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STABILITY RESULT OF LAMINATED BEAM WITH INTERNAL DISTRIBUTED DELAY

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Abstract. In this paper, we consider a laminated Timoshenko beam system with frictional damping and an internal distributed delay feedback on the effective rotational angle. Under appropriate assumptions on the weight of the delay term and wave speeds of the first two equations of the system, we prove that the dissipation through the frictional damping is enough to stabilize the system exponentially.

1. Introduction

In this paper, we consider the following laminated beam system with a frictional damping and an internal distributed delay feedback acting on the effective rotational angle:

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x = 0, \\ I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) \\ + \mu_1(3s_t - \psi_t) + \int_{\tau_1}^{\tau_2} \mu_2(r)(3s_t - \psi_t)(x, t - r)dr = 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s = 0, \end{cases} \tag{1}$$

where $(x,r) \in (0,1) \times (\tau_1, \tau_2)$ and $t \geqslant 0$. Here $w = w(x,t)$ is the transverse displacement, $\psi = \psi(x,t)$ is the rotation angle, $s = s(x,t)$ is proportional to the amount of slip along the interface and, $3s - \psi$ denotes the effective rotation angle. The positive parameters ρ, I_ρ, G, D and γ are the density, mass moment of inertia, shear stiffness, flexural rigidity and adhesive stiffness respectively. The positive constant μ_1 is the frictional damping coefficient and, $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function, with τ_1 and τ_2 being positive real numbers satisfying $0 \leqslant \tau_1 < \tau_2$. System (1) is subject to the following boundary and initial conditions:

$$\begin{cases} w(0,t) = s_x(0,t) = \psi_x(0,t) = 0, \quad t \geqslant 0, \\ w_x(1,t) = s(1,t) = \psi(1,t) = 0, \quad t \geqslant 0, \\ w(x,0) = w_0, \ w_t(x,0) = w_1, \ \psi(x,0) = \psi_0, \quad x \in (0,1), \\ \psi_t(x,0) = \psi_1, \ s(x,0) = s_0, \ s_t(x,0) = s_1, \quad x \in (0,1), \\ (3s_t - \psi_t)(x,-t) = f_0(x,t), \quad x \in (0,1), \quad t \in (0,\tau_2). \end{cases} \tag{2}$$

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The initial data $(w_0, w_1, \psi_0, \psi_1, s_0,\ s_1,\ f_0)$ belongs to a suitable functional space.

Introduced by Hansen et al. [18], the laminated beam model describes a vibrating structure of two-layered beams of the same thickness, stuck together by an adhesive layer of negligible mass and thickness, causing a small amount of slip while they are continuously in contact with each other. Such structures are of substantial importance in engineering applications, for instance in Glued-laminated timber (GLT) beam, PVB-laminated glass components, among others. The laminated beam model consists of three coupled hyperbolic equations and, without any interfering forces, the model takes the following form:

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x = 0, \\ I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t = 0. \end{cases} \tag{3}$$

The subscripted t and x denote differentiation with respect to time and to the longitudinal spatial variable respectively. The first two equations are derived on the assumption of Timoshenko beam theory, coupled with the third equation of (3) describes the dynamics of slip. Moreover, if s is identically zero, the standard Timoshenko model is restored. In the presence of structural damping ($\beta \neq 0$), the adhesion at the interface produces a restorative comparable force to counteract the interfacial slip. Otherwise, the third equation of (3) describes the dynamics of slip of the coupled laminated beams without structural damping.

Time delay effects are inevitable in most physical problems, and they may occur in form of lags between the input and processing the output, or lags in attaining or restoring the desired system stability after perturbations due to internal or external factors, among others. Thus, in recent times, control PDEs with time delay effects have attracted attention of researchers. Even though the voluntary inclusion of time delay can stabilize a control system, see [1, 34], in most cases, time delay is diagnosed as a source of instability or deterioration in system performance. In modeling systems where propagation and transport of material and/or information is assumed to reach from one unit to another without being affected by the past history of the received information, discrete delay representation may be sufficient. However, this is not always the case. For example, if Laminated beam structures are subjected to external factors such as radiation, heat, moisture, etc, there is a possibility of gradual degeneration over time. It may be in form of adhesive softening, wear and tear on the individual beams, among others. If this translates into time lags in equilibrium restoration of the structure, then time delay which incorporates memory is a more appropriate and realistic representation. In this work, we assume that such delay significantly acts through the effective rotation angle $3s - \psi$, implying that the system (1) can be considered as a problem with a memory acting only on the time interval $(t - \tau_2, t - \tau_1)$, and indeed with change of variable, we note that,

$$\int_{\tau_1}^{\tau_2} \mu_2(r)(3s_t - \psi_t)(x, t - r)dr = \int_{t-\tau_2}^{t-\tau_1} \mu_2(t - r)(3s_t - \psi_t)(x, r)dr.$$

The exponential behavior of (1) with $\mu_2 = 0$ (absence of delay) was studied by Apalara et al. [7]. The authors established uniform stability due to frictional damping acting on the effective rotation angle without any other kind of internal or boundary controls. Similar result was reached with only structural damping, see [8]. Aside from this work, system (3) has been greatly investigated by mathematicians and considerable stability results have been established by employing different damping mechanisms to the system. We cite some of the most related results.

Regarding stabilization through boundary feedback controls, we mention the work of Wang et al. [36]. The authors considered (3) with cantilever boundary conditions

$$\begin{cases} w(0,t) = \psi(0,t) = s(0,t) = 0, \\ s_x(1,t) = 0, \quad \psi(1,t) - w_x(1,t) = k_1 w_t(1,t), \\ (3s_x - \psi_x)(1,t) = -k_2(3s_t - \psi_t)(1,t), \end{cases}$$

and asserted that the system decays only polynomially in case of $k_1 = k_2 = 0$, otherwise exponential stability is possible if $r_1 = \sqrt{\frac{\rho}{G}} \neq \sqrt{\frac{I_\rho}{D}} = r_2,\ k_i \neq r_i (i = 1,2)$. Later, Cao et al. [9] gave a simpler test method of verifying the exponential stability of the closed loop system by designing a control law to compel laminated beams back to their equilibrium position. In the same line, Tatar [35] and Mustafa [26] improved the result in [36] by establishing the exponential stability under better assumptions on the system's parameters ρ, G, I_ρ, and D. In [2], authors proved that, if boundary feedback controls are coupled with structural damping, then exponential decay requires no further dissipation or restrictions on parameters, otherwise the assumption of equal wave speeds is necessary.

Apart from boundary control stabilization, researchers have considered other damping mechanisms in order to achieve the desired decay results. For instance, using internal linear frictional damping terms, Raposo [32] established exponential stability results, and the case of non-linear frictional damping was later investigated in [13, 8]. For interesting results regarding dissipation through thermal effects, see [3, 14, 6, 20, 21], and [25, 10, 22, 23, 15, 24] for viscoelastic damping mechanisms. In addition to material dissipation, authors exploited structural and/or frictional damping with some restrictions of parameters to reach the desired stability results.

Concerning distributed delay effect on stability, Nicaise et al. [30] investigated a wave equation with frictional damping and an internal distributed delay

$$u_{tt} - \Delta u + \mu_0 u_t + \int_{\tau_1}^{\tau_2} a(x)\mu(s)u_t(t-s)ds = 0 \quad \text{in} \quad \Omega \times (0,\infty)$$

with initial, mixed Dirichlet-Neumann boundary conditions and a is a function belonging to an appropriate space. Assuming,

$$\mu_0 > \|a\|_\infty \int_{\tau_1}^{\tau_2} \mu(s)ds,$$

the authors established exponential stability of the solution. Similarly, Apalara [4] studied a Timoshenko system with linear frictional damping and a distributed delay acting on the displacement equation. He established a well-posedness and an exponential decay result of the system under suitable assumptions. For further results pertaining distributed delay, the reader is referred to [5, 16, 17, 27, 28].

With regard to laminated beam system with delay, we proceed by mentioning the work of Feng [12], in which he considered a laminated beam with three internal constant delays

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x + a_1 w_t(x,t-\tau) = 0, \\ I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) + a_2(3s_t - \psi_t)(x,t-\tau) = 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + a_3 s_t(x,t-\tau) = 0, \end{cases}$$

together with three boundary feedback controls and, established the well-posedness as well as exponential decay result of the solution with some conditions the parameters. Seghour et al. [33] on the other hand, investigated a thermoelastic laminated beam with neutral delay in dynamics of slip equation. In addition to the dissipation through thermal effect, the authors introduced a linear frictional damping in the transverse displacement and established exponential stability in case of $\rho = GI_\rho$ and, polynomial decay otherwise. In a similar development, Choucha et al. [11], considered a thermoelastic laminated Timoshenko beam with distributed delay term in the third equation

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x + \delta\theta_x = 0, \\ I_\rho(3s - \psi)_{tt} - (3s - \psi)_{xx} - G(\psi - w_x) = 0, \\ I_\rho s_{tt} - Ds_{xx} + G(\psi - w_x) + \frac{4}{3}\gamma s + \frac{4}{3}\beta s_t + \frac{4}{3}\int_{\tau_1}^{\tau_2} |\mu_2(\sigma)| s_t(x,t-\sigma)\,d\sigma = 0, \\ \rho_3\theta_t + q_x + \delta w_{tx} = 0, \\ \tau q_t + \alpha q + \theta_x = 0, \end{cases} \tag{4}$$

with mixed Neumann-Dirichlet boundary conditions. Using structural and thermoelastic damping coupled by setting

$$\chi = \tau\delta^2 D - (D\rho - GI_\rho)\left(\frac{\tau\rho_3 D}{I_\rho} - 1\right),$$

the authors established exponential and polynomial decay results for $\chi = 0$ and $\chi \neq 0$ respectively, provided $\beta > \int_{\tau_1}^{\tau_2} |\mu_2(\sigma)|d\sigma$.

For results regarding asymptotic behavior of laminated beam system subject to constant delay, with dissipation through frictional and structure damping, see [24].

From the above work, it is evident that for Timoshenko laminated beam with delay so far, authors have exploited boundary controls or material dissipation, coupled with either structural or frictional damping in addition to restrictions on delay weight and system parameters, to achieve the desired stability results. Taking into account all this in addition to results in [7], we find it wanting to investigate system (3) with distributed delay term and a single friction damping as the only source of dissipation. Precisely, we consider (1)–(2) and establish an exponential decay result under equal propagation wave speed provided that

$$\mu_1 > \int_{\tau_1}^{\tau_2} |\mu_2(r)|dr. \tag{5}$$

The rest of the article is organized as follows. In section 2, we present some preliminaries which include a necessary transformation and state the well-posedness result without proof. In Section 3, we state and prove some technical lemmas. Section 4 focuses on the statement and proof of our main result.

2. Preliminaries

We proceed as in [29] by introducing the following new variable

$$z(x,\sigma,r,t) = (3s_t - \psi_t)(x,t-\sigma r) \qquad \text{in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty). \tag{6}$$

It simply follows that z satisfies

$$rz_t(x,\sigma,r,t) + z_\sigma(x,\sigma,r,t) = 0 \qquad \text{in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty). \tag{7}$$

Consequently, the system (1)–(2) is equivalent to

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x = 0 \quad \text{in } (0,1)\times(0,\infty), \\ I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) \\ +\mu_1(3s_t - \psi_t) + \displaystyle\int_{\tau_1}^{\tau_2} \mu_2(r)z(x,1,r,t)dr = 0 \quad \text{in } (0,1)\times(0,\infty), \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s = 0 \quad \text{in } (0,1)\times(0,\infty), \\ rz_t(x,\sigma,r,t) + z_\sigma(x,\sigma,r,t) = 0 \quad \text{in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty), \end{cases} \tag{8}$$

with the following boundary and initial conditions:

$$\begin{cases} z(x,0,r,t) = (3s_t - \psi_t)(x,t) \quad \text{in } (0,1)\times(\tau_1,\tau_2)\times[0,\infty), \\ w(0) = w_0,\ s(0) = s_0,\ \psi(0) = \psi_0 \quad \text{in } (0,1), \\ w_t(0) = w_1,\ s_t(0) = s_1,\ \psi_t(0) = \psi_1 \quad \text{in } (0,1), \\ w(0,t) = s_x(0,t) = \psi_x(0,t) = 0 \quad \text{in } [0,\infty), \\ w_x(1,t) = s(1,t) = \psi(1,t) = 0 \quad \text{in } [0,\infty), \\ z(x,\sigma,r,0) = f_0(x,\sigma r) \quad \text{in } (0,1)\times(0,1)\times(0,\tau_2). \end{cases} \tag{9}$$

Henceforth, we consider (8)–(9) instead of (1)–(2) and $z(\sigma)$ to mean $z(x,\sigma,r,t)$.

We define the energy functional of the solution of problem (8)–(9) as follows

$$\begin{aligned} E(t) = &\frac{1}{2}\int_0^1 \left[\rho w_t^2 + I_\rho(3s_t - \psi_t)^2 + D(3s_x - \psi_x)^2 + 3I_\rho s_t^2 + 3Ds_x^2 + 4\gamma s^2\right]dx \\ &+ \frac{1}{2}\int_0^1 \left[G(\psi - w_x)^2 + \int_0^1\int_{\tau_1}^{\tau_2} r|\mu_2(r)|z^2(\sigma)drd\sigma\right]dx. \end{aligned} \tag{10}$$

Concerning the existence, uniqueness, and smoothness of solution of problem (8)–(9), we introduce the vector function $\Phi = (w,u,\xi,v,s,y,z)^T$; $u = w_t$, $\xi = 3s - \psi$, $v = \xi_t$, and $y = s_t$, and thereby transform system (8)–(9) to

$$\begin{cases} \frac{d}{dt}\Phi(t) = \mathscr{A}\Phi(t), \qquad t > 0, \\ \Phi(0) = \Phi_0 = (w_0, w_1, 3s_0 - \psi_0, 3s_1 - \psi_1, s_0, s_1, f_0)^T, \end{cases} \tag{11}$$

where the operator $\mathscr{A}$ is defined by

$$\mathscr{A}\Phi = \begin{pmatrix} u \\ -\frac{1}{\rho}\left(G(3s-\xi-w_x)_x\right) \\ v \\ \frac{1}{I_\rho}\left(D\xi_{xx}+G(3s-\xi-w_x)-\mu_1 v-\int_{\tau_1}^{\tau_2}\mu_2(r)z(1)dr\right) \\ y \\ \frac{1}{I_\rho}\left(Ds_{xx}-G(3s-\xi-w_x)-\frac{4\gamma}{3}s\right) \\ -\frac{1}{r}z_\sigma(\sigma) \end{pmatrix}.$$

We now consider the following spaces

$$H_a^1=\{\varphi:\varphi\in H^1(0,1)\ :\varphi(0)=0\},\quad H_b^1=\{\varphi:\varphi\in H^1(0,1)\ :\varphi(1)=0\}.$$

Let

$$\begin{aligned}\mathscr{H}:=&H_a^1(0,1)\times L^2(0,1)\times H_b^1(0,1)\times L^2(0,1)\times H_b^1(0,1)\times L^2(0,1)\\ &\times L^2\left((0,1)\times(0,1)\times(\tau_1,\tau_2)\right)\end{aligned}$$

be the Hilbert space equipped with the following inner product

$$\begin{aligned}(\Phi,\tilde{\Phi})_{\mathscr{H}}=&\rho\int_0^1 u\tilde{u}dx+G\int_0^1(3s-\xi-w_x)\left(3\tilde{s}-\tilde{\xi}-\tilde{w}_x\right)dx+I_\rho\int_0^1 v\tilde{v}dx\\ &+3I_\rho\int_0^1 y\tilde{y}dx+D\int_0^1\xi_x\tilde{\xi}_xdx+4\gamma\int_0^1 s\tilde{s}dx+3D\int_0^1 s_x\tilde{s}_xdx\\ &+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} r|\mu_2(r)|z(\sigma)\tilde{z}(\sigma)drd\sigma dx.\end{aligned}\tag{12}$$

The domain of $\mathscr{A}$ is given by

$$D(\mathscr{A})=\left\{\begin{array}{l}\Phi\in\mathscr{H}\mid w\in H^2(0,1)\cap H_a^1(0,1),\quad \xi,s\in H^2(0,1)\cap H_b^1(0,1),\\ u\in H_a^1(0,1),\quad v,y\in H_b^1(0,1),\quad z,z_\sigma\in L^2\left((0,1)\times(0,1)\times(\tau_1,\tau_2)\right),\\ w_x(1)=\xi_x(0)=s_x(0)=0\end{array}\right\}.$$

Note that $D(\mathscr{A})$ is independent of time $t>0$. Furthermore, it is obvious that $D(\mathscr{A})$ is dense in $\mathscr{H}$. We have the following well-posedness result.

THEOREM 1. *Assume* (5) *holds, then for any* $\Phi_0\in\mathscr{H}$, *there exits a unique weak solution* $\Phi\in C(\mathbb{R}^+,\mathscr{H})$ *of problem* (11). *Moreover, if* $\Phi_0\in D(\mathscr{A})$, *then* $\Phi\in C(\mathbb{R}^+, D(\mathscr{A}))\cap C^1(\mathbb{R}^+,\mathscr{H})$.

REMARK 1. The proof of Theorem 1 can be established using the standard semigroup method as in [3, 4].

3. Technical lemmas

In this section, we state and prove some technical lemmas necessary in the proof of our stability result.

LEMMA 1. *If (w,ψ,s,z) is a solution of (8)–(9), then the energy functional (10) satisfies*

$$\frac{d}{dt}E(t) \leqslant -m_0\int_0^1 (3s_t-\psi_t)^2 dx \qquad \forall t \geqslant 0, \tag{13}$$

for some positive constant m_0.

Proof. By multiplying the first three equations in (8) by $w_t, (3s_t-\psi_t)$ and s_t respectively, then integrate by parts over $(0,1)$ using the boundary conditions (9), we deduce that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\left[\rho w_t^2+I_\rho(3s_t-\psi_t)^2+D(3s_x-\psi_x)^2+3I_\rho s_t^2+3Ds_x^2+4\gamma s^2+G(\psi-w_x)^2\right]dx\\&=-\mu_1\int_0^1(3s_t-\psi_t)^2dx-\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(r)(3s_t-\psi_t)z(1)drdx.\end{aligned} \tag{14}$$

Next, multiplying the last equation in (8) by $|\mu_2(r)|z$, integrating the product over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$, and using the fact that $z(x,0,r,t)=(3s_t-\psi_t)(x,t)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} r|\mu_2(r)|z^2(\sigma)drd\sigma dx = &-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(r)|z^2(1)drdx\\&+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(r)|(3s_t-\psi_t)^2drdx.\end{aligned} \tag{15}$$

Combining (14) with (15) and using (10) leads to

$$\begin{aligned}\frac{d}{dt}E(t)=&-\left(\mu_1-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(r)|dr\right)\int_0^1(3s_t-\psi_t)^2dx\\&-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(r)|z^2(1)drdx-\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(r)(3s_t-\psi_t)z(1)drdx.\end{aligned} \tag{16}$$

By using Young's inequality, we easily observe that last the term in (16) satisfies

$$\begin{aligned}-\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(r)(3s_t-\psi_t)z(1)drdx \leqslant &\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(r)|dr\int_0^1(3s_t-\psi_t)^2dx\\&+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(r)|z^2(1)drdx.\end{aligned} \tag{17}$$

We complete the proof of (13) by substituting (17) in (16), and using (5). □

LEMMA 2. *If* (w,ψ,s,z) *is a solution of* (8)–(9), *then the functional* F_1, *defined by*

$$F_1(t) := -\rho \int_0^1 w w_t dx$$

satisfies, for any $\varepsilon_1 > 0$, *the estimate*

$$\begin{aligned} \frac{d}{dt}F_1(t) \leqslant & -\rho \int_0^1 w_t^2 dx + \frac{G}{2}\int_0^1 (3s_x - \psi_x)^2 dx + \varepsilon_1 \int_0^1 s_x^2 dx \\ & + \left(\frac{3G}{2} + \frac{9G^2}{4\varepsilon_1}\right)\int_0^1 (\psi - w_x)^2 dx. \end{aligned} \tag{18}$$

Proof. Differentiating F_1 and using the first equation in (8), we get

$$\begin{aligned} \frac{d}{dt}F_1(t) = & -\rho \int_0^1 w_t^2 dx + G\int_0^1 (\psi - w_x)^2 dx + G\int_0^1 (3s-\psi)(\psi - w_x)dx \\ & - 3G\int_0^1 (\psi - w_x) s dx. \end{aligned} \tag{19}$$

It follows from Young's and Poincaré's inequalities that

$$\begin{aligned} G\int_0^1 (3s-\psi)(\psi-w_x)dx & \leqslant \frac{G}{2}\int_0^1 (3s-\psi)^2 dx + \frac{G}{2}\int_0^1 (\psi - w_x)^2 dx \\ & \leqslant \frac{G}{2}\int_0^1 (3s_x-\psi_x)^2 dx + \frac{G}{2}\int_0^1 (\psi - w_x)^2 dx \end{aligned} \tag{20}$$

and for any $\varepsilon_1 > 0$,

$$\begin{aligned} -3G\int_0^1 (\psi - w_x) s dx & \leqslant \varepsilon_1 \int_0^1 s^2 dx + \frac{9G^2}{4\varepsilon_1}\int_0^1 (\psi - w_x)^2 dx \\ & \leqslant \varepsilon_1 \int_0^1 s_x^2 dx + \frac{9G^2}{4\varepsilon_1}\int_0^1 (\psi - w_x)^2 dx. \end{aligned} \tag{21}$$

Consequently, from (19), (20) and (21), we obtain (18). □

LEMMA 3. *If* (w,ψ,s,z) *is a solution of* (8)–(9), *then the functional* F_2, *defined by*

$$F_2(t) := I_\rho \int_0^1 (3s_t - \psi_t)(3s-\psi)dx + \frac{\mu_1}{2}\int_0^1 (3s-\psi)^2 dx - \rho\int_0^1 w_t \int_0^x (3s-\psi)(y) dy dx$$

satisfies, for any $\varepsilon_2 > 0$, *the estimate*

$$\begin{aligned} \frac{d}{dt}F_2(t) \leqslant & -\frac{3D}{4}\int_0^1 (3s_x - \psi_x)^2 dx + \varepsilon_2 \int_0^1 w_t^2 dx + \frac{\mu_1}{D}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(r)| z^2(1) dr dx \\ & + \left(I_\rho + \frac{\rho^2}{4\varepsilon_2}\right)\int_0^1 (3s_t - \psi_t)^2 dx. \end{aligned} \tag{22}$$

Proof. We differentiate F_2, use the second equation in (8), and integrate by parts the terms involving $(\psi - w_x)_x$ and $(3s_{xx} - \psi_{xx})$ to obtain

$$\begin{aligned}\frac{d}{dt}F_2(t) = &- D\int_0^1 (3s_x - \psi_x)^2 dx - \rho\int_0^1 w_t \int_0^x (3s_t - \psi_t)(y)dydx \\ &+ I_\rho \int_0^1 (3s_t - \psi_t)^2 dx - \int_0^1 (3s-\psi)\int_{\tau_1}^{\tau_2} \mu_2(r)z(1)drdx.\end{aligned} \tag{23}$$

Exploiting Young's, Poincaré's, Cauchy-Schwarz inequalities and using (5), we end up with

$$\begin{aligned}-\int_0^1 (3s-\psi)\int_{\tau_1}^{\tau_2}\mu_2(r)z(1)drdx &\leqslant \frac{1}{D}\int_0^1\left(\int_{\tau_1}^{\tau_2}\mu_2(r)z(1)dr\right)^2 dx + \frac{D}{4}\int_0^1 (3s-\psi)^2 dx \\ &\leqslant \frac{1}{D}\int_{\tau_1}^{\tau_2}|\mu_2(r)|dr\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(r)|z^2(1)drdx \\ &\quad + \frac{D}{4}\int_0^1 (3s_x-\psi_x)^2 dx \\ &\leqslant \frac{\mu_1}{D}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(r)|z^2(1)drdx + \frac{D}{4}\int_0^1 (3s_x-\psi_x)^2 dx\end{aligned} \tag{24}$$

and for any $\varepsilon_2 > 0$,

$$\begin{aligned}-\rho\int_0^1 w_t\int_0^x (3s_t-\psi_t)(y)dydx &\leqslant \varepsilon_2\int_0^1 w_t^2 dx + \frac{\rho^2}{4\varepsilon_2}\int_0^1\left(\int_0^x (3s_t-\psi_t)(y)dy\right)^2 dx \\ &\leqslant \varepsilon_2\int_0^1 w_t^2 dx + \frac{\rho^2}{4\varepsilon_2}\int_0^1 (3s_t-\psi_t)^2 dx.\end{aligned} \tag{25}$$

Finally, substituting (24) and (25) into (23) completes the proof. □

LEMMA 4. *If (w,ψ,s,z) is a solution of (8)–(9), then functional F_3, defined by*

$$F_3(t) := 3I_\rho\int_0^1 s_t s dx + 3\rho\int_0^1 w_t\int_0^x s(y)dydx$$

satisfies, for any $\varepsilon_3 > 0$, the estimate

$$\frac{d}{dt}F_3(t) \leqslant -3D\int_0^1 s_x^2 dx - 4\gamma\int_0^1 s^2 dx + \varepsilon_3\int_0^1 w_t^2 dx + \left(3I_\rho + \frac{9\rho^2}{4\varepsilon_3}\right)\int_0^1 s_t^2 dx. \tag{26}$$

Proof. Direct computations yield

$$\frac{d}{dt}F_3(t) = -3D\int_0^1 s_x^2 dx - 4\gamma\int_0^1 s^2 dx + 3I_\rho\int_0^1 s_t^2 dx + 3\rho\int_0^1 w_t\int_0^x s_t(y)dydx. \tag{27}$$

Using Young's, Cauchy-Schawarz and Poincaré's, we obtain

$$\begin{aligned} 3\rho\int_0^1 w_t\int_0^x s_t(y)dydx &\leqslant \varepsilon_3\int_0^1 w_t^2dx+\frac{9\rho^2}{4\varepsilon_3}\int_0^1\left(\int_0^x s_t(y)dy\right)^2dx \\ &\leqslant \varepsilon_3\int_0^1 w_t^2dx+\frac{9\rho^2}{4\varepsilon_3}\int_0^1 s_t^2dx \end{aligned} \tag{28}$$

for any $\varepsilon_3>0$. The combination of (26) and (27) gives (28). □

The assumption of equal wave speeds $GI_\rho=\rho D$ plays a paramount role in the next two lemmas.

LEMMA 5. *If* (w,ψ,s,z) *is a solution of* (8)–(9)*, then the functional* F_4*, defined by*

$$F_4(t):=\int_0^1(3s_t-\psi_t)(w_x-3s)dx+\int_0^1(3s_x-\psi_x)w_tdx-\frac{\mu_1}{2I_\rho}\int_0^1(3s-\psi)^2dx$$

satisfies, for any $\varepsilon_4>0$*, the estimate*

$$\begin{aligned} \frac{d}{dt}F_4(t)\leqslant &-\frac{G}{2I_\rho}\int_0^1(\psi-w_x)^2dx+\left(\frac{2D}{I_\rho}+\frac{3G}{2I_\rho}\right)\int_0^1(3s_x-\psi_x)^2dx \\ &+\varepsilon_4\int_0^1 s_t^2dx+\left(\frac{3\mu_1}{2GI_\rho}+\frac{\mu_1}{4DI_\rho}\right)\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(r)|z^2(1)drdx \\ &+\left(\frac{3\mu_1^2}{2GI_\rho}+\frac{9}{4\varepsilon_4}\right)\int_0^1(3s_t-\psi_t)^2dx. \end{aligned} \tag{29}$$

Proof. By differentiating F_4, and using the first two equations (8), the boundary conditions (9) and the fact that $w_x=-(\psi-w_x)-(3s-\psi)+3s$, we arrive at

$$\begin{aligned} \frac{d}{dt}F_4(t)=&-\frac{G}{I_\rho}\int_0^1(\psi-w_x)^2dx+\frac{D}{I_\rho}\int_0^1(3s_x-\psi_x)^2dx-3\int_0^1(3s_t-\psi_t)s_tdx \\ &-\frac{G}{I_\rho}\int_0^1(\psi-w_x)(3s-\psi)dx+\frac{1}{I_\rho}\int_0^1(\psi-w_x)\int_{\tau_1}^{\tau_2}\mu_2(r)z(1)drdx \\ &+\frac{\mu_1}{I_\rho}\int_0^1(3s_t-\psi_t)(\psi-w_x)dx+\frac{1}{I_\rho}\int_0^1(3s-\psi)\int_{\tau_1}^{\tau_2}\mu_2(r)z(1)drdx. \end{aligned} \tag{30}$$

Exploiting Young's, Cauchy-Schawarz and Poincaré's inequalities and using (5), we have

$$\begin{aligned} -\frac{G}{I_\rho}\int_0^1(\psi-w_x)(3s-\psi)dx &\leqslant \frac{G}{6I_\rho}\int_0^1(\psi-w_x)^2dx+\frac{3G}{2I_\rho}\int_0^1(3s-\psi)^2dx \\ &\leqslant \frac{G}{6I_\rho}\int_0^1(\psi-w_x)^2dx+\frac{3G}{2I_\rho}\int_0^1(3s_x-\psi_x)^2dx, \end{aligned}$$

$$\begin{aligned}
&\frac{1}{I_\rho}\int_0^1 (\psi - w_x)\int_{\tau_1}^{\tau_2} \mu_2(r)z(1)drdx \\
&\leqslant \frac{G}{6I_\rho}\int_0^1 (\psi - w_x)^2 dx + \frac{3}{2GI_\rho}\int_0^1 \left(\int_{\tau_1}^{\tau_2} \mu_2(r)z(1)dr\right)^2 dx \\
&\leqslant \frac{G}{6I_\rho}\int_0^1 (\psi - w_x)^2 dx + \frac{3\mu_1}{2GI_\rho}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(r)|z^2(1)drdx,
\end{aligned}$$

$$\begin{aligned}
&\frac{\mu_1}{I_\rho}\int_0^1 (3s_t - \psi_t)(\psi - w_x)dx \\
&\leqslant \frac{G}{6I_\rho}\int_0^1 (\psi - w_x)^2 dx + \frac{3\mu_1^2}{2GI_\rho}\int_0^1 (3s_t - \psi_t)^2 dx,
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{I_\rho}\int_0^1 (3s - \psi)\int_{\tau_1}^{\tau_2} \mu_2(r)z(1)drdx \\
&\leqslant \frac{D}{I_\rho}\int_0^1 (3s - \psi)^2 dx + \frac{1}{4DI_\rho}\int_0^1 \left(\int_{\tau_1}^{\tau_2} \mu_2(r)z(1)dr\right)^2 dx \\
&\leqslant \frac{D}{I_\rho}\int_0^1 (3s_x - \psi_x)^2 dx + \frac{\mu_1}{4DI_\rho}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(r)|z^2(1)drdx,
\end{aligned}$$

and for any $\varepsilon_4 > 0$,

$$3\int_0^1 (3s_t - \psi_t)s_t dx \leqslant \frac{9}{4\varepsilon_4}\int_0^1 (3s_t - \psi_t)^2 dx + \varepsilon_4 \int_0^1 s_t^2 dx.$$

By substituting the above four estimates in (30), estimate (29) is established. □

LEMMA 6. *If (w,ψ,s,z) is a solution of (8)–(9), then the functional F_5, defined by*

$$F_5(t) := -\int_0^1 (\psi - w_x)s_t dx + \int_0^1 w_t s_x dx$$

satisfies, for any $\varepsilon_5 > 0$, the estimate

$$\begin{aligned}
\frac{d}{dt}F_5(t) \leqslant & -2\int_0^1 s_t^2 dx + \varepsilon_5 \int_0^1 s_x^2 dx + \left(\frac{G}{I_\rho} + \frac{4\gamma^2}{9I_\rho^2\varepsilon_5}\right)\int_0^1 (\psi - w_x)^2 dx \\
& + \frac{1}{4}\int_0^1 (3s_t - \psi_t)^2 dx.
\end{aligned} \tag{31}$$

Proof. Direct computations, using (8)–(9) and the fact that $w_x = -(\psi - w_x) - (3s - \psi) + 3s$, yield

$$\frac{d}{dt}F_5(t) = -\int_0^1 (\psi - w_x)s_{tt}dx + \int_0^1 w_{tt}s_x dx + \int_0^1 (3s_t - \psi_t)s_t dx - 3\int_0^1 s_t^2 dx.$$

In consideration of the above, the first and third equations in (8), followed by a simple integration by parts over $(0,1)$ the term containing s_{xx}, we note that

$$\frac{d}{dt}F_5(t) = -3\int_0^1 s_t^2 dx + \frac{G}{I_\rho}\int_0^1 (\psi - w_x)^2 dx + \frac{4\gamma}{3I_\rho}\int_0^1 s(\psi - w_x)dx + \int_0^1 (3s_t - \psi_t)s_t dx. \tag{32}$$

Using Young's and Poincaré's inequalities, the last two terms on the right hand side of (32) give

$$\int_0^1 (3s_t - \psi_t)s_t dx \leqslant \frac{1}{4}\int_0^1 (3s_t - \psi_t)^2 dx + \int_0^1 s_t^2 dx,$$

and for any $\varepsilon_5 > 0$,

$$\begin{aligned}\frac{4\gamma}{3I_\rho}\int_0^1 s(\psi - w_x)dx &\leqslant \varepsilon_5\int_0^1 s^2 dx + \frac{4\gamma^2}{9I_\rho^2\varepsilon_5}\int_0^1 (\psi - w_x)^2 dx \\ &\leqslant \varepsilon_5\int_0^1 s_x^2 dx + \frac{4\gamma^2}{9I_\rho^2\varepsilon_5}\int_0^1 (\psi - w_x)^2 dx.\end{aligned}$$

Consequently, the assertion of the lemma follows by substituting the above two estimates into (32). □

LEMMA 7. *If (w,ψ,s,z) is a solution of (8)–(9), then the functional F_6, defined by*

$$F_6(t) := \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} re^{-\sigma r}|\mu_2(r)|z^2(\sigma)drd\sigma dx$$

satisfies, for $m_1 > 0$, the estimate:

$$\frac{d}{dt}F_6(t) \leqslant -m_1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(r)|z^2(1)drdx + \mu_1\int_0^1 (3s_t - \psi_t)^2 dx - m_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} r|\mu_2(r)|z^2(\sigma)drd\sigma dx. \tag{33}$$

Proof. We proceed by differentiating F_6, then use $(8)_4$ and the fact that $z(x,0,r,t) = (3s_t - \psi_t)(x,t)$ as follows

$$\begin{aligned}\frac{d}{dt}F_6(t) &= 2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} re^{-\sigma r}|\mu_2(r)|z(\sigma)z_t(\sigma)drd\sigma dx \\ &= -2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-\sigma r}|\mu_2(r)|z(\sigma)z_\sigma(\sigma)drd\sigma dx \\ &= -\frac{d}{d\sigma}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-\sigma r}|\mu_2(r)|z^2(\sigma)drd\sigma dx \\ &\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} re^{-\sigma r}|\mu_2(r)|z^2(\sigma)drd\sigma dx\end{aligned}$$

$$
\begin{aligned}
&= -\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(r)| \left[e^{-r} z^2(1) - z^2(0)\right] drdx \\
&\quad - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} re^{-\sigma r} |\mu_2(r)| z^2(\sigma) drd\sigma dx \\
&= -\int_0^1 \int_{\tau_1}^{\tau_2} e^{-r} |\mu_2(r)| z^2(1) drdx + \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(r)| (3s_t - \psi_t)^2 drdx \\
&\quad - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} re^{-\sigma r} |\mu_2(r)| z^2(\sigma) drd\sigma dx.
\end{aligned}
$$

By the virtue of $-e^{-r}$ being an increasing function, we have $-e^{-r} \leqslant -e^{-\tau_2}$, for all $r \in [\tau_1, \tau_2]$. Finally, letting $m_1 = e^{-\tau_2}$ and using the fact that $-e^{-r} \leqslant -e^{-\sigma r} \leqslant 1$ for all $\sigma \in [0,1]$, we end up with (33). □

4. Exponential stability

This section is dedicated to the statement and proof of our stability result. We prove that a given Lyapunov functional is equivalent to the energy functional.

LEMMA 8. *Let $N, N_k, k = 1, \cdots, 6$, be positive constants. The functional defined by*

$$
\mathscr{L}(t) := NE(t) + \sum_{k=1}^{6} N_k F_k(t), \quad t > 0, \tag{34}
$$

satisfies the equivalence relation $\mathscr{L} \sim E$, that is

$$
c_1 E(t) \leqslant \mathscr{L}(t) \leqslant c_2 E(t), \quad \forall t > 0, \tag{35}
$$

for some positive constants c_1 and c_2.

Proof.

$$
\begin{aligned}
|\mathscr{L}(t) - NE(t)| \leqslant\ & \rho N_1 \int_0^1 |ww_t|\, dx + I_\rho N_2 \int_0^1 |(3s_t - \psi_t)(3s - \psi)|\, dx \\
& + \frac{\mu_1 N_2}{2} \int_0^1 (3s - \psi)^2 dx + \rho N_2 \int_0^1 \left| w_t \int_0^x (3s - \psi)(y) dy \right| dx \\
& + 3\rho N_3 \int_0^1 \left| w_t \int_0^x s(y) dy \right| dx + 3 I_\rho N_3 \int_0^1 |s_t s|\, dx \\
& + N_4 \int_0^1 |(3s_t - \psi_t)(w_x - 3s)|\, dx + N_4 \int_0^1 |(3s_x - \psi_x) w_t|\, dx \\
& + \frac{\mu_1 N_4}{2 I_\rho} \int_0^1 (3s - \psi)^2 dx + N_5 \int_0^1 |(\psi - w_x) s_t|\, dx \\
& + N_5 \int_0^1 |w_t s_x|\, dx + N_6 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} re^{-\sigma r} |\mu_2(r)| z^2(\sigma) drd\sigma dx.
\end{aligned}
$$

It follows from Young's, Poincaré's, and Cauchy–Schwarz inequalities, (10), coupled with the fact that $w_x = -(\psi - w_x) - (3s - \psi) + 3s$ and $e^{-\sigma\tau} \leqslant 1$ for all $\sigma \in (0,1)$, that for some constant $\eta > 0$, we have

$$
\begin{aligned}
|\mathscr{L}(t) - NE(t)| \leqslant \eta \int_0^1 &\left[w_t^2 + (3s_t - \psi_t)^2 + (3s_x - \psi_x)^2 + s_t^2 + s_x^2 + (\psi - w_x)^2 \right] dx \\
&+ \eta \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} r|\mu_2(r)| z^2(\sigma) dr d\sigma dx, \\
\leqslant\ & \eta E(t).
\end{aligned}
$$

Therefore,

$$
(N - \eta)E(t) \leqslant \mathscr{L}(t) \leqslant (N + \eta)E(t).
$$

Taking N is sufficiently large, the assertion of (35) follows accordingly. □

Our exponential stability result reads as follows:

THEOREM 2. (Main) *Let (w, ψ, s, z) be a solution of (8)–(9), assume that $\frac{G}{\rho} = \frac{D}{I_\rho}$ and (5) hold, then energy functional (10) satisfies,*

$$
E(t) \leqslant k_0 e^{-k_1 t}, \quad \forall t \geqslant 0, \tag{36}
$$

where k_0 and k_1 are positive constants.

Proof. Differentiate (34), substitute the estimates (13), (18), (22), (26), (29), (31) and (33) and choose $N_1 = N_3 = 1$, $\varepsilon_1 = D$ and $\varepsilon_3 = \frac{\rho}{2}$, we end up with,

$$
\begin{aligned}
\mathscr{L}'(t) \leqslant & -\left[m_0 N - cN_2 \left(1 + \frac{1}{\varepsilon_2}\right) - cN_4 \left(1 + \frac{1}{\varepsilon_4}\right) - \frac{N_5}{4} - \mu_1 N_6 \right] \int_0^1 (3s_t - \psi_t)^2 dx \\
& - \left[\frac{\rho}{2} - \varepsilon_2 N_2 \right] \int_0^1 w_t^2 dx - \left[\frac{3DN_2}{4} - \frac{G}{2} - cN_4 \right] \int_0^1 (3s_x - \psi_x)^2 dx - 4\gamma \int_0^1 s^2 dx \\
& - [2D - \varepsilon_5 N_5] \int_0^1 s_x^2 dx - \left[\frac{GN_4}{2I_\rho} - cN_5 \left(1 + \frac{1}{\varepsilon_5}\right) - c \right] \int_0^1 (\psi - w_x)^2 dx \\
& - [2N_5 - c - \varepsilon_4 N_4] \int_0^1 s_t^2 dx - [m_1 N_6 - cN_4 - cN_2] \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(r)| z^2(1) dr dx \\
& - m_1 N_6 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} r|\mu_2(r)| z^2(\sigma) dr d\sigma dx,
\end{aligned}
$$

for some constant $c > 0$.

At this point, we choose N_5 large enough such that

$$
\lambda := 2N_5 - c > 0.
$$

With N_5 fixed, we choose ε_5 small enough so that

$$
2D - \varepsilon_5 N_5 > 0.
$$

Next, we choose N_4 large enough such that

$$\frac{GN_4}{2I_\rho} - cN_3\left(1+\frac{1}{\varepsilon_5}\right) - c > 0.$$

Once N_4 is fixed, we proceed to choose ε_4 sufficiently small and N_2 large enough such that

$$\lambda - \varepsilon_4 N_4 > 0 \quad \text{and} \quad \frac{3DN_2}{4} - c - cN_4 > 0,$$

respectively. Fixing N_2 permits us to choose ε_2 small enough so that

$$\frac{\rho}{2} - \varepsilon_2 N_2 > 0.$$

Next, we choose N_6 adequately large such that

$$m_1 N_6 - cN_4 - cN_2 > 0.$$

Lastly, we choose N sufficiently larger so that (35) remains valid

$$m_0 N - cN_2\left(1+\frac{1}{\varepsilon_2}\right) - cN_4\left(1+\frac{1}{\varepsilon_4}\right) - \frac{N_5}{4} - \mu_1 N_6 > 0.$$

It then follows that for some $\alpha_0 > 0$,

$$\begin{aligned} \mathscr{L}'(t) \leqslant & -\alpha_0 \int_0^1 \left[w_t^2 + s_t^2 + (3s_t - \psi_t)^2 + (3s_x - \psi_x)^2 + s_x^2 + s^2 + (\psi - w_x)^2\right] dx \\ & - \alpha_0 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(r)| z^2(1) dr dx - \alpha_0 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} r|\mu_2(r)| z^2(\sigma) dr d\sigma dx. \end{aligned} \tag{37}$$

Hence from (10), we have

$$\mathscr{L}'(t) \leqslant -\alpha_0 E(t), \quad \forall t \geqslant 0. \tag{38}$$

In view of (35) and (38), we note that

$$\mathscr{L}'(t) \leqslant -k_1 \mathscr{L}(t), \quad \forall t \geqslant 0, \tag{39}$$

where $k_1 = \frac{\alpha_0}{c_2}$. A simple integration of (39) over $(0,t)$ yields

$$\mathscr{L}(t) \leqslant \mathscr{L}(0) e^{-k_1 t}, \quad \forall t \geqslant 0. \tag{40}$$

Lastly, the relation (36) follows by virtue of (40) and (35) with $k_0 = \frac{c_2 E(0)}{c_1}$. □

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