same way.

It is worth noting that the above two denotations are equivalent. And both of them are used in this paper.

If $\alpha = 0$ or 1, partial Diamond-Alpha dynamic derivatives change into the combinations of partial delta and nabla derivatives, such as

$$\frac{\partial^k f(v_1, v_2, \cdots, v_n)}{\Delta_j v_j \nabla_j v_j \cdots \Delta_k v_k},$$

for convenience we call it partial Delta-Nabla derivatives.

According to the definition of diamond-alpha differential, the formula next point out the link between partial Diamond-Alpha and Delta-Nabla dynamic derivatives.

$$\frac{\partial^n \phi(v_1, v_2, \cdots, v_n)}{\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n}
= \sum_{\omega_1, \omega_2, \cdots, \omega_n = 0 \text{ or } 1} \psi_1(\omega_1) \psi_2(\omega_2) \cdots \psi_n(\omega_n) \frac{\partial^n \phi(v_1, v_2, \cdots, v_n)}{\diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n},$$

(2.1)

where $\omega_1, \cdots, \omega_n = 0$ or 1 means all non-repetitive possibilities in binary and functions $\psi_i : \{0, 1\} \to [0, 1](i = 1, 2, \cdots, n)$ defined as follows,

$$\psi_i(x) = \begin{cases} \alpha_i & \text{if } x = 1, \\ 1 - \alpha_i & \text{if } x = 0. \end{cases}$$

In the same way, [12] gave the following theorem.

**THEOREM 2.1.** (see [12]) Suppose $\phi(v_1, v_2, \cdots, v_n)$ is diamond-$\alpha_i$ ($i = 1, \cdots, n$) integrable on $T_1 \times T_2 \times \cdots \times T_n$, then

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \phi(v_1, \cdots, v_n) \diamond_{\alpha_1} v_1 \cdots \diamond_{\alpha_n} v_n
= \sum_{\omega_1, \cdots, \omega_n = 0 \text{ or } 1} \psi_1(\omega_1) \cdots \psi_n(\omega_n) \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \phi(v_1, \cdots, v_n) \diamond_{\omega_1} v_1 \cdots \diamond_{\omega_n} v_n,$$

where $\omega_1, \cdots, \omega_n = 0$ or 1 means all non-repetitive possibilities in binary and functions $\psi_i : \{0, 1\} \to [0, 1](i = 1, 2, \cdots, n)$ defined as follows,

$$\psi_i(x) = \begin{cases} \alpha_i & \text{if } x = 1, \\ 1 - \alpha_i & \text{if } x = 0. \end{cases}$$
3. Left-Opial Diamond-Alpha inequality of one variable

In this section, one of our aims is giving the proof of Theorem 1.10. Meantime we will give a special case as well.

Proof of Theorem 1.10. Noting that

\[
\frac{\partial f^2(t)}{\Delta t} = \alpha \frac{\partial f^2(t)}{\Delta t} + (1 - \alpha) \frac{\partial f^2(t)}{\Delta t}
\]

Left hand side equal to

\[
\int_0^h \left| \frac{\partial f^2(t)}{\Delta t} \right| \Delta t + (1 - \alpha) \int_0^h \left| \frac{\partial f^2(t)}{\Delta t} \right| \nabla t
\]

\[
= \alpha \int_0^h \left| \alpha f^2(t)(f^\sigma(t) + f(t)) + (1 - \alpha)f^\nabla(t)(f^\rho(t) + f(t)) \right| \Delta t
\]

\[
+ (1 - \alpha) \int_0^h \left| \alpha f^2(t)(f^\sigma(t) + f(t)) + (1 - \alpha)f^\nabla(t)(f^\rho(t) + f(t)) \right| \nabla t
\]

\[
\leq \alpha^2 \int_0^h \left| f^2(t)(f^\sigma(t) + f(t)) \right| \Delta t + \alpha(1 - \alpha) \int_0^h \left| f^\nabla(t)(f^\rho(t) + f(t)) \right| \Delta t
\]

\[
+ \alpha(1 - \alpha) \int_0^h \left| f^\nabla(t)(f^\rho(t) + f(t)) \right| \nabla t + (1 - \alpha)^2 \int_0^h \left| f^\nabla(t)(f^\rho(t) + f(t)) \right| \nabla t.
\]

Based on the Theorems 1.1 and 1.2, the following two inequalities have been proved.

\[
\int_0^h \left| f^\Delta(t)(f^\sigma(t) + f(t)) \right| \Delta t \leq h \int_0^h \left| f^\Delta(t) \right|^2 \Delta t,
\]

(3.1)

\[
\int_0^h \left| f^\nabla(t)(f^\rho(t) + f(t)) \right| \nabla t \leq h \int_0^h \left| f^\nabla(t) \right|^2 \nabla t.
\]

(3.2)

Since \( f^\Delta, f^\nabla \) are continuous functions, we have

\[
f^\Delta(x) = f^\nabla(\sigma(x)), \hspace{1cm} f^\nabla(x) = f^\Delta(\rho(x)) \hspace{1cm} \forall x \in \mathbb{T}.
\]

Hence we get

\[
\int_0^h \left| f^\nabla(t)(f^\rho(t) + f(t)) \right| \Delta t = \int_0^h \left| \frac{\partial f^2(t)}{\nabla t} \right| \Delta t = \int_0^h \left| \frac{\partial f^2(\rho(t))}{\Delta t} \right| \Delta t.
\]

Noting that \( f(\rho(0)) = f(0) = 0 \), then we use Theorem 1.1

\[
\int_0^h \left| \frac{\partial f^2(\rho(t))}{\Delta t} \right| \Delta t \leq h \int_0^h \left| \frac{\partial f(\rho(t))}{\Delta t} \right|^2 \Delta t.
\]
So we have
\[ \int_{0}^{h} |f^{\alpha}(t)(f^{\sigma}(t) + f(t))| \Delta t \leq h \int_{0}^{h} \left| \frac{\partial f(\rho(t))}{\Delta t} \right|^{2} \Delta t. \quad (3.3) \]

In the same manner, we can get
\[ \int_{0}^{h} |f^{\alpha}(t)(f^{\sigma}(t) + f(t))| \nabla t \leq h \int_{0}^{h} \left| \frac{\partial f(\sigma(t))}{\nabla t} \right|^{2} \nabla t. \quad (3.4) \]

According to (3.1), (3.2), (3.3) and (3.4), we can complete the proof
\[
\begin{align*}
\int_{0}^{h} \left| \frac{\partial f^{2}(t)}{\alpha \nabla t} \right| \Delta t &= \alpha^{2} \int_{0}^{h} \left| f^{\alpha}(t)(f^{\sigma}(t) + f(t)) \right| \Delta t + \alpha(1 - \alpha) \int_{0}^{h} \left| f^{\alpha}(t)(f^{\rho}(t) + f(t)) \right| \Delta t \\
&+ \alpha(1 - \alpha) \int_{0}^{h} \left| f^{\alpha}(t)(f^{\sigma}(t) + f(t)) \right| \nabla t + (1 - \alpha)^{2} h \int_{0}^{h} \left| f^{\alpha}(t)(f^{\rho}(t) + f(t)) \right| \nabla t \\
&\leq \alpha^{2} h \int_{0}^{h} \left| \frac{\partial f(t)}{\Delta t} \right|^{2} \Delta t + \alpha(1 - \alpha) h \int_{0}^{h} \left| \frac{\partial f(\rho(t))}{\Delta t} \right|^{2} \Delta t \\
&+ \alpha(1 - \alpha) h \int_{0}^{h} \left| \frac{\partial f(\sigma(t))}{\nabla t} \right|^{2} \nabla t + (1 - \alpha)^{2} h \int_{0}^{h} \left| \frac{\partial f(t)}{\nabla t} \right|^{2} \nabla t \\
&= \alpha^{2} h \int_{0}^{h} \left| \frac{\partial f(t)}{\Delta t} \right|^{2} \Delta t + \alpha(1 - \alpha) h \int_{0}^{h} \left| \frac{\partial f(t)}{\nabla t} \right|^{2} \nabla t \\
&+ \alpha(1 - \alpha) h \int_{0}^{h} \left| \frac{\partial f(t)}{\nabla t} \right|^{2} \nabla t + (1 - \alpha)^{2} h \int_{0}^{h} \left| \frac{\partial f(t)}{\nabla t} \right|^{2} \nabla t. \quad \square
\end{align*}
\]

We can find that if \( \alpha = 0 \) or 1, Theorem 1.10 reduces to Theorems 1.1 and 1.2. The right hand side of (1.9) equivalent to
\[
\begin{align*}
\alpha^{2} h \int_{0}^{h} \left| \frac{\partial f(t)}{\Delta t} \right|^{2} \Delta t + \alpha(1 - \alpha) h \int_{0}^{h} \left| \frac{\partial f(t)}{\nabla t} \right|^{2} \Delta t \\
+ \alpha(1 - \alpha) h \int_{0}^{h} \left| \frac{\partial f(t)}{\nabla t} \right|^{2} \nabla t + (1 - \alpha)^{2} h \int_{0}^{h} \left| \frac{\partial f(t)}{\nabla t} \right|^{2} \nabla t \\
= \alpha h \int_{0}^{h} |f^{\alpha}(t)|^{2} \Delta t + (1 - \alpha) h \int_{0}^{h} |f^{\alpha}(t)|^{2} \nabla t \\
= h \int_{0}^{h} \alpha |f^{\alpha}(t)|^{2} + (1 - \alpha) |f^{\alpha}(t)|^{2} \nabla t. \quad (3.5)
\end{align*}
\]

If taking \( \alpha = \frac{1}{2} \), then we have the following corollary.

**Corollary 3.1.** Suppose \( 0, h \in \mathbb{T} \) and \( f : \mathbb{T} \to \mathbb{R} \) with \( f(\sigma(0)) = f(0) = \)
\[ f(\rho(0)) = 0, \text{ and } f^\Delta, f^\nabla \text{ are continuous functions, then} \]
\[
\int_0^h \left| \frac{\partial f^2(t)}{\partial \frac{\tau}{2}} \right|^{\frac{\tau}{2}} t \leq \frac{1}{4} h \left( \int_0^h \left| \frac{\partial f(t)}{\Delta t} \right|^2 \Delta t + \int_0^h \left| \frac{\partial f(t)}{\nabla t} \right|^2 \nabla t \right)
+ \int_0^h \left| \frac{\partial f(t)}{\Delta t} \right|^2 \nabla t + \int_0^h \left| \frac{\partial f(t)}{\nabla t} \right|^2 \nabla t \right)
= \frac{h}{2} \int_0^h |f^\Delta(t)|^2 + |f^\nabla(t)|^2 \frac{\tau}{2} t. \tag{3.6}
\]

4. Opial inequalities of \( n \) variables on time scales

In [13, Theorem 6.80, 6.94], the authors give Delta and nabla dynamic derivatives Leibniz Formula. They also point Delta-Nabla dynamic derivatives Leibniz Formula can be defined in the same manner in the end of chapter 6, that’s means, they explore the form of

\[
(f(u_1, u_2, \cdots, u_n)g(u_1, u_2, \cdots, u_n))^{\omega_1}_{\omega_2} \cdots \omega_n , \tag{4.1}
\]

where \( \omega_i = 0 \) or 1.

And we will explore the following formula in the next,

\[
(f(u_1, u_2, \cdots, u_n)g(u_1, u_2, \cdots, u_n))^{\omega_1}_{\omega_2} \cdots \omega_n , \tag{4.2}
\]

where \( \omega_i = 0 \) or 1 for all \( 1 \leq i \leq n \) and \( i_j \neq i_k \) when \( j \neq k \) for all \( 1 \leq j, k \leq n \). In this paper, we only take

\[
(f(u_1, u_2, \cdots, u_n)g(u_1, u_2, \cdots, u_n))^{\omega_1}_{\omega_2} \cdots \omega_n \tag{4.2}
\]
as an example.

To simplify statement, we introduce the notation

\[ f^\tau_1,\tau_2,\cdots,\tau_n (u_1, u_2, \cdots, u_n), \tag{4.3} \]

where \( \tau_i = 1, 2, \cdots, n \) in \( \{\Delta, \nabla, \sigma, \rho, O\} \). And if \( \tau_i \in \{\Delta, \nabla\} \), then we differentiate with respect to the \( u_i \), else if \( \tau_i \in \{\sigma, \rho\}, \) then replace \( u_i \) with \( \tau_i (u_i) \), and we do nothing to \( u_i \) when \( \tau_i = O \). For examples, we set

\[ f^{\Delta,\sigma,\Delta,\sigma}(u_1, u_2, u_3, u_4, u_5) := \frac{\partial^2 f(u_1, \sigma(u_2), \sigma(u_3), u_4, u_5)}{\Delta_1 u_1 \Delta_3 u_4}, \]

and

\[ f^{\rho,\nabla,\rho,\rho,\rho}(u_1, u_2, u_3, u_4, u_5, u_6) := \frac{\partial f(u_1, u_2, u_3, \rho(u_4), u_5, \rho(u_6))}{\nabla_2 u_2}. \]

One thing we should emphasize is that every variable only correspond to one \( \tau \) in this notation.
Lemma 4.1. If \( f(u_1,u_2,\cdots,u_n) \), \( g(u_1,u_2,\cdots,u_n) \) are \( \Delta_i \)-differential \( (i=1,2,\cdots,n) \) on \( \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n \), then

\[
\frac{\partial^n(f(u_1,u_2,\cdots,u_n)g(u_1,u_2,\cdots,u_n))}{\Delta_1 u_1 \Delta_2 u_2 \cdots \Delta_n u_n} = \frac{1}{2^n} \sum_{(\tau_1,\tau_2,\cdots,\tau_n) \in L_0^{(n)}} f_{1,2,\cdots,n}^{\tau_1,\tau_2,\cdots,\tau_n}(u_1,u_2,\cdots,u_n)g_{1,2,\cdots,n}^{\tau_{n+1},\tau_{n+2},\cdots,\tau_2n}(u_1,u_2,\cdots,u_n),
\]

where \( L_0^{(n)} \) defined as follows,

\[
L_0^{(n)} = \{ (\tau_1,\tau_2,\cdots,\tau_n) : \text{one of } \tau_i \text{ and } \tau_{n+i} \text{ is } \Delta, \text{another is } O \text{ or } \sigma \}. \tag{4.5}
\]

Proof. We will use induction. Clearly, we have

\[
(f(s)g(s))^\Delta = f^\Delta(s)g(s) + f^\sigma(s)g^\Delta(s) = f^\Delta(s)g^\sigma(s) + f(s)g^\Delta(s) = \frac{1}{2}(f(s)g^\Delta(s) + f^\sigma(s)g^\Delta(s) + f^\Delta(s)g^\sigma(s) + f(s)g^\Delta(s)). \tag{4.6}
\]

On the other hand, when \( n = 1 \), \( L_0^{(1)} = \{ (O,\Delta), (\sigma,\Delta), (\Delta, O), (\Delta, \sigma) \} \). So (4.4) holds when \( n = 1 \).

Then we suppose it is true for \( n = k \), that’s means

\[
\frac{\partial^k(f(u_1,u_2,\cdots,u_k)g(u_1,u_2,\cdots,u_k))}{\Delta_1 u_1 \Delta_2 u_2 \cdots \Delta_k u_k} = \frac{1}{2^k} \sum_{(\tau_1,\tau_2,\cdots,\tau_k) \in L_0^{(k)}} f_{1,2,\cdots,k}^{\tau_1,\tau_2,\cdots,\tau_k}(u_1,u_2,\cdots,u_k)g_{1,2,\cdots,k}^{\tau_{k+1},\tau_{k+2},\cdots,\tau_{2k}}(u_1,u_2,\cdots,u_k). \tag{4.7}
\]

When \( n = k + 1 \), we have

\[
\frac{\partial^{k+1}(f(u_1,\cdots,u_k,u_{k+1})g(u_1,\cdots,u_k,u_{k+1}))}{\Delta_1 u_1 \cdots \Delta_k u_k \Delta_{k+1} u_{k+1}} = \frac{\partial}{\Delta_{k+1} u_{k+1}} \left( \frac{\partial^k(f(u_1,\cdots,u_k,u_{k+1})g(u_1,\cdots,u_k,u_{k+1}))}{\Delta_1 u_1 \cdots \Delta_k u_k} \right) \]

\[
= \frac{1}{2^k} \frac{\partial}{\Delta_{k+1} u_{k+1}} \left( \sum_{(\tau_1,\tau_2,\cdots,\tau_k) \in L_0^{(k)}} f_{1,2,\cdots,k+1}^{\tau_1,\cdots,\tau_k,O}(u_1,\cdots,u_k,u_{k+1})g_{1,2,\cdots,k+1}^{\tau_{k+1},\cdots,\tau_{2k},O}(u_1,\cdots,u_k,u_{k+1}) \right)
\]

\[
= \frac{1}{2^{k+1}} \sum_{(\tau_1,\tau_2,\cdots,\tau_k) \in L_0^{(k)}} \left( f_{1,2,\cdots,k+1}^{\tau_1,\cdots,\tau_k,\Delta}(u_1,\cdots,u_k,u_{k+1})g_{1,2,\cdots,k+1}^{\tau_{k+1},\cdots,\tau_{2k},\sigma}(u_1,\cdots,u_k,u_{k+1}) + f_{1,2,\cdots,k+1}^{\tau_1,\cdots,\tau_k}(u_1,\cdots,u_k,u_{k+1})g_{1,2,\cdots,k+1}^{\tau_{k+1},\cdots,\tau_{2k}}(u_1,\cdots,u_k,u_{k+1}) \right).
\]
LEMMA 4.1. and \\
So we can get where $L$ \\
REMARK 4.2. We give an example when $\omega_i (i = 1, 2, \ldots, n) = 0$ or 1, then $\omega = V$ or $\Delta$. Noting that $\Delta$ and $V$ have the same algorithms, so if replace $\Delta_i$ with $\omega_i$, we can get the following lemma. And in order to simplify notation, we define \\
$$\Psi(x) = \begin{cases} \\
\sigma, & \text{if } x = 1, \\
\rho, & \text{if } x = 0, \\
\end{cases}$$
and \\
$$p = \sum_{i=1}^{n} \omega_i 2^{i-1}.$$ \\
LEMMA 4.2. If $f(u_1, u_2, \ldots, u_n)$, $g(u_1, u_2, \ldots, u_n)$ are $\Delta_i$-differential and $\nabla_i$-differential ($i = 1, 2, \ldots, n$) on $\mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$, then \\
$$\frac{\partial^n}{\partial \omega_1 u_1 \partial \omega_2 u_2 \cdots \partial \omega_n u_n} f(u_1, u_2, \ldots, u_n) g(u_1, u_2, \ldots, u_n)$$
$$= \frac{1}{2^n} \sum_{(\tau_1, \tau_2, \ldots, \tau_{2n}) \in L_p^{(n)}} f^{\tau_1 \tau_2 \cdots \tau_n} (u_1, u_2, \ldots, u_n) g^{\tau_{n+1} \tau_{n+2} \cdots \tau_{2n}} (u_1, u_2, \ldots, u_n),$$
where $L_p^{(n)}$ defined as follows, \\
$$L_p^{(n)} = \{ (\tau_1, \tau_2, \ldots, \tau_{2n}) : \text{one of } \tau_i \text{ and } \tau_{n+i} \text{ is } \omega_i, \text{ another is } O \text{ or } \Psi(\omega_i) \}. $$
REMARK 4.1. If $p = 0$, then every $\omega_i = 1 (i = 1, 2, \ldots, n)$, lemma 4.2 reduces to lemma 4.1. \\
REMARK 4.2. We give an example when $n = 2, p = 1$, then \\
$$L_1^{(2)} = \{ (\Delta, \rho, \sigma, V), (\Delta, V, \sigma, \rho), (\sigma, V, \Delta, \rho), (\sigma, \rho, \Delta, V), \\
(\Delta, O, O, V), (\Delta, V, O, O), (O, V, \Delta, O), (O, O, \Delta, V), \\
(\Delta, \rho, O, V), (\Delta, V, \sigma, O), (\sigma, V, \Delta, O), (\sigma, O, \Delta, V), \\
(\Delta, O, \sigma, V), (\Delta, V, O, \rho), (O, V, \Delta, \rho), (O, \rho, \Delta, V) \}. $$
So we can get \\
$$\frac{\partial^2}{\Delta_1 s_1 V_2 s_2} f(s_1, s_2) g(s_1, s_2)$$
$$= \frac{1}{4} \left( f_{1,2}^{\Delta \rho} (s_1, s_2) g_{1,2}^{\sigma, V} (s_1, s_2) + f_{1,2}^{\sigma, V} (s_1, s_2) g_{1,2}^{\Delta \rho} (s_1, s_2) + f_{1,2}^{\Delta V} (s_1, s_2) g_{1,2}^{\sigma, \rho} (s_1, s_2) \right)$$
where $L_{1,2}^\Delta (s_1, s_2) g_{1,2}^\Delta (s_1, s_2) + f_{1,2}^\Delta (s_1, s_2) g_{1,2}^\Delta (s_1, s_2)$
\[+ f_{1,2}^\Delta (s_1, s_2) g_{1,2}^\Delta (s_1, s_2) + f_{1,2}^\Delta (s_1, s_2) g_{1,2}^\Delta (s_1, s_2) + f_{1,2}^\Delta (s_1, s_2) g_{1,2}^\Delta (s_1, s_2) \]
Noting that if $g = f$, then lemma 4.2 reduces to
\[\text{COROLLARY 4.1. If } f(u_1, u_2, \ldots, u_n) \text{ is } \Delta_i\text{-differential and } \nabla_i\text{-differential} (i = 1, 2, \ldots, n) \text{ on } \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n, \text{ then} \]
\[
\frac{\partial^n (f^2(u_1, u_2, \ldots, u_n))}{\Diamond u_1 \Diamond u_2 \cdots \Diamond u_n u_n} = \frac{1}{2^n} \sum_{\langle \tau_1, \tau_2, \cdots, \tau_{2n} \rangle \in L_p^{(n)}} f_{1,2,\ldots,n}^{\tau_1, \tau_2, \cdots, \tau_{2n}} (u_1, u_2, \ldots, u_n), \quad (4.11) \]
where $L_p^{(n)}$ is defined as follows,
\[L_p^{(n)} = \{ \langle \tau_1, \tau_2, \cdots, \tau_{2n} \rangle : \text{one of } \tau_i \text{ and } \tau_{n+i} \text{ is } \Diamond \omega_k \text{, another is } O \text{ or } \Psi(\omega_k) \}. \quad (4.12) \]

In order to prove Theorem 1.12, we give the proof of following theorem first.

**Proof of Theorem 1.11.** Set
\[m(u_1, u_2, \ldots, u_n) = \int_0^{u_n} \int_0^{u_{n-1}} \cdots \int_0^{u_1} \left[ \frac{\partial^n f(v_1, v_2, \ldots, v_n)}{\Diamond v_1 \Diamond v_2 \cdots \Diamond v_n v_n} \right] \Diamond v_1 \Diamond v_2 \cdots \Diamond v_n v_n. \]

Hence we get
\[\frac{\partial^n m(u_1, u_2, \ldots, u_n)}{\Diamond u_1 \Diamond u_2 \cdots \Diamond u_n u_n} = \left[ \frac{\partial^n f(u_1, u_2, \ldots, u_n)}{\Diamond u_1 \Diamond u_2 \cdots \Diamond u_n u_n} \right]. \]

And then we suppose set $S = \{ j_1, j_2, \cdots, j_s \} \subset N = \{ 1, 2, \cdots, n \}$, that's means $j_1, j_2, \cdots, j_s$ are arbitrary $s$ unequal numbers in the set $N$ where $s < n$. We also set $N \setminus S = \{ j_{s+1}, j_{s+2}, \cdots, j_n \}$.

\[\frac{\partial^s m(u_1, u_2, \ldots, u_n)}{\Diamond u_1 \Diamond u_2 \cdots \Diamond u_s u_s} = \left[ \frac{\partial^s f(v_1, \cdots, v_n)}{\Diamond v_1 \cdots \Diamond v_n v_n} \right] \Diamond v_1 \cdots \Diamond v_n v_n \]

\[= \frac{\partial^s}{\Diamond u_1 \Diamond u_2 \cdots \Diamond u_s u_s} \int_0^{u_n} \int_0^{u_{n-1}} \cdots \int_0^{u_1} \left[ \frac{\partial^s f(v_1, \cdots, v_n)}{\Diamond v_1 \cdots \Diamond v_n v_n} \right] \Diamond v_1 \cdots \Diamond v_n v_n \]
In particular, if we let the set \( S = \emptyset \), then
\[
m(u_1, u_2, \cdots, u_n) \geq |f(u_1, u_2, \cdots, u_n)|.
\] (4.14)

Then using corollary 4.1, we have
\[
\int_0^{h_1} \cdots \int_0^{h_1} \int_0^{h_1} \cdots \int_0^{h_1} \int_0^{h_1} \cdots \int_0^{h_1} \cdots \frac{\partial^n f(v_1, \cdots, v_n)}{\partial v_1 \cdots \partial v_n} \partial v_1 = u_j \partial u_j
\]

\[
= \left| \frac{\partial^2 f(u_1, u_2, \cdots, u_n)}{\partial u_j \partial u_j} \right|
\] (4.13)

\[
m(u_1, u_2, \cdots, u_n) \geq |f(u_1, u_2, \cdots, u_n)|.
\] (4.14)
then differentiable function $f$ is

Further, we can give the proof of Theorem 1.12.

**COROLLARY 4.2.** Suppose for all $i \in \mathbb{N}$, $0, h_i \in T_i$, diamond-$\alpha_i$ integral and differentiable function $f : [0, h_1]^{\mathbb{T}_1} \times [0, h_2]^{\mathbb{T}_2} \times \cdots \times [0, h_n]^{\mathbb{T}_n} \to \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ satisfies that $f(v_1, v_2, \cdots, v_n) = 0$ when one of $v_i$ ($i = 1, 2, \cdots, n$) is $0$ and $\Delta$-integrable, then

$$
\int_0^{h_n} \int_0^{h_{n-1}} \cdots \int_0^{h_1} \frac{\partial^n f^2(v_1, v_2, \cdots, v_n)}{\Delta v_1 \Delta v_2 \cdots \Delta v_n} \Delta v_1 \Delta v_2 \cdots \Delta v_n.
$$

Taking $\omega_i$ ($i = 1, 2, \cdots, n$) = 0, hence we get

**COROLLARY 4.3.** Suppose $0, h_i \in T_i$, diamond-$\alpha_i$ ($i = 1, 2, \cdots, n$) integral and differentiable function $f : [0, h_1]^{\mathbb{T}_1} \times [0, h_2]^{\mathbb{T}_2} \times \cdots \times [0, h_n]^{\mathbb{T}_n} \to \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ satisfies that $f(v_1, v_2, \cdots, v_n) = 0$ when one of $v_i$ ($i = 1, 2, \cdots, n$) is $0$ and $\nabla$-integrable, then

$$
\int_0^{h_n} \int_0^{h_{n-1}} \cdots \int_0^{h_1} \frac{\partial^n f^2(v_1, v_2, \cdots, v_n)}{\nabla v_1 \nabla v_2 \cdots \nabla v_n} \nabla v_1 \nabla v_2 \cdots \nabla v_n.
$$

Further, we can find that Theorem 1.1 and 1.2 is the cases of $n = 1$ in Corollary 4.2 and 4.3.

Based on Theorem 1.11, we can give the proof of Theorem 1.12.

**Proof of Theorem 1.12.** Firstly, based on the condition (ii), we have

$$
\left( \psi_1(\omega_1) \psi_2(\omega_2) \cdots \psi_n(\omega_n) f_{1,\cdots,n}^{\omega_1,\cdots,\omega_n} \right)^2
\leq \left( \sum_{\omega_1,\cdots,\omega_n=0 \text{ or } 1} \psi_1(\omega_1) \psi_2(\omega_2) \cdots \psi_n(\omega_n) f_{1,\cdots,n}^{\omega_1,\cdots,\omega_n} \right)^2
= \left( \frac{\partial^n f(v_1, v_2, \cdots, v_n)}{\Delta \omega_1, \omega_2, \cdots, \omega_n} \right)^2.
$$

(4.18)
And then using Theorem 1.11, we have

\[ \sum_{\omega_1, \ldots, \omega_n = 0 \text{ or } 1} \psi_i^3 (\omega_1) \cdots \psi_n^3 (\omega_n) \int_0^{h_n} \cdots \int_0^{h_1} \left| \frac{\partial^n (f^2(v_1, \ldots, v_n))}{\partial \omega_1 v_1 \cdots \partial \omega_n v_n} \right| \omega_1 v_1 \cdots \omega_n v_n \]

\[ \leq \sum_{\omega_1, \ldots, \omega_n = 0 \text{ or } 1} \psi_i^3 (\omega_1) \cdots \psi_n^3 (\omega_n) \sum_{i=1}^{n} h_i \int_0^{h_n} \cdots \int_0^{h_1} \left( \frac{\partial^n f(v_1, \ldots, v_n)}{\partial \omega_1 v_1 \cdots \partial \omega_n v_n} \right)^2 \omega_1 v_1 \cdots \omega_n v_n \]

\[ = \sum_{\omega_1, \ldots, \omega_n = 0 \text{ or } 1} \psi_i (\omega_1) \cdots \psi_n (\omega_n) \sum_{i=1}^{n} h_i \int_0^{h_n} \cdots \int_0^{h_1} \left( \frac{\partial^n f(v_1, v_2, \ldots, v_n)}{\partial \omega_1 v_1 \partial \omega_2 v_2 \cdots \partial \omega_n v_n} \right)^2 \omega_1 v_1 \partial \omega_2 v_2 \cdots \partial \omega_n v_n. \quad (4.19) \]

We have the following Corollary when \( \alpha = \frac{1}{2} \) in Theorem 1.12,

**COROLLARY 4.4.** Suppose \( 0, h_i \in \mathbb{T}_i (\forall i \in \mathbb{N}) \), diamond- \( \alpha_i \) (\( i = 1, 2, \ldots, n \)) integrable and differentiable function \( f : [0, h_1]_{\mathbb{T}_1} \times [0, h_2]_{\mathbb{T}_2} \times \cdots \times [0, h_n]_{\mathbb{T}_n} \rightarrow \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \) satisfies

(i) \( f(v_1, v_2, \ldots, v_n) = 0 \) when one of \( v_i \) (\( i = 1, 2, \ldots, n \)) = 0;

(ii) \( f^{(\alpha_1, \alpha_2, \ldots, \alpha_n)} (v_1, v_2, \ldots, v_n) \geq 0 \) for all \( \omega_i = 0 \) or 1;

then

\[ \sum_{\omega_1, \ldots, \omega_n = 0 \text{ or } 1} \int_0^{h_n} \cdots \int_0^{h_1} \left| \frac{\partial^n (f^2(v_1, v_2, \ldots, v_n))}{\partial \omega_1 v_1 \partial \omega_2 v_2 \cdots \partial \omega_n v_n} \right| \omega_1 v_1 \partial \omega_2 v_2 \cdots \partial \omega_n v_n \]

\[ \leq 8^n \sum_{i=1}^{n} h_i \int_0^{h_n} \cdots \int_0^{h_1} \left( \frac{\partial^n f(v_1, v_2, \ldots, v_n)}{\partial \omega_1 v_1 \partial \omega_2 v_2 \cdots \partial \omega_n v_n} \right)^2 \omega_1 v_1 \partial \omega_2 v_2 \cdots \partial \omega_n v_n, \quad (4.20) \]

where \( \omega_i (i = 1, 2, \ldots, n) = 0 \) or 1.

Theorem 1.12 reduces to Theorem 1.8 when \( n = 1 \), and if taking \( n = 2 \), we have the following Corollary.

**COROLLARY 4.5.** Suppose \( 0, h_1 \in \mathbb{T}_1, 0, h_2 \in \mathbb{T}_2 \), diamond- \( \alpha_i \) (\( i = 1, 2 \)) integrable and differentiable function \( f : [0, h_1]_{\mathbb{T}_1} \times [0, h_2]_{\mathbb{T}_2} \rightarrow \mathbb{R} \times \mathbb{R} \) satisfies

(i) \( f(v_1, v_2) = 0 \) when one of \( v_i \) (\( i = 1, 2 \)) = 0;

(ii) \( f^{(\alpha_1, \alpha_2)} (v_1, v_2) \geq 0 \) for all \( \omega_i \) (\( i = 1, 2 \)) = 0 or 1;

then
\[
\sum_{\omega_1, \omega_2=0}^{1} \psi_1^3(\omega_1) \psi_2^3(\omega_2) \int_0^{h_2} \int_0^{h_1} \left| \frac{\partial^2 (f^2(v_1, v_2))}{\Delta_1 v_1 \Delta_2 v_2} \right| \Delta_1 v_1 \Delta_2 v_2
\]
\[= \alpha_1^3 \alpha_2^3 \int_0^{h_2} \int_0^{h_1} \left| \frac{\partial^2 (f^2(v_1, v_2))}{\Delta_1 v_1 \Delta_2 v_2} \right| \Delta_1 v_1 \Delta_2 v_2
\]
\[+ \alpha_1^3 (1 - \alpha_2)^3 \int_0^{h_2} \int_0^{h_1} \left| \frac{\partial^2 (f^2(v_1, v_2))}{\Delta_1 v_1 \nabla_2 v_2} \right| \Delta_1 v_1 \nabla_2 v_2
\]
\[+ (1 - \alpha_1)^3 \alpha_2^3 \int_0^{h_2} \int_0^{h_1} \left| \frac{\partial^2 (f^2(v_1, v_2))}{\nabla_1 v_1 \Delta_2 v_2} \right| \nabla_1 v_1 \Delta_2 v_2
\]
\[+ (1 - \alpha_1)^3 (1 - \alpha_2)^3 \int_0^{h_2} \int_0^{h_1} \left| \frac{\partial^2 (f^2(v_1, v_2))}{\nabla_1 v_1 \nabla_2 v_2} \right| \nabla_1 v_1 \nabla_2 v_2 \]
\[\leq h_1 h_2 \int_0^{h_2} \int_0^{h_1} \left| \frac{\partial^2 f(v_1, v_2)}{\Delta_1 v_1 \Delta_2 v_2} \right|^2 \Delta_1 v_1 \Delta_2 v_2, \tag{4.21}\]

where \(\omega_i \ (i=1,2) = 0 \) or 1.

Next we explore the Left-Opial Diamond-Alpha Inequality of \(n\) variable. First of all, suppose \(f(v_1, v_2, \cdots, v_n)\) satisfies
\[
f_i^{\Delta_1}(v_1, v_2, \cdots, v_i, \cdots, v_n) = f_i^{\nabla_1}(v_1, v_2, \cdots, \sigma(v_i), \cdots, v_n),
\]
and
\[
f_i^{\nabla_1}(v_1, v_2, \cdots, v_i, \cdots, v_n) = f_i^{\Delta_1}(v_1, v_2, \cdots, \rho(v_i), \cdots, v_n),
\]
for all \(i=1,2,\cdots,n\).

And we set
\[
\kappa_x = \begin{cases} 
\rho & \text{if } x = -1, \\
\sigma & \text{if } x = 1, \\
O & \text{if } x = 0,
\end{cases} \tag{4.22}
\]
and
\[
f^\kappa(v_1, v_2, \cdots, v_n) := f_{1,2,\cdots,n}^{\kappa_{\omega_1^{-\omega_1}} \cdots \kappa_{\omega_n^{-\omega_n}}}(v_1, v_2, \cdots, v_n). \tag{4.23}\]

Under the conditions and notations above, the following lemma holds.

**Lemma 4.3.** Suppose \(f: \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n \rightarrow \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}\), and function \(f\) satisfies
\[
f_i^{\Delta_1}(v_1, v_2, \cdots, v_i, \cdots, v_n) = f_i^{\nabla_1}(v_1, v_2, \cdots, \sigma(v_i), \cdots, v_n),
\]
and
\[
f_i^{\nabla_1}(v_1, v_2, \cdots, v_i, \cdots, v_n) = f_i^{\Delta_1}(v_1, v_2, \cdots, \rho(v_i), \cdots, v_n),
\]
for all \(i=1,2,\cdots,n\), then
\[
\frac{\partial^n f(v_1, v_2, \cdots, v_n)}{\Delta_1 v_1 \Delta_2 v_2 \cdots \Delta_n v_n} = \frac{\partial^n f^\kappa(v_1, v_2, \cdots, v_n)}{\Delta_1 v_1 \Delta_2 v_2 \cdots \Delta_n v_n}, \tag{4.24}
\]
where both $\omega_i'$ and $\omega_i$ equal to 0 or 1 for all $i = 1, 2, \ldots, n$.

Hence we can give the proof of Theorem 1.13.

**Proof of Theorem 1.13.** Noting that

$$
\frac{\partial^n f^2(v_1, v_2, \ldots, v_n)}{\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n}
= \sum_{\omega_1, \omega_2, \ldots, \omega_n = 0 \text{ or } 1} \psi_1(\omega_1) \psi_2(\omega_2) \cdots \psi_n(\omega_n) \frac{\partial^n f^2(v_1, v_2, \ldots, v_n)}{\diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n},
$$

then, the left hand side of (1.12) can be rewritten as follows,

$$
\int_0^{h_n} \int_0^{h_{n-1}} \cdots \int_0^{h_1} \left| \frac{\partial^n f^2(v_1, v_2, \ldots, v_n)}{\diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n} \right| \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n
= \sum_{\omega_1, \omega_2, \ldots, \omega_n = 0 \text{ or } 1} \psi_1(\omega_1) \psi_2(\omega_2) \cdots \psi_n(\omega_n) \int_0^{h_n} \int_0^{h_{n-1}} \cdots \int_0^{h_1} \\
\left| \sum_{\omega_1', \omega_2', \ldots, \omega_n' = 0 \text{ or } 1} \psi_1(\omega_1') \psi_2(\omega_2') \cdots \psi_n(\omega_n') \frac{\partial^n f^2(v_1, v_2, \ldots, v_n)}{\diamond_{\omega_1'} v_1 \diamond_{\omega_2'} v_2 \cdots \diamond_{\omega_n'} v_n} \right| \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n
\leq \sum_{\omega_1, \omega_2, \ldots, \omega_n = 0 \text{ or } 1} \psi_1(\omega_1) \psi_2(\omega_2) \cdots \psi_n(\omega_n) \int_0^{h_n} \int_0^{h_{n-1}} \cdots \int_0^{h_1} \\
\sum_{\omega_1', \omega_2', \ldots, \omega_n' = 0 \text{ or } 1} \psi_1(\omega_1') \psi_2(\omega_2') \cdots \psi_n(\omega_n') \left| \frac{\partial^n f^2(v_1, v_2, \ldots, v_n)}{\diamond_{\omega_1'} v_1 \diamond_{\omega_2'} v_2 \cdots \diamond_{\omega_n'} v_n} \right| \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n
= \sum_{\omega_1, \omega_2, \ldots, \omega_n = 0 \text{ or } 1} \psi_1(\omega_1) \psi_2(\omega_2) \cdots \psi_n(\omega_n) \sum_{\omega_1', \omega_2', \ldots, \omega_n' = 0 \text{ or } 1} \psi_1(\omega_1') \psi_2(\omega_2') \cdots \psi_n(\omega_n')}
$$
\[
\sum_{\omega_1, \omega_2, \ldots, \omega_n = 0 \text{ or } 1} \psi_1(\omega_1) \psi_2(\omega_2) \cdots \psi_n(\omega_n) \sum_{\omega'_1, \omega'_2, \ldots, \omega'_n = 0 \text{ or } 1} \psi_1(\omega'_1) \psi_2(\omega'_2) \cdots \psi_n(\omega'_n)
\]

\[
= \prod_{i=1}^{n} h_i \int_0^{h_n} \int_0^{h_n-1} \cdots \int_0^{h_1} \left| \frac{\partial^n f(v_1, v_2, \ldots, v_n)}{\omega_1 v_1 \omega_2 v_2 \cdots \omega_n v_n} \right|^2 \omega_1 v_1 \omega_2 v_2 \cdots \omega_n v_n
\]

In particular, if setting \( \alpha = \frac{1}{2} \) in Theorem 1.13 and using (4.27), then we have the following corollary.

**Corollary 4.6.** Suppose \( 0, h_i \in \mathbb{T}_i \), diamond-\( \alpha_i \) (\( i = 1, 2, \ldots, n \)) integrable and differentiable function \( f : \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n \to \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \) satisfies

(i) \( f(v_1, v_2, \ldots, v_n) = 0 \) when one of \( v_i \) (\( i = 1, 2, \ldots, n \)) = 0;
(ii) $f(v_1, v_2, \ldots, v_n) = 0$ when one of $v_i$ ($i = 1, 2, \ldots, n$) is $\rho(0)$;  
(iii) $f(v_1, v_2, \ldots, v_n) = 0$ when one of $v_i$ ($i = 1, 2, \ldots, n$) is $\sigma(0)$;  
(iv) $f^{\Delta_i}(v_1, v_2, \ldots, v_i, \ldots, v_n) = f^{\Delta_i}(v_1, v_2, \ldots, \sigma(v_i), \ldots, v_n)$ for all $i = 1, 2, \ldots, n$;  
(v) $f^{\nabla_i}(v_1, v_2, \ldots, v_i, \ldots, v_n) = f^{\Delta_i}(v_1, v_2, \ldots, \rho(v_i), \ldots, v_n)$ for all $i = 1, 2, \ldots, n$; then

$$
\int_0^{h_1} \int_0^{h_1} \cdots \int_0^{h_1} \frac{\partial^2 f^2(v_1, v_2, \ldots, v_n)}{\sigma^2 v_1 v_2 \cdots v_n} \sigma v_1 v_2 \cdots v_n \leq \frac{1}{2^n} \prod_{i=1}^{n} h_i \int_0^{h_i} \int_0^{h_i} \cdots \int_0^{h_i} \sum_{\omega_1, \omega_2, \ldots=0, 1} \left| \frac{\partial^2 f(v_1, \ldots, v_n)}{\omega_1 v_1 \cdots \omega_n v_n} \right|^2 \omega_1 v_1 \cdots \omega_n v_n. \quad (4.28)
$$

It can be found that Theorem 1.13 reduces to Theorem 1.10 when $n = 1$, and using (4.27) we have following corollary when $n = 2$.

**Corollary 4.7.** Suppose $0, h_i \in \mathbb{T}$ ($i = 1, 2$), diamond-$\alpha_i$ ($i = 1, 2$) integrable and differentiable function $f: \mathbb{T} \times \mathbb{T} \to \mathbb{R} \times \mathbb{R}$ satisfies

(i) $f(v_1, v_2) = 0$ when one of $v_i$ ($i = 1, 2$) is $\rho(0)$;  
(ii) $f(v_1, v_2) = 0$ when one of $v_i$ ($i = 1, 2$) is $\sigma(0)$;  
(iii) $f(v_1, v_2) = 0$ when one of $v_i$ ($i = 1, 2$) is $\rho(0)$;  
(iv) $f^{\Delta_i}(v_1, v_2) = f^{\nabla_i}(v_1, v_2)$, $f^{\Delta_2}(v_1, v_2) = f^{\nabla_2}(v_1, \sigma(v_2))$;  
(v) $f^{\nabla_i}(v_1, v_2) = f^{\Delta_i}(v_1, \rho(v_2))$; then

$$
\int_0^{h_2} \int_0^{h_1} \frac{\partial^2 f^2(v_1, v_2)}{\sigma^2 v_1 v_2} \sigma v_1 \sigma v_2 \leq h_1 h_2 \sum_{\omega_1, \omega_2 = 0 \text{ or } 1} \psi_1(\omega_1) \psi_2(\omega_2) \times \sum_{\omega_1, \omega_2 = 0 \text{ or } 1} \psi_1(\omega_1) \psi_2(\omega_2) \int_0^{h_2} \int_0^{h_1} \frac{\partial^2 f(v_1, v_2)}{\sigma^2 v_1 v_2} \sigma v_1 \sigma v_2 \leq h_1 h_2 \sum_{\omega_1, \omega_2 = 0 \text{ or } 1} \psi_1(\omega_1) \psi_2(\omega_2) \int_0^{h_2} \int_0^{h_1} \frac{\partial^2 f^2(v_1, v_2)}{\Delta_1 v_1 \Delta_2 v_2} \Delta_1 v_1 \Delta_2 v_2 + \alpha_1 (1 - \alpha_2) \left( \frac{\partial^2 f(v_1, v_2)}{\Delta_1 v_1 \nabla_2 v_2} \right)^2 + (1 - \alpha_1) \alpha_2 \left( \frac{\partial^2 f(v_1, v_2)}{\nabla_1 v_1 \Delta_2 v_2} \right)^2 \psi_1(\omega_1) \psi_2(\omega_2). \quad (4.29)
$$

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**References**


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