A GENERALIZATION OF $S$–NEKRASOV MATRICES

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Abstract. The class of $H$-matrices plays an important role in various scientific disciplines. In this paper, we introduce a new subclass of $H$-matrices, called generalized $S$-Nekrasov matrices. We prove that this class contains the class of $S$-Nekrasov matrices. We also present a sufficient condition for a weak Nekrasov matrix to be an $H$-matrix.

1. Introduction

$H$-matrices and its subclasses play a significant role in many fields of science such as computational mathematics, mathematical physics and control theory; see [6, 7, 10, 11, 12] and the references therein. In 2009, Cvetković, Kostić, and Rauški [2] introduced a new subclass of $H$-matrices: $S$-Nekrasov matrices. This matrix class has been extensively studied; see [1, 3, 4, 5, 8, 9]. In this paper, we introduce a new subclass of $H$-matrices, called generalized $S$-Nekrasov matrices. In addition, we find that every $S$-Nekrasov matrix belongs to this new matrix class.

To present our result, we need the following notations and definitions. Let $\langle n \rangle = \{1, 2, \cdots, n\}$ and let $M_n$ be the set of all $n \times n$ complex matrices. For $A = (a_{ij}) \in M_n$, denote

$P_i(A) = \sum_{j \in \langle n \rangle, j \neq i} |a_{ij}|, \quad \forall \ i \in \langle n \rangle;$

$R_1(A) = P_1(A), \quad R_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{R_j(A)}{|a_{jj}|} + \sum_{j=i+1}^{n} |a_{ij}|, \quad 2 \leq i \leq n;$

$I_1(A) = 0, \quad I_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{R_j(A)}{|a_{jj}|}, \quad 2 \leq i \leq n.$

DEFINITION 1.1. Let $A = (a_{ij}) \in M_n$. Then $A$ is a (row) diagonally dominant matrix ($D_n$) if

$|a_{ii}| \geq P_i(A), \quad \forall \ i \in \langle n \rangle. \quad (1.1)$

$A$ is a strictly diagonally dominant matrix ($SD_n$) if all representative inequalities in (1.1) are strict. If there exists a positive diagonal matrix $X$ such that $AX \in SD_n$, $A$ is said to be a generalized strictly diagonally dominant matrix (i.e., nonsingular $H$-matrix).


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Definition 1.2. Let $A = (a_{ij}) \in M_n$. Then $A$ is a weak Nekrasov matrix if
\[
|a_{ii}| \geq R_i(A), \quad \forall \, i \in \langle n \rangle.
\]
(1.2)

$A$ is a Nekrasov matrix ($N_n$) if all the inequalities in (1.2) are strict.

A matrix $A$ is a nonsingular $H$-matrix if there exists a diagonal matrix $D$ such that $AD$ is strictly diagonally dominant. For each Nekrasov matrix $B$, it is a nonsingular $H$-matrix and then there is a diagonal matrix $D$ such that $BD$ is strictly diagonally dominant. Hence, if a matrix can be scaled to a Nekrasov matrix by a diagonal matrix from the right side, this matrix is a nonsingular $H$-matrix. The initial purpose of this paper is to find some practical and efficient criteria for $H$-matrices in this way. To our surprise, every $S$-Nekrasov matrix satisfies the sufficient condition we get.

Denote by $S$ a nonempty subset of $\langle n \rangle$, and $\overline{S}$ the complement set of $S$ in $\langle n \rangle$.

We also need the following notations.

\[
R^S_i(A) = \sum_{j \in S, j \neq i} |a_{ij}|, \quad R^S_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{R^S_j(A)}{|a_{jj}|} + \sum_{j=i+1, j \in S}^{n} |a_{ij}|, \quad 2 \leq i \leq n;
\]

\[
l^S_i(A) = 0, \quad l^S_i(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{R^S_j(A)}{|a_{jj}|}, \quad 2 \leq i \leq n;
\]

\[r_s = \max_{i \in S} \frac{R^S_i(A)}{|a_{ii}|}, \quad \delta^S_i(A) = \frac{R^S_i + r_s R^S_i(A)}{|a_{ii}|}, \quad \forall \, i \in S;
\]

\[N_1(A) = \{i \in \langle n \rangle \mid |a_{ii}| \leq R_i(A)\}, \quad N_2(A) = \{i \in \langle n \rangle \mid |a_{ii}| > R_i(A)\}.
\]

It is easy to conclude that the following equality holds for $i \in \langle n \rangle$.

\[f_i(A) = f^S_i(A) + f^\overline{S}_i(A), \quad (1.3)
\]

where $f \in \{l, R\}$. Now we introduce a new matrix class as follows.

Definition 1.3. Let $A = (a_{ij}) \in M_n$ with $n \geq 2$. Given an arbitrary $S \subseteq N_2(A)$, $A$ is said to be a generalized $S$-Nekrasov matrix ($GSN$) if the following inequalities hold.

\[|a_{ii}| > R^S_i(A) + r_s l^S_i(A) + \sum_{j>i+1, j \in S} |a_{ij}| \delta^S_j(A), \quad \forall \, i \in N_1(A).
\]

(1.4)

In the second section, we will prove that $GSN$ is a subclass of $H$-matrices. Moreover, $GSN$ contains the class of $S$-Nekrasov matrices.
2. A generalization of $S$-Nekrasov matrices

We need the following lemmas.

**Lemma 2.1.** Let $A = (a_{ij}) \in M_n$. If $S \subseteq N_2(A)$, then

$$r_S \geq \max_{i \in S} \delta_i^S(A).$$  \hspace{1cm} (2.1)

**Proof.** By the definition of $N_2(A)$, $i \in N_2(A)$ leads to $|a_{ii}| > 0$. To the contrary, suppose that there exists $i \in S$ such that $r_S < \delta_i^S(A)$. By the definition of $\delta_i^S(A)$, we get

$$r_S \leq \frac{R_i^S(A) + r_S R_i^S(A)}{|a_{ii}|}.$$

It is equivalent with

$$r_S |a_{ii}| < R_i^S(A) + r_S R_i^S(A).$$

Collect the terms of $r_S$ and then we get

$$r_S < \frac{R_i^S(A)}{|a_{ii}| - R_i^S(A)},$$

which contradicts the definition of $r_S$. □

**Lemma 2.2.** Let $A = (a_{ij}) \in M_n$, and let $X$ be a positive diagonal matrix with all its entries less than or equal to 1. Given any $S \subseteq \langle n \rangle$, then

$$R_i^S(AX) \leq R_i^S(A), \quad \forall i \in \langle n \rangle.$$  \hspace{1cm} (2.2)

**Proof.** Let $B = AX$. Let $X = \text{diag}(x_1, \cdots, x_n)$ with $x_i \leq 1$ for $i \in \langle n \rangle$. For $i = 1$, we have

$$R_1^S(B) = \sum_{j>1, j \in S} |b_{1,j}| \leq \sum_{j>1, j \in S} |a_{1,j}| = R_1^S(A).$$

Assume that for $i = 2, 3, \ldots, k$, the inequalities in (2.2) hold. Now consider the case $i = k + 1$. We have

$$R_{k+1}^S(B) = \sum_{j=1}^k |b_{k+1,j}| \frac{R_j^S(B)}{|b_{jj}|} + \sum_{j \geq k+2, j \in S} |b_{k+1,j}|$$

$$\leq \sum_{j=1}^k |b_{k+1,j}| \frac{R_j^S(A)}{|b_{jj}|} + \sum_{j=k+2, j \in S} |b_{k+1,j}|$$

$$\leq \sum_{j=1}^k |a_{k+1,j}| \frac{R_j^S(A)}{|a_{jj}|} + \sum_{j=k+2, j \in S} |a_{k+1,j}| = R_{k+1}^S(A).$$

Hence, we get (2.2). □

The proof of the following lemma follows the same idea as in the proofs of [2, Theorem 2]. We state the details for completeness.
LEMMA 2.3. Let \( A = (a_{ij}) \in M_n \), let \( S \subseteq \langle n \rangle \) and let \( X = \text{diag}(x_1, \cdots, x_n) \), where
\[
x_i = \begin{cases} 
1, & i \in \bar{S} \\
\gamma, & i \in S. 
\end{cases}
\]
The following equalities hold for all \( i \in \langle n \rangle \).

(i) \( f_{i}^{\bar{S}}(AX) = \gamma f_{i}^{\bar{S}}(A) \);

(ii) \( f_{i}^{S}(AX) = f_{i}^{S}(A) \),

where \( f \in \{l, R\} \).

Proof. Let \( B = AX \). First we consider \( f = R \). We use induction on the row index \( i \). For \( i = 1 \), we have
\[
R_{1}^{S}(B) = \sum_{j>1, j \in S} |b_{1j}| = \sum_{j>1, j \in S} \gamma |a_{1j}| = \gamma R_{1}^{S}(A).
\]
Assume that (i) holds for \( i = 2, 3, \ldots, k \), then we consider the case \( i = k + 1 \).
\[
R_{k+1}^{S}(B) = \sum_{j=1}^{k} |b_{k+1,j}| \frac{R_{j}^{S}(B)}{|b_{jj}|} + \sum_{j=k+2, j \in S} |b_{k+1,j}|
\]
\[
= \sum_{j=1}^{k} |a_{k+1,j}| \frac{\gamma R_{j}^{S}(A)}{|a_{jj}|} + \sum_{j=k+2, j \in S} \gamma |a_{k+1,j}|
\]
\[
= \gamma R_{k+1}^{S}(A)
\]
Hence we get (i). Similarly, we can get the equality in (ii).
Now assume \( f = l \). For \( i = 1 \), (i) holds trivially. For \( i \geq 2 \), we have
\[
l_{i}^{S}(B) = \sum_{j=1}^{i-1} |b_{ij}| \frac{R_{j}^{S}(B)}{|b_{jj}|} = \sum_{j=1}^{i-1} |a_{ij}| \frac{\gamma R_{j}^{S}(A)}{|a_{jj}|} = \gamma l_{i}^{S}(A).
\]
Applying the same argument we get \( l_{i}^{S}(B) = l_{i}^{S}(A) \). \( \square \)

LEMMA 2.4. Let \( A \in M_n \), let \( S \subseteq \langle n \rangle \), and let \( X = \text{diag}(x_1, \cdots, x_n) \), where
\[
x_i = \begin{cases} 
1, & i \in \bar{S} \\
\gamma, & i \in S. 
\end{cases}
\]
The following inequalities hold for all \( i \in \langle n \rangle \).

(i) \( R_{i}(AX) \leq \gamma R_{i}^{S}(A) + R_{i}^{\bar{S}}(A) \);

(ii) \( l_{i}^{S}(AX) \leq \gamma l_{i}^{S}(A) \),
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where \( \gamma = \max_{i \in S} \gamma_i \).

**Proof.** Let \( C = AX \). Let \( B \) be the same matrix as defined in Lemma 2.3. By (1.3) and Lemma 2.3, we have

\[
R_i(B) = \gamma R_i^S(A) + R_i^S(A), \quad \forall \ i \in \langle n \rangle.
\]

Let \( X^* = \text{diag}(x_1^*, x_2^*, \ldots, x_n^*) \), where

\[
x_i^* = \begin{cases} 
1, & i \in \overline{S} \\
\gamma, & i \in S
\end{cases}.
\]

It is clear that \( C = BX^* \). Since \( \gamma_i/\gamma \leq 1 \), by Lemma 2.2 we get \( R_i(C) \leq R_i(B) \), which leads to (i).

For the second part, by Lemma 2.2 and Lemma 2.3 (i) we have

\[
l_i^S(C) = \sum_{j=1, j \in S}^{i-1} \frac{R_j^S(C)}{|c_{jj}|} \leq \sum_{j=1, j \in S}^{i-1} \frac{a_{ij} R_j^S(B)}{|a_{jj}|}
\]

\[
= \sum_{j=1, j \in S}^{i-1} \frac{a_{ij} \gamma R_j^S(A)}{|a_{jj}|}
\]

\[
= \gamma l_i^S(A). \quad \square
\]

Now we are ready to prove that \( GSN \) is a subclass of \( H \)-matrices.

**Theorem 2.5.** Let \( A \in M_n \). Given any \( S \subseteq N_2(A) \), if \( A \in GSN \), then \( A \) is a nonsingular \( H \)-matrix.

**Proof.** Let

\[
\varepsilon_i = \frac{|a_{ii}| - R_i^S(A) - r_S l_i^S(A) - \sum_{j>i+1, j \in S} |a_{ij}| \delta^S_j(A)}{R_i(A)}.
\]

If \( R_i(A) = 0 \), we let \( \varepsilon_i = \infty \). By (1.4), we get \( \varepsilon_i > 0 \) for all \( i \in N_1(A) \). Since \( S \subseteq N_2(A) \), we get \( r_S < 1 \). It follows from Lemma 2.1 that \( \delta^S_i(A) < 1 \) for all \( i \in S \). Let \( \varepsilon \) be a positive number satisfying

\[
\varepsilon < \min \{ \min_{i \in N_1(A)} \varepsilon_i, \min_{i \in S} (1 - \delta^S_i(A)) \}.
\]

Then we have

\[
\delta^S_i + \varepsilon < 1, \quad \forall \ i \in S, \quad (2.3)
\]
\[ |a_{ii}| > R_i^\delta(A) + (r_S + \varepsilon)l_i^\delta(A) + \sum_{j>i+1, j \in S} |a_{ij}|(\delta_j^S(A) + \varepsilon), \quad \forall i \in N_1(A). \quad (2.4) \]

Let \( X = \text{diag}(x_1, \ldots, x_n) \), where
\[
x_i = \begin{cases} 1, & i \in \bar{S} \\ \delta_i^S(A) + \varepsilon, & i \in S. \end{cases}
\]

Let \( B = (b_{ij}) = AX \). We consider the following cases.

**Case 1.** \( i \in N_1(A) \). Combining (2.4), Lemma 2.1, Lemma 2.2 and Lemma 2.4 (ii), we have
\[
|b_{ii}| = |a_{ii}| > R_i^\delta(A) + (r_S + \varepsilon)l_i^\delta(A) + \sum_{j>i+1, j \in S} |a_{ij}|(\delta_j^S(A) + \varepsilon)
\geq R_i^\delta(B) + l_i^\delta(B) + \sum_{j>i+1, j \in S} |b_{ij}|
= R_i^\delta(B) + R_i^S(B) = R_i(B).
\]

**Case 2.** \( i \in N_2(A) \setminus S \). By Lemma 2.2 and the definition of \( N_2(A) \), we have
\[
|b_{ii}| = |a_{ii}| > R_i(A) \geq R_i(B).
\]

**Case 3.** \( i \in S \). By the definition of \( \delta_i^S(A) \) and Lemma 2.4 (i), we have
\[
|b_{ii}| = |a_{ii}|(\delta_i^S(A) + \varepsilon)
= R_i^\delta(A) + r_S R_i^S(A) + |a_{ii}| \varepsilon
= R_i^\delta(A) + (r_S + \varepsilon) R_i^S(A) + (|a_{ii}| - R_i^S(A)) \varepsilon
> R_i^\delta(A) + (r_S + \varepsilon) R_i^S(A) \geq R_i(B)
\]

Hence \( B \in N_n \). It follows that \( A \) is a nonsingular \( H \)-matrix. This completes the proof. \( \square \)

By Theorem 2.5, we can get the following corollary immediately.

**COROLLARY 2.6.** If \( A \) is a weak Nekrasov matrix of order \( n \) and
\[
R_i^{N_2(A)}(A) \neq 0, \quad \forall i \in N_1(A),
\]
then \( A \) is a nonsingular \( H \)-matrix.

Next we will show the relationship between \( GSN \) and the \( S \)-Nekrasov matrices. Let recall the definition of the \( S \)-Nekrasov matrices.
**Definition 2.7.** Let $A = (a_{ij}) \in M_n$ with $n \geq 2$. Given any nonempty subset $S$ of $\langle n \rangle$, $A$ is an $S$-Nekrasov matrix if

$$|a_{ii}| > R_i^S(A), \quad |a_{jj}| > R_j^S(A)$$

and

$$\left[|a_{ii}| - R_i^S(A)\right] \left[|a_{jj}| - R_j^S(A)\right] > R_i^S(A)R_j^S(A) \quad (2.5)$$

for all $i \in S$ and $j \in \bar{S}$ hold.

**Remark 2.8.** Given an arbitrary $S$-Nekrasov matrix $A$, then either $S$ or $\bar{S}$ is a subset of $N_2(A)$. Otherwise, $N_1(A) \cap S \neq \emptyset$ and $N_1(A) \cap \bar{S} \neq \emptyset$. There exist $i \in S$ and $j \in \bar{S}$ such that $i \in N_1(A)$ and $j \in N_1(A)$. By the definition of $N_1(A)$, we get

$$\left[|a_{ii}| - R_i^S(A)\right] \left[|a_{jj}| - R_j^S(A)\right] \leq R_i^S(A)R_j^S(A),$$

which contradicts (2.5). Without loss of generality, we assume $S \subseteq N_2(A)$. We turn (2.5) into

$$|a_{jj}| - R_j^S(A) > \frac{R_j^S(A)}{|a_{ii}| - R_i^S(A)}R_i^S(A), \quad \forall i \in S, j \in \bar{S}.$$

By the definition of $r_S$, we have

$$|a_{jj}| - R_j^S(A) > r_j^S(A), \quad \forall j \in \bar{S}.$$

On the other hand, by Lemma 2.1 we obtain

$$r_j^S(A) = r_j^S(A) + r_S \sum_{k > j + 1, k \in S} |a_{kj}| \geq r_j^S(A) + \sum_{k > j + 1, k \in S} |a_{kj}| \delta_k^S(A), \quad \forall j \in \bar{S}.$$

Note that $N_1(A) \subseteq \bar{S}$. Hence $A$ is a $GSN$. We have proved that every $S$-Nekrasov matrix is in $GSN$. But not vice versa.

**Example 2.9.** Consider the matrix as follows.

$$A = \begin{bmatrix}
40 & 1 & 3 & 2 & 2 \\
0 & 8 & 4 & 4 & 6 \\
20 & 2 & 15 & 4 & 8 \\
0 & 4 & 5 & 18 & 2 \\
40 & 4 & 0 & 0 & 40
\end{bmatrix}.$$

We get $N_1(A) = \{2, 3\}$ and $N_2(A) = \{1, 4, 5\}$. For $S \in \{\{1\}, \{4\}, \{5\}, \{1, 4\}, \{1, 5\}, \{4, 5\}, \{1, 4, 5\}\}$, $A$ is not an $S$-Nekrasov matrix and hence it is not an $S$-Nekrasov matrix for any nonempty subset $S$ of $\langle n \rangle$. But for $S = \{1, 4, 5\}$, we can get $A \in GSN$ and hence it is a nonsingular $H$-matrix.
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