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RECONSTRUCTION OF TWO APPROXIMATION PROCESSES IN ORDER TO REPRODUCE e^{ax} AND e^{2ax}, $a > 0$

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Abstract. We propose two modifications for Gauss-Weierstrass operators and moment-type operators which fix e^{ax} and e^{2ax} with $a > 0$. First, we present moment identities for new operators. Then, we discuss weighted approximation and prove Voronovskaya-type theorems for them in exponentially weighted spaces. Using modulus of continuity in exponentially weighted spaces, we obtain some global smoothness preservation properties. We give a comparison result for Gauss-Weierstrass operators. Finally, we provide some graphical illustrations that show that modified operators perform better than classical ones.

1. Introduction

After Weierstrass's famous theorem on approximation, Bohman-Korovkin theorem brought a new vision to the scientific community who wants to specialize in the field of positive linear operators. For years, many theorems have been proved by using Bohman-Korovkin theorem and its various generalizations, and the related information can be found in monograph of Altomare and Campiti [5].

Another theorem which has importance in terms of theoretical diversity is a theorem so-called Voronovskaya-type theorem. It should be noted that after Voronovskaya [27] expressed asymptotic form of approximation for Bernstein polynomials, this type of theorems generated a special interest among approximation theory researchers. Later on quantitative versions of this theorem, Voronovskaya-type theorems in terms of appropriate modulus of continuity, were also proved in various papers. For further reading about Voronovskaya-type approximation and its applications, we refer the interested reader to [3, 12, 13, 14, 16, 17] and the references therein.

Some positive linear operators are known by the names of those who constructed them, such as Gauss-Weierstrass integral operators, and they are accepted as great inventions on behalf of scientific progress. Letting $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{N} = \{1, 2, \ldots\}$ throughout this manuscript, Gauss-Weierstrass operators are expressed in classical sense as follows:

$$W_n(f;x) = \frac{\sqrt{n}}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x+t)\, e^{-nt^2} dt, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{1.1}$$

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For further information, we refer the reader to the monograph by Butzer and Nessel [15].

In recent years, the most popular topic among others in this field can be seen as reconstructing the operators in order to preserve some functions, such as polynomial functions and exponential functions. Very recently, operators of type (1.1) have created interest among researchers working on the preservation of functions. For further reading regarding this topic, we refer the reader to [1, 2, 4, 9, 21] and the references therein.

In the current manuscript, inspiring by the work of Aral [8], in order to fix the functions e^{ax} and e^{2ax} with $a>0$, we will introduce the specific modifications for Gauss-Weierstrass operators (1.1) and moment-type operators:

$$T_n(f;x)=\int_{-\infty}^{\infty} f(x+t)\,n\varkappa_{\left[0,\frac{1}{n}\right]}(t)dt,\quad x\in\mathbb{R},\quad n\in\mathbb{N}, \tag{1.2}$$

where $\varkappa_{[.,.]}(.)$ denotes the characteristic function of the set $\left[0,\frac{1}{n}\right]$. The operators of type (1.2) and similar versions were considered in [22] and [26, 28], respectively. For further information about some moment-type operators, we refer the reader to [10, 11]. The kernels of these operators are of type approximate identity as are Gauss-Weierstrass kernels (see, e.g., [15]).

This manuscript contributes to theory of exponential approximation, and in this study, our main motivation is introducing two new modifications to the theory, as well as to observe what will happen, even with limited examples, by putting an operator sequence which is relatively easy to compute next to the Gauss-Weierstrass operators. The proposed modifications for the operators defined in (1.1) and (1.2) are respectively given as follows:

$$W_n^{\circ}(f;x)=\frac{\sqrt{n}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-a\left(\beta_n^{\circ}(x)+t\right)}e^{ax}f\left(\beta_n^{\circ}(x)+t\right)e^{-nt^2}dt,\quad x\in\mathbb{R},\quad n\in\mathbb{N} \tag{1.3}$$

and

$$T_n^{\circ}(f;x)=\int_{-\infty}^{\infty} e^{-a\left(\lambda_n^{\circ}(x)+t\right)}e^{ax}f\left(\lambda_n^{\circ}(x)+t\right)n\varkappa_{\left[0,\frac{1}{n}\right]}(t)dt,\quad x\in\mathbb{R},\quad n\in\mathbb{N}, \tag{1.4}$$

where $\beta_n^{\circ}(x)=x-\frac{a}{4n}$, $\lambda_n^{\circ}(x)=x-a^{-1}\ln\left(\frac{n(e^{\frac{a}{n}}-1)}{a}\right)$ and $a>0$. Here, $W_n^{\circ}\to W_n$ and $T_n^{\circ}\to T_n$ as $a\to 0^+$. In this work, as in [9], $\exp_a(t)$ and $\log_a(.)$ stand for e^{at} with $t\in\mathbb{R}$ and logarithmic function with base e^a with $a>0$, respectively.

We obtain weighted convergence of the operators defined in (1.3) and (1.4) in polynomial weighted space. Then, we prove quantitative and Voronovskaya-type theorems for them in exponentially weighted space. Using a modulus of continuity defined for exponentially weighted space, we present some global smoothness preservation properties of these operators. We give a comparison result for Gauss-Weierstrass operators. Finally, we provide some graphical illustrations.

2. Auxiliary results

Now, we give some lemmas related to moment identities of integral operators defined in (1.3) and (1.4) which will be used in the sequel.

LEMMA 1. *Let* $\mathfrak{e}_i = t^i,\ i = 0,\ 1,\ 2$ *be test functions on* $\mathbb{R}$. *For each* $n \in \mathbb{N}$ *and* $x \in \mathbb{R}$, *the following identities hold there:*

$$\begin{aligned} W_n^{\circ}(\mathfrak{e}_0;x) &= \psi_{11}(n) \\ W_n^{\circ}(\mathfrak{e}_1;x) &= \psi_{11}(n)x + \phi_{11}(n) \\ W_n^{\circ}(\mathfrak{e}_2;x) &= \psi_{11}(n)x^2 + \phi_{12}(n)x + \phi_{13}(n) \end{aligned}$$

and

$$\begin{aligned} W_n^{\circ}(\exp_a;x) &= e^{ax} \\ W_n^{\circ}(\exp_{2a};x) &= e^{2ax} \\ W_n^{\circ}(\exp_{3a};x) &= \psi_{11}(n)e^{3ax} \\ W_n^{\circ}(\exp_{4a};x) &= \psi_{12}(n)e^{4ax}. \end{aligned}$$

Here, $\psi_{11}(n) = e^{\frac{a^2}{2n}}$, $\psi_{12}(n) = e^{\frac{3a^2}{2n}}$, $\phi_{11}(n) = -\frac{3a}{4n}e^{\frac{a^2}{2n}}$, $\phi_{12}(n) = -\frac{3a}{2n}e^{\frac{a^2}{2n}}$ *and* $\phi_{13}(n) = \left(\frac{8n+9a^2}{16n^2}\right)e^{\frac{a^2}{2n}}$. *Note that* $\lim_{n\to\infty}\psi_{1i}(n) = 1$ *and* $\lim_{n\to\infty}\phi_{1j}(n) = 0$, *where* $i = 1,\ 2$ *and* $j = 1,\ 2,\ 3$.

LEMMA 2. *Let* $\mathfrak{e}_i = t^i,\ i = 0,\ 1,\ 2$ *be test functions on* $\mathbb{R}$. *For each* $n \in \mathbb{N}$ *and* $x \in \mathbb{R}$, *the following identities hold there:*

$$\begin{aligned} T_n^{\circ}(\mathfrak{e}_0;x) &= \psi_{21}(n) \\ T_n^{\circ}(\mathfrak{e}_1;x) &= \psi_{22}(n)x + \phi_{21}(n) \\ T_n^{\circ}(\mathfrak{e}_2;x) &= \psi_{22}(n)x^2 + \phi_{22}(n)x + \phi_{23}(n) \end{aligned}$$

and

$$\begin{aligned} T_n^{\circ}(\exp_a;x) &= e^{ax} \\ T_n^{\circ}(\exp_{2a};x) &= e^{2ax} \\ T_n^{\circ}(\exp_{3a};x) &= \psi_{23}(n)e^{3ax} \\ T_n^{\circ}(\exp_{4a};x) &= \psi_{24}(n)e^{4ax}. \end{aligned}$$

Here, $\psi_{21}(n) = \frac{2n^2}{a^2}\left(-1+\cosh\left(\frac{a}{n}\right)\right)$, $\psi_{22}(n) = \frac{n^2(1-e^{\frac{-a}{n}})(-1+e^{\frac{a}{n}})}{a^2}$, $\psi_{23}(n) = \frac{a}{2n}\coth(\frac{a}{2n})$, $\psi_{24}(n) = \frac{a^2(-1+e^{\frac{3a}{n}})}{3n^2(-1+e^{\frac{a}{n}})^3}$ *and*

$$\phi_{21}(n) = \frac{\left(-1+e^{\frac{a}{n}}\right)n^2\left(1-e^{\frac{-a}{n}}-\frac{a}{n}e^{\frac{-a}{n}}+\left(-1+e^{\frac{-a}{n}}\right)\ln\left(\frac{n}{a}\left(-1+e^{\frac{a}{n}}\right)\right)\right)}{a^3}$$

$$\phi_{22}(n)=\frac{2\left(-1+e^{\frac{a}{n}}\right)n^{2}\left(1-e^{\frac{-a}{n}}-\frac{a}{n}e^{\frac{-a}{n}}+\ln\left(\frac{n}{a}\left(-1+e^{\frac{a}{n}}\right)\right)\left(-1+e^{\frac{-a}{n}}\right)\right)}{a^{3}}$$

and

$$\phi_{23}(n)=\frac{\left(-1+e^{\frac{a}{n}}\right)n^{2}\left(2-e^{\frac{-a}{n}}\left(2+2\frac{a}{n}+\frac{a^{2}}{n^{2}}\right)+2\ln\left(\frac{n}{a}\left(-1+e^{\frac{a}{n}}\right)\right)\left(-1+e^{\frac{-a}{n}}\right)\right)}{a^{4}}$$
$$+\frac{\left(-1+e^{\frac{a}{n}}\right)n^{2}\ln\left(\frac{n}{a}\left(-1+e^{\frac{a}{n}}\right)\right)\left(\frac{2a}{n}e^{\frac{-a}{n}}+\ln\left(\frac{n}{a}\left(-1+e^{\frac{a}{n}}\right)\right)-e^{\frac{-a}{n}}\ln\left(\frac{n}{a}\left(-1+e^{\frac{a}{n}}\right)\right)\right)}{a^{4}}.$$

Note that $\lim_{n\to\infty}\psi_{2i}(n)=1$ *and* $\lim_{n\to\infty}\phi_{2j}(n)=0$, *where* $i=1,\ 2,\ 3,\ 4$ *and* $j=1,\ 2,\ 3$.

3. Weighted approximation

First, we give definions of some weighted spaces defined in [20]. Using the weight function ρ which is expressed as $\rho(x)=1+\varphi^{2}(x)$ with $\lim_{x\to\pm\infty}\rho(x)=\infty$, where φ is a strictly increasing and continuous function on $\mathbb{R}$, Gadžiev [20] introduced the weighted spaces $B_{\rho}(\mathbb{R})$, $C_{\rho}(\mathbb{R})$ and $C_{\rho}^{0}(\mathbb{R})$ as follows:

$$B_{\rho}(\mathbb{R}):=\left\{f:|f(x)|\leqslant M_{f}\rho(x),\ \ x\in\mathbb{R}\right\},$$

where M_{f} is a constant which only depends on the function f, and

$$C_{\rho}(\mathbb{R}):=\left\{f:f\in B_{\rho}(\mathbb{R}),\ f\text{ is continuous on }\mathbb{R}\right\}$$
$$C_{\rho}^{0}(\mathbb{R}):=\left\{f:f\in C_{\rho}(\mathbb{R}),\lim_{x\to\pm\infty}\frac{f(x)}{\rho(x)}=K_{f}\text{ exists finitely}\right\},$$

where K_{f} is a constant which only depends on the function f. Here, the space $B_{\rho}(\mathbb{R})$ is equipped with the norm defined by

$$\|f\|_{\rho}=\sup_{-\infty<x<\infty}\frac{|f(x)|}{\rho(x)}.$$

Here, the spaces $C_{\rho}(\mathbb{R})$ and $C_{\rho}^{0}(\mathbb{R})$ are associated with the same norm. The general version of the following theorems was proved as Theorem 2 in [20]. Similar result was proved in [1] for Szász-Mirakyan-type operators preserving some exponential functions.

THEOREM 1. *Let* $\rho(x)=1+x^{2}$, $x\in\mathbb{R}$. *For the operators* W_{n}° *defined in* (1.3), *we have*

$$\lim_{n\to\infty}\left\|W_{n}^{\circ}f-f\right\|_{\rho}=0$$

for every function $f\in C_{\rho}^{0}(\mathbb{R})$.

Proof. The hypotheses of Theorem 2 in [20] will be used in order to prove the theorem. Clearly, $\overset{\circ}{W}_n$ is a positive linear operator acting from $C_\rho(\mathbb{R})$ to $B_\rho(\mathbb{R})$. Therefore, it is sufficient to show that the following conditions hold:

$$\lim_{n\to\infty}\left\|\overset{\circ}{W}_n\varphi^v-\varphi^v\right\|_\rho=0,\quad v=0,\ 1,\ 2,$$

where $\varphi(x)=x$ since $\rho(x)=1+x^2=1+\varphi^2(x)$ according to hypothesis.

Let $v=0$. By Lemma 1, we can write

$$\begin{aligned}\left\|\overset{\circ}{W}_n(\mathfrak{e}_0;x)-1\right\|_\rho &= \sup_{-\infty<x<\infty}\frac{\left|\overset{\circ}{W}_n(\mathfrak{e}_0;x)-1\right|}{1+x^2}\\ &= \sup_{-\infty<x<\infty}\frac{|\psi_{11}(n)-1|}{1+x^2}\\ &\leqslant |\psi_{11}(n)-1|.\end{aligned}$$

Since $\frac{a^2}{2n}>0$ for all n and $\lim\limits_{n\to\infty}\psi_{11}(n)=1$, the result follows.

Let $v=1$. Again using Lemma 1, we obtain

$$\begin{aligned}\left\|\overset{\circ}{W}_n(\mathfrak{e}_1;x)-x\right\|_\rho &= \sup_{-\infty<x<\infty}\frac{\left|\overset{\circ}{W}_n(\mathfrak{e}_1;x)-x\right|}{1+x^2}\\ &\leqslant \sup_{-\infty<x<\infty}\frac{|x|}{1+x^2}|\psi_{11}(n)-1|+\sup_{-\infty<x<\infty}\frac{|\phi_{11}(n)|}{1+x^2}\\ &\leqslant \sup_{-\infty<x<\infty}\frac{|x|}{1+x^2}|\psi_{11}(n)-1|+|\phi_{11}(n)|.\end{aligned}$$

Since $\frac{|x|}{1+x^2}<1$ for all $x\in\mathbb{R}$, we have

$$\left\|\overset{\circ}{W}_n(\mathfrak{e}_1;x)-x\right\|_\rho\leqslant|\psi_{11}(n)-1|+|\phi_{11}(n)|.$$

$\lim\limits_{n\to\infty}\psi_{11}(n)=1$ and $\lim\limits_{n\to\infty}\phi_{11}(n)=0$, the result follows.

Lastly, let $v=2$. Using similar considerations, we get

$$\begin{aligned}\left\|\overset{\circ}{W}_n(\mathfrak{e}_2;x)-x^2\right\|_\rho &= \sup_{-\infty<x<\infty}\frac{\left|\overset{\circ}{W}_n(\mathfrak{e}_2;x)-x^2\right|}{1+x^2}\\ &\leqslant \sup_{-\infty<x<\infty}\frac{x^2}{1+x^2}|\psi_{11}(n)-1|\\ &\quad+\sup_{-\infty<x<\infty}\frac{|x|}{1+x^2}|\phi_{12}(n)|+\sup_{-\infty<x<\infty}\frac{|\phi_{13}(n)|}{1+x^2}.\end{aligned}$$

Since $\frac{x^2}{1+x^2}<1$ for all $x\in\mathbb{R}$, we are able to write

$$\left\|\overset{\circ}{W}_n(\mathfrak{e}_2;x)-x^2\right\|_\rho\leqslant|\psi_{11}(n)-1|+|\phi_{12}(n)|+|\phi_{13}(n)|.$$

Keeping in mind that $\lim_{n\to\infty}\psi_{11}(n)=1$, $\lim_{n\to\infty}\phi_{12}(n)=0$ and $\lim_{n\to\infty}\phi_{13}(n)=0$, the result follows. Thus the proof is completed. $\square$

Using similar arguments and Lemma 2, one may prove the following result.

THEOREM 2. *Let* $\rho(x)=1+x^2$, $x\in\mathbb{R}$. *For the operators* T_n° *defined in* (1.4), *we have*

$$\lim_{n\to\infty}\left\|T_n^{\circ}f-f\right\|_{\rho}=0$$

for every function $f\in C_{\rho}^{0}(\mathbb{R})$.

4. Quantitative estimates

In this section, we give analogous results which were given in [8].

Let $\rho_1(x)=e^{a|x|}$ and $C_{\rho_1}(\mathbb{R})$ denote the space of all continuous functions f for which $\|f\|_{\rho_1}=\sup_{-\infty<x<\infty}\frac{|f(x)|}{\rho_1(x)}\leqslant\mathscr{B}_f$, where $\mathscr{B}_f$ is a positive constant depending on only f (cf. [20]). In order to measure rate of convergence in exponentially weighted space and to obtain an appropriate estimate, following [23, 25], we use weighted modulus of continuity defined as

$$\widetilde{\omega}(f;\delta)=\sup_{|h|\leqslant\delta}\left[\sup_{-\infty<x<\infty}\frac{|f(x+h)-f(x)|}{e^{a|x|}}\right], \tag{4.1}$$

where $\delta>0$ and $a>0$. For a number $\zeta>0$, this weighted modulus of continuity has the following properties (see [18]; see also [25]):

$$\widetilde{\omega}(f;\zeta\delta)\leqslant(1+\zeta)e^{a\zeta\delta}\widetilde{\omega}(f;\delta) \text{ and } \lim_{\delta\to 0^+}\widetilde{\omega}(f;\delta)=0.$$

THEOREM 3. *For* $f\in C_{\rho_1}(\mathbb{R})$, *there holds*

$$\left\|W_n^{\circ}f-f\right\|_{\rho_1}\leqslant\widetilde{\omega}\left(f;\frac{1}{\sqrt{n}}\right)e^{\frac{a^2}{2n}}\left(\frac{1}{\sqrt{\pi}}+\frac{(5a+4\sqrt{n})}{2\sqrt{n}}e^{\frac{a^2}{n}}\right)+\|f\|_{\rho_1}\left|e^{\frac{a^2}{2n}}-1\right|.$$

Proof. Since

$$\left\|W_n^{\circ}f\right\|_{\rho_1}\leqslant\mathscr{B}_f e^{\frac{a^2}{2n}}\left(1+2e^{\frac{a^2}{n}}\right)$$

for any function $f\in C_{\rho_1}(\mathbb{R})$, W_n° is the sequence of positive linear operators acting on $C_{\rho_1}(\mathbb{R})$ (see [19]). In view of Lemma 1, we have

$$\begin{aligned}
&\left|W_n^{\circ}(f;x)-f(x)\right|\\
&\leqslant\left|\frac{\sqrt{n}}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-a\left(\beta_n^{\circ}(x)+t\right)}e^{ax}f\left(\beta_n^{\circ}(x)+t\right)e^{-nt^2}dt-f(x)\frac{\sqrt{n}}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-a\left(\beta_n^{\circ}(x)+t\right)}e^{ax}e^{-nt^2}dt\right|\\
&+|f(x)|\left|W_n^{\circ}(\mathfrak{e}_0;x)-1\right|
\end{aligned}$$

$$= \left| \frac{\sqrt{n}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left[f\left(\beta_n^{\circ}(x)+t\right) - f(x) \right] e^{-a\left(-\frac{a}{4n}+t\right)} e^{-nt^2} dt \right| + |f(x)| \left| W_n^{\circ}(\mathfrak{e}_0;x) - 1 \right|.$$

Since

$$\left| f\left(\beta_n^{\circ}(x)+t\right) - f(x) \right| = \left| f\left(x - \frac{a}{4n} + t\right) - f(x) \right| \leqslant e^{a|x|} \widetilde{\omega}\left(f; \left|t - \frac{a}{4n}\right|\right),$$

we have

$$\begin{aligned}
&\left| W_n^{\circ}(f;x) - f(x) \right| \\
&\leqslant \frac{\sqrt{n}}{\sqrt{\pi}} e^{a|x|} e^{\frac{a^2}{4n}} \int_{-\infty}^{\infty} \widetilde{\omega}\left(f; \left|t - \frac{a}{4n}\right|\right) e^{a|t|} e^{-nt^2} dt + |f(x)| \left| W_n^{\circ}(\mathfrak{e}_0;x) - 1 \right| \\
&\leqslant \frac{\sqrt{n}}{\sqrt{\pi}} e^{a|x|} \widetilde{\omega}(f;\delta) e^{\frac{a^2}{4n}} \int_{-\infty}^{\infty} \left(1 + \frac{\left|t - \frac{a}{4n}\right|}{\delta}\right) e^{a|t|} e^{a\left|t - \frac{a}{4n}\right|} e^{-nt^2} dt + |f(x)| \left| W_n^{\circ}(\mathfrak{e}_0;x) - 1 \right| \\
&\leqslant \frac{\sqrt{n}}{\sqrt{\pi}} e^{a|x|} \widetilde{\omega}(f;\delta) e^{\frac{a^2}{2n}} \int_{-\infty}^{\infty} \left(1 + \frac{a}{4\delta n} + \frac{|t|}{\delta}\right) e^{2a|t|} e^{-nt^2} dt + |f(x)| \left| W_n^{\circ}(\mathfrak{e}_0;x) - 1 \right|.
\end{aligned}$$

After straightforward evaluations, we have

$$\begin{aligned}
&\left| W_n^{\circ}(f;x) - f(x) \right| \\
&\leqslant e^{a|x|} \widetilde{\omega}(f;\delta) e^{\frac{a^2}{2n}} \left(\frac{1}{\delta\sqrt{n}\sqrt{\pi}} + \left(\frac{5a}{4\delta n} + 1\right) \left(e^{\frac{a^2}{n}} + e^{\frac{a^2}{n}} \frac{2}{\sqrt{\pi}} \int_{0}^{\frac{a}{\sqrt{n}}} e^{-t^2} dt \right) \right) \\
&\quad + |f(x)| \left| W_n^{\circ}(\mathfrak{e}_0;x) - 1 \right| \\
&\leqslant e^{a|x|} \widetilde{\omega}(f;\delta) e^{\frac{a^2}{2n}} \left(\frac{1}{\delta\sqrt{n}\sqrt{\pi}} + \left(\frac{5a}{4\delta n} + 1\right) 2e^{\frac{a^2}{n}} \right) + |f(x)| \left| e^{\frac{a^2}{2n}} - 1 \right|.
\end{aligned}$$

Considering $\delta := \frac{1}{\sqrt{n}}$, we obtain the desired conclusion, that is,

$$\left\| W_n^{\circ} f - f \right\|_{\rho_1} \leqslant \widetilde{\omega}\left(f; \frac{1}{\sqrt{n}}\right) e^{\frac{a^2}{2n}} \left(\frac{1}{\sqrt{\pi}} + \frac{(5a + 4\sqrt{n})}{2\sqrt{n}} e^{\frac{a^2}{n}} \right) + \|f\|_{\rho_1} \left| e^{\frac{a^2}{2n}} - 1 \right|. \quad \square$$

By Theorem 3 and properties of weighted modulus of continuity defined in (4.1), we have the following deduction.

COROLLARY 1. *Let* $f \in C_{\rho_1}(\mathbb{R})$. *Then*

$$\lim_{n \to \infty} \left\| W_n^{\circ} f - f \right\|_{\rho_1} = 0.$$

THEOREM 4. *For $f \in C_{\rho_1}(\mathbb{R})$, there holds*

$$\left\|T_n^{\circ} f - f\right\|_{\rho_1} \leqslant \widetilde{\omega}\left(f; \left(-1+e^{\frac{a}{n}}\right)^2 n\left(a+2n\ln\left(\frac{n(e^{\frac{a}{n}}-1)}{a}\right)\right)\right) \times \left(\frac{\left(-1+e^{\frac{a}{n}}\right)^2 n^2}{a^2} + \frac{1}{2a^3}\right) + \|f\|_{\rho_1}\left|\frac{2n^2}{a^2}\left(-1+\cosh\left(\frac{a}{n}\right)\right)-1\right|.$$

Proof. It is easy to see that T_n° is the sequence of positive linear operators acting on $C_{\rho_1}(\mathbb{R})$. Let $\alpha_a(n) := a^{-1}\ln\left(\frac{n(e^{\frac{a}{n}}-1)}{a}\right) > 0$ by the fact that $e^{\frac{a}{n}} > 1+\frac{a}{n}$ for any fixed $n \in \mathbb{N}$ and $a > 0$. In view of Lemma 2, we have

$$\begin{aligned}
&\left|T_n^{\circ}(f;x) - f(x)\right| \\
&\leqslant \left|\int_{-\infty}^{\infty} e^{-a\left(\lambda_n^{\circ}(x)+t\right)} e^{ax} f\left(\lambda_n^{\circ}(x)+t\right) n\varkappa_{[0,\frac{1}{n}]}(t)dt - f(x)\int_{-\infty}^{\infty} e^{-a\left(\lambda_n^{\circ}(x)+t\right)} e^{ax} n\varkappa_{[0,\frac{1}{n}]}(t)dt\right| \\
&+|f(x)|\left|T_n^{\circ}(\mathfrak{e}_0;x)-1\right| \\
&\leqslant \left|\int_0^{\frac{1}{n}} e^{-a(-\alpha_a(n)+t)}\left[f\left(\lambda_n^{\circ}(x)+t\right)-f(x)\right] n dt\right| + |f(x)|\left|T_n^{\circ}(\mathfrak{e}_0;x)-1\right|.
\end{aligned}$$

Since

$$\left|f\left(\lambda_n^{\circ}(x)+t\right)-f(x)\right| \leqslant e^{a|x|}\widetilde{\omega}\left(f;|t-\alpha_a(n)|\right),$$

we have

$$\begin{aligned}
&\left|T_n^{\circ}(f;x)-f(x)\right| \\
&\leqslant n e^{a|x|} e^{a\alpha_a(n)} \int_0^{\frac{1}{n}} \widetilde{\omega}\left(f;|t-\alpha_a(n)|\right) e^{-at} dt + |f(x)|\left|T_n^{\circ}(\mathfrak{e}_0;x)-1\right| \\
&\leqslant n^3 e^{a|x|}\left(\frac{(e^{\frac{a}{n}}-1)}{a}\right)^2 \widetilde{\omega}(f;\delta)\int_0^{\frac{1}{n}}\left(1+\frac{\ln\left(\frac{n(e^{\frac{a}{n}}-1)}{a}\right)}{a\delta}+\frac{t}{\delta}\right)e^{-at}e^{at}dt \\
&\quad + |f(x)|\left|T_n^{\circ}(\mathfrak{e}_0;x)-1\right| \\
&= e^{a|x|}\widetilde{\omega}(f;\delta)\left(\frac{\left(-1+e^{\frac{a}{n}}\right)^2 n^2}{a^2} + \frac{\left(-1+e^{\frac{a}{n}}\right)^2 n\left(a+2n\ln\left(\frac{n(e^{\frac{a}{n}}-1)}{a}\right)\right)}{2a^3\delta}\right) \\
&\quad + |f(x)|\left|\frac{2n^2}{a^2}\left(-1+\cosh\left(\frac{a}{n}\right)\right)-1\right|.
\end{aligned}$$

Considering $\delta := \left(-1+e^{\frac{a}{n}}\right)^2 n(a+2n\ln(\frac{n(e^{\frac{a}{n}}-1)}{a}))$ with

$$\lim_{n\to\infty}\left(-1+e^{\frac{a}{n}}\right)^2 n(a+2n\ln(\frac{n(e^{\frac{a}{n}}-1)}{a}))=0$$

and

$$\lim_{n\to\infty}\frac{\left(-1+e^{\frac{a}{n}}\right)^2 n^2}{a^2}=1$$

we get the result, that is,

$$\left\|T_n^{\circ} f-f\right\|_{\rho_1} \leqslant \widetilde{\omega}\left(f;\left(-1+e^{\frac{a}{n}}\right)^2 n(a+2n\ln(\frac{n(e^{\frac{a}{n}}-1)}{a}))\right)\left(\frac{\left(-1+e^{\frac{a}{n}}\right)^2 n^2}{a^2}+\frac{1}{2a^3}\right)$$
$$+\|f\|_{\rho_1}\left|\frac{2n^2}{a^2}\left(-1+\cosh\left(\frac{a}{n}\right)\right)-1\right|. \quad \square$$

By Theorem 4 and properties of weighted modulus of continuity defined in (4.1), we have the following deduction.

COROLLARY 2. *Let $f\in C_{\rho_1}(\mathbb{R})$. Then*

$$\lim_{n\to\infty}\left\|T_n^{\circ} f-f\right\|_{\rho_1}=0.$$

5. Voronovskaya-type asymptotic relations

In this section, as in [8] and [9], using exponential moments instead of polynomial moments, we will prove Voronovskaya-type theorems for the operators $(W_n^{\circ})_{n\in\mathbb{N}}$ and $(T_n^{\circ})_{n\in\mathbb{N}}$.

THEOREM 5. *Let $f\in C_{\rho_1}(\mathbb{R})$. If f'' exists finitely at a point $x\in\mathbb{R}$, then there holds*

$$\lim_{n\to\infty} n\left[W_n^{\circ}(f;x)-f(x)\right]=\frac{a^2}{2}f(x)-\frac{3a}{4}f'(x)+\frac{1}{4}f''(x). \tag{5.1}$$

Proof. We use local Taylor formula as follows:

$$\begin{aligned}
& f(\beta_n^{\circ}(x)+t)\\
&=(f\circ\log_a)(e^{a(\beta_n^{\circ}(x)+t)})\\
&=(f\circ\log_a)(e^{ax})+(f\circ\log_a)'(e^{ax})\left(e^{a(\beta_n^{\circ}(x)+t)}-e^{ax}\right)\\
&\quad+\frac{1}{2}(f\circ\log_a)''(e^{ax})\left(e^{a(\beta_n^{\circ}(x)+t)}-e^{ax}\right)^2+r_x(u)\left(e^{a(\beta_n^{\circ}(x)+t)}-e^{ax}\right)^2,
\end{aligned}$$

where $r_x,\ r_x(u) := r_x(u(t))$ with $u(t) = t - \frac{a}{4n}$, is a function in $C_{\rho_1}(\mathbb{R})$ with $\lim_{u\to 0} r_x(u) = 0$. Implementing the operators to the formula, we have

$$\begin{aligned}
&W_n^{\circ}(f;x) - f(x)\\
&= f(x)\left(W_n^{\circ}(\mathfrak{e}_0;x) - 1\right)\\
&\quad + \frac{\sqrt{n}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-a\left(\beta_n^{\circ}(x)+t\right)} e^{ax} \left[(f\circ \log_a)'(e^{ax}) \left(e^{a(\beta_n^{\circ}(x)+t)} - e^{ax} \right) \right] e^{-nt^2} dt\\
&\quad + \frac{\sqrt{n}}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-a\left(\beta_n^{\circ}(x)+t\right)} e^{ax} \left[(f\circ \log_a)''(e^{ax}) \left(e^{a(\beta_n^{\circ}(x)+t)} - e^{ax} \right)^2 \right] e^{-nt^2} dt\\
&\quad + \frac{\sqrt{n}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-a\left(\beta_n^{\circ}(x)+t\right)} e^{ax} \left[r_x(u) \left(e^{a(\beta_n^{\circ}(x)+t)} - e^{ax} \right)^2 \right] e^{-nt^2} dt.
\end{aligned}$$

In view of $(f\circ \log_a)'(e^{ax}) = a^{-1}e^{-ax}f'(x)$ and $(f\circ \log_a)''(e^{ax}) = e^{-2ax}(a^{-2}f''(x) - a^{-1}f'(x))$, there holds

$$\begin{aligned}
&W_n^{\circ}(f;x) - f(x)\\
&= f(x)\left(W_n^{\circ}(\mathfrak{e}_0;x) - 1\right)\\
&\quad + \frac{\sqrt{n}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-a\left(\beta_n^{\circ}(x)+t\right)} e^{ax} \left[a^{-1}e^{-ax}f'(x) \left(e^{a(\beta_n^{\circ}(x)+t)} - e^{ax} \right) \right] e^{-nt^2} dt\\
&\quad + \frac{\sqrt{n}}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-a\left(\beta_n^{\circ}(x)+t\right)} e^{ax} \left[e^{-2ax}a^{-2}f''(x) \left(e^{a(\beta_n^{\circ}(x)+t)} - e^{ax} \right)^2 \right] e^{-nt^2} dt\\
&\quad - \frac{\sqrt{n}}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-a\left(\beta_n^{\circ}(x)+t\right)} e^{ax} \left[e^{-2ax}a^{-1}f'(x) \left(e^{a(\beta_n^{\circ}(x)+t)} - e^{ax} \right)^2 \right] e^{-nt^2} dt\\
&\quad + \frac{\sqrt{n}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-a\left(\beta_n^{\circ}(x)+t\right)} e^{ax} \left[r_x(u) \left(e^{a(\beta_n^{\circ}(x)+t)} - e^{ax} \right)^2 \right] e^{-nt^2} dt\\
&:= \gamma_{11}(n) + \gamma_{12}(n) + \gamma_{13}(n) + \gamma_{14}(n) + \gamma_{15}(n).
\end{aligned}$$

Let us consider $\gamma_{11}(n)$. Multiplying both sides by n and passing to the limit as n tends to ∞, we have

$$\begin{aligned}
\lim_{n\to\infty} n\gamma_{11}(n) &= \lim_{n\to\infty} n\left[f(x)W_n^{\circ}(\mathfrak{e}_0;x) - 1)\right]\\
&= \lim_{n\to\infty} nf(x)\left[e^{\frac{a^2}{2n}} - 1\right]\\
&= \frac{a^2}{2} f(x).
\end{aligned}$$

If we continue with $\gamma_{12}(n)$, we get

$$\begin{aligned}\lim_{n\to\infty} n\gamma_{12}(n) &= \lim_{n\to\infty} na^{-1}e^{-ax}f'(x)\left[W_n^{\circ}(\exp_a;x) - e^{ax}W_n^{\circ}(\mathfrak{e}_0;x)\right]\\ &= \lim_{n\to\infty} na^{-1}e^{-ax}f'(x)\left[e^{ax} - e^{ax}e^{\frac{a^2}{2n}}\right]\\ &= \lim_{n\to\infty} na^{-1}e^{-ax}f'(x)e^{ax}\left[1 - e^{\frac{a^2}{2n}}\right]\\ &= -\frac{a}{2}f'(x).\end{aligned}$$

Further for $\gamma_{13}(n)$, we get

$$\begin{aligned}\gamma_{13}(n) &= \frac{e^{-2ax}a^{-2}f''(x)}{2}\left[W_n^{\circ}(\exp_{2a};x) - 2e^{ax}W_n^{\circ}(\exp_a;x) + e^{2ax}W_n^{\circ}(\mathfrak{e}_0;x)\right]\\ &= \frac{e^{-2ax}a^{-2}f''(x)}{2}\left[e^{2ax} - 2e^{ax}e^{ax} + e^{2ax}e^{\frac{a^2}{2n}}\right].\end{aligned}$$

Therefore, we obtain

$$\lim_{n\to\infty} n\gamma_{13}(n) = \frac{1}{4}f''(x).$$

Similarly, for $\gamma_{14}(n)$, we get

$$\lim_{n\to\infty} n\gamma_{14}(n) = -\frac{a}{4}f'(x).$$

In order to complete the proof, it is sufficient to show that

$$\lim_{n\to\infty} n\gamma_{15}(n) = 0.$$

Using Cauchy-Schwarz inequality, we see that

$$|n\gamma_{15}(n)| \leqslant \left(W_n^{\circ}(r_x^2(u);x)\right)^{1/2}\left(n^2\frac{\sqrt{n}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-a\left(\beta_n^{\circ}(x)+t\right)}e^{ax}\left(e^{a(\beta_n^{\circ}(x)+t)} - e^{ax}\right)^4 e^{-nt^2}dt\right)^{1/2}.$$

Observe that

$$\begin{aligned}&n^2\frac{\sqrt{n}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-a\left(\beta_n^{\circ}(x)+t\right)}e^{ax}\left(e^{a(\beta_n^{\circ}(x)+t)} - e^{ax}\right)^4 e^{-nt^2}dt\\ &= n^2(W_n^{\circ}(\exp_{4a};x) - 4e^{ax}W_n^{\circ}(\exp_{3a};x) + 6e^{2ax}W_n^{\circ}(\exp_{2a};x))\\ &\quad + n^2(-4e^{3ax}W_n^{\circ}(\exp_a;x) + e^{4ax}W_n^{\circ}(\mathfrak{e}_0;x))\\ &= n^2(e^{\frac{3a^2}{2n}}e^{4ax} - 4e^{ax}e^{\frac{a^2}{2n}}e^{3ax} + 6e^{2ax}e^{2ax})\\ &\quad + n^2(-4e^{3ax}e^{ax} + e^{4ax}e^{\frac{a^2}{2n}}).\end{aligned}$$

Further evaluations give that the right hand side of last equality tends to $\frac{3}{4}(a^4)(e^{4ax})$ as n tends to ∞. We infer from Theorem 3 that $\lim_{n\to\infty} W_n^{\circ}(r_x^2(u);x) = r_x^2(0) = 0$. Hence

$$\lim_{n\to\infty} n\gamma_{15}(n) = 0.$$

This completes the proof. □

THEOREM 6. *Let $f \in C_{\rho_1}(\mathbb{R})$. If f'' exists finitely at a point $x \in \mathbb{R}$, then there holds*

$$\lim_{n\to\infty} n^2 \left[(T_n^{\circ}(f;x) - f(x)\right] = \frac{a^2}{12} f(x) - \frac{a}{8} f'(x) + \frac{1}{24} f''(x).$$

Proof. We use local Taylor formula as follows:

$$\begin{aligned} f(\lambda_n^{\circ}(x)+t) &= (f \circ \log_a)\left(e^{a(\lambda_n^{\circ}(x)+t)}\right) \\ &= (f \circ \log_a)(e^{ax}) + (f \circ \log_a)'(e^{ax})\left(e^{a(\lambda_n^{\circ}(x)+t)} - e^{ax}\right) \\ &\quad + \frac{1}{2}(f \circ \log_a)''(e^{ax})\left(e^{a(\lambda_n^{\circ}(x)+t)} - e^{ax}\right)^2 \\ &\quad + h_x(u)\left(e^{a(\lambda_n^{\circ}(x)+t)} - e^{ax}\right), \end{aligned}$$

where h_x, $h_x(u) := h_x(u(t))$ with $u(t) = t - a^{-1}\ln\left(\frac{n(e^{\frac{a}{n}}-1)}{a}\right)$, is a function in $C_{\rho_1}(\mathbb{R})$ with $\lim_{u\to 0} h_x(u) = 0$. Using Local Taylor formula and implementing the operators to it, we have

$$\begin{aligned} T_n^{\circ}(f;x) - f(x) &= f(x)\left(T_n^{\circ}(\mathfrak{e}_0;x) - 1\right) \\ &\quad + n\int_0^{\frac{1}{n}} e^{-a\left(\lambda_n^{\circ}(x)+t\right)} e^{ax}\left[a^{-1}e^{-ax} f'(x)\left(e^{a(\lambda_n^{\circ}(x)+t)} - e^{ax}\right)\right] dt \\ &\quad + \frac{n}{2}\int_0^{\frac{1}{n}} e^{-a\left(\lambda_n^{\circ}(x)+t\right)} e^{ax}\left[a^{-2}e^{-2ax} f''(x)\left(e^{a(\lambda_n^{\circ}(x)+t)} - e^{ax}\right)^2\right] dt \\ &\quad - \frac{n}{2}\int_0^{\frac{1}{n}} e^{-a\left(\lambda_n^{\circ}(x)+t\right)} e^{ax}\left[a^{-1}e^{-ax} f'(x)\left(e^{a(\lambda_n^{\circ}(x)+t)} - e^{ax}\right)^2\right] dt \\ &\quad + n\int_0^{\frac{1}{n}} e^{-a\left(\lambda_n^{\circ}(x)+t\right)} e^{ax}\left[h_x(u)\left(e^{a(\lambda_n^{\circ}(x)+t)} - e^{ax}\right)^2\right] dt \\ &:= \gamma_{21}(n) + \gamma_{22}(n) + \gamma_{23}(n) + \gamma_{24}(n) + \gamma_{25}(n). \end{aligned}$$

Clearly,

$$\lim_{n\to\infty} n^2\gamma_{21}(n) = \lim_{n\to\infty} n^2 f(x)\left[\frac{2n^2}{a^2}\left(-1+\cosh\left(\frac{a}{n}\right)\right)-1\right]$$
$$= \frac{a^2}{12} f(x).$$

For $\gamma_{22}(n)$, we have

$$\gamma_{22}(n) = a^{-1}e^{-ax}f'(x)\left[T_n^{\circ}(\exp_a;x) - e^{ax}T_n^{\circ}(\mathfrak{e}_0;x)\right]$$
$$= a^{-1}e^{-ax}f'(x)\left[e^{ax} - e^{ax}\frac{2n^2}{a^2}\left(-1+\cosh\left(\frac{a}{n}\right)\right)\right].$$

Therefore, $\lim_{n\to\infty} n^2\gamma_{22}(n) = -\frac{a}{12}f'(x)$. Similar considerations yield: $\lim_{n\to\infty} n^2\gamma_{23}(n) = \frac{1}{24}f''(x)$ and $\lim_{n\to\infty} n^2\gamma_{24}(n) = -\frac{a}{24}f'(x)$.

Using Cauchy-Schwarz inequality, we observe that

$$\left|n^2\gamma_{25}(n)\right| \leqslant \left(T_n^{\circ}(h_x^2(u);x)\right)^{1/2}\left(n^4 n\int_0^{\frac{1}{n}} e^{-a\left(\lambda_n^{\circ}(x)+t\right)}e^{ax}\left(e^{a(\lambda_n^{\circ}(x)+t)}-e^{ax}\right)^4 dt\right)^{1/2}.$$

Proceeding as in the previous proof, we have

$$\lim_{n\to\infty} n^4 n\int_0^{\frac{1}{n}} e^{-a\left(\lambda_n^{\circ}(x)+t\right)}e^{ax}\left(e^{a(\lambda_n^{\circ}(x)+t)}-e^{ax}\right)^4 dt = \frac{1}{80}a^4e^{4ax}.$$

We infer from Theorem 4 that $\lim_{n\to\infty} T_n^{\circ}(h_x^2(u);x) = h_x^2(0) = 0$. Hence

$$\lim_{n\to\infty} n^2\gamma_{25}(n) = 0.$$

This completes the proof. □

6. Global smoothness preservation

Now, we will establish some estimates concerning global smoothness preservation properties of $(W_n^{\circ})_{n\in\mathbb{N}}$ and $(T_n^{\circ})_{n\in\mathbb{N}}$ using modulus of continuity given in (4.1). For some related information, we refer the reader to [6, 7].

THEOREM 7. *Let $f\in C_{\rho_1}(\mathbb{R})$. If $\widetilde{\omega}(f;\delta)<\infty$ for $\delta>0$ and $x\in\mathbb{R}$, then there holds*

$$\widetilde{\omega}(W_n^{\circ}(f;.);\delta) \leqslant e^{\frac{a^2}{2n}}\widetilde{\omega}(f;\delta).$$

Proof. Under the hypotheses of the theorem, for $h>0,$ we have

$$\begin{aligned}
& e^{-a|x|}\left(W_n^{\circ}(f;x+h)-W_n^{\circ}(f;x)\right) \\
&= e^{-a|x|}\frac{\sqrt{n}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-a\left(\beta_n^{\circ}(x+h)+t\right)}e^{a(x+h)}f\left(\beta_n^{\circ}(x+h)+t\right)e^{-nt^2}dt \\
&\quad -e^{-a|x|}\frac{\sqrt{n}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-a\left(\beta_n^{\circ}(x)+t\right)}e^{ax}f\left(\beta_n^{\circ}(x)+t\right)e^{-nt^2}dt \\
&= e^{-a|x|}\frac{\sqrt{n}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-a\left(x+h-\frac{a}{4n}+t\right)}e^{a(x+h)}f\left(\beta_n^{\circ}(x+h)+t\right)e^{-nt^2}dt \\
&\quad -e^{-a|x|}\frac{\sqrt{n}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-a\left(x-\frac{a}{4n}+t\right)}e^{ax}f\left(\beta_n^{\circ}(x)+t\right)e^{-nt^2}dt.
\end{aligned}$$

Further, the following inequality holds:

$$\begin{aligned}
& e^{-a|x|}\left|W_n^{\circ}(f;x+h)-W_n^{\circ}(f;x)\right| \\
&\leqslant e^{-a|x|}\frac{\sqrt{n}}{\sqrt{\pi}}\int_{-\infty}^{\infty}\left|f\left(\beta_n^{\circ}(x+h)+t\right)-f\left(\beta_n^{\circ}(x)+t\right)\right|e^{-at}e^{\frac{a^2}{4n}}e^{-nt^2}dt \\
&\leqslant \widetilde{\omega}(f;h)\frac{\sqrt{n}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-at}e^{\frac{a^2}{4n}}e^{-nt^2}dt.
\end{aligned}$$

Thus, we have the result, that is,

$$\widetilde{\omega}\left(W_n^{\circ}(f;.);h\right)\leqslant \widetilde{\omega}(f;h)\,e^{\frac{a^2}{2n}}. \quad \square$$

THEOREM 8. *Let $f\in C_{\rho_1}(\mathbb{R})$. If $\widetilde{\omega}(f;\delta)<\infty$ for $\delta>0$ and $x\in\mathbb{R}$, then there holds*

$$\widetilde{\omega}\left(T_n^{\circ}(f;.);\delta\right)\leqslant \frac{2n^2}{a^2}\left|-1+\cosh\left(\frac{a}{n}\right)\right|\widetilde{\omega}(f;\delta).$$

7. Comparison result for Gauss-Weierstrass operators

Following [9], we prove following comparison result.

THEOREM 9. *Let $f\in C_{\rho_1}^2(\mathbb{R})$. Suppose that there exists $n_0\in\mathbb{N}$ and $a>0$ for which there holds*

$$f(x)\leqslant W_n^{\circ}(f;x)\leqslant W_n(f;x)$$

for all $n\geqslant n_0$ and $x\in\mathbb{R}$. Then

$$0\leqslant 3af'(x)-2a^2f(x)\leqslant f''(x),\quad x\in\mathbb{R}. \tag{7.1}$$

Conversely, if inequality (7.1) *holds as a strict inequality at a point* $x \in \mathbb{R}$, *then there exists* $n_1 \in \mathbb{N}$ *for which there holds*

$$f(x) < W_n^{\circ}(f;x) < W_n(f;x)$$

for all $n \geqslant n_1$ *and* $x \in \mathbb{R}$.

Proof. Since

$$f(x) \leqslant W_n^{\circ}(f;x) \leqslant W_n(f;x)$$

for all $n \geqslant n_0$ and $x \in \mathbb{R}$, using $\lim_{n\to\infty} n\left[W_n(f;x) - f(x)\right] = \frac{f''(x)}{4}$ (see [24]), we have

$$0 \leqslant n\left[W_n^{\circ}(f;x) - f(x)\right] \leqslant n\left[W_n(f;x) - f(x)\right],$$

and by (5.1) as n tends to infinity there holds:

$$0 \leqslant \frac{f''(x)}{4} + \frac{a^2 f(x)}{2} - \frac{3af'(x)}{4} \leqslant \frac{f''(x)}{4}.$$

Thus, the result easily follows.

Conversely, if inequality (7.1) is strict at a point $x \in \mathbb{R}$, we have

$$0 < \frac{f''(x)}{4} + \frac{a^2 f(x)}{2} - \frac{3af'(x)}{4} < \frac{f''(x)}{4}$$

If we reverse the process, we obtain the required result. Thus, the proof is completed. □

8. Graphical examples

Modified Gauss-Weierstrass operators: In Figure (1-A)-Figure (2-B), dotted, dashed and colored curves belong to $W_n(f;.)$, $W_n^{\circ}(f;.)$ and original function f, respectively. We can understand from the graphs how the approximation changes depending on the choice of parameters in the related subtitles.

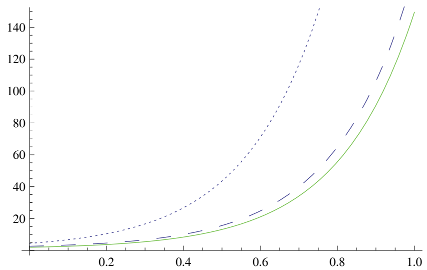

Figure 1-A: $f(x) = 1 + e^{5x}$, $a = 2$, $n = 5$

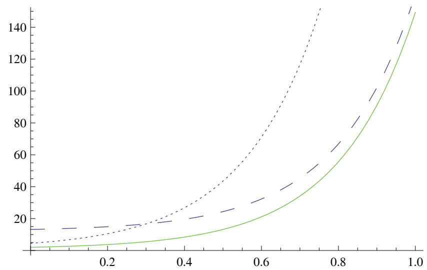

Figure 1-B: $f(x) = 1 + e^{5x}$, $a = 5$, $n = 5$

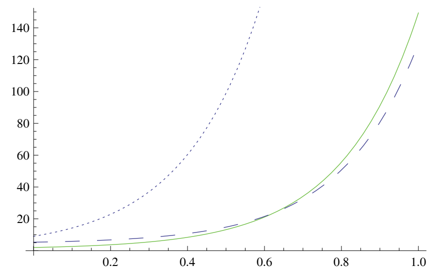

Figure 2-A: $f(x) = 1 + e^{5x}$, $a = 3$, $n = 5$

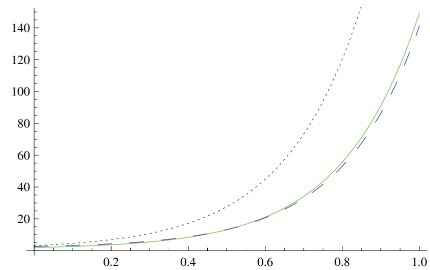

Figure 2-B: $f(x) = 1 + e^{5x}$, $a = 3$, $n = 8$

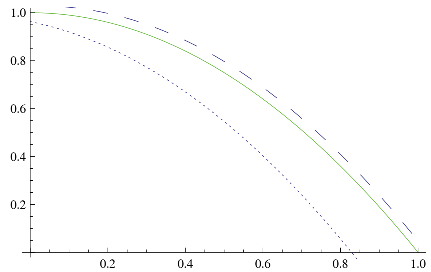

Figure 3-A: $f(x) = 1 - x^2$, $a = 2$, $n = 3$

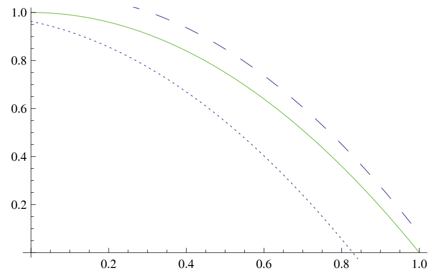

Figure 3-B: $f(x) = 1 - x^2$, $a = 3$, $n = 3$

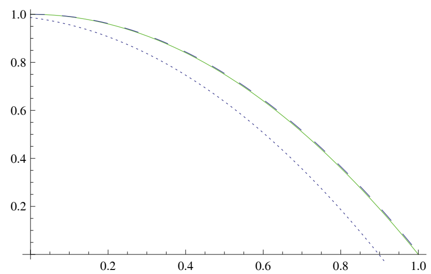

Figure 4-A: $f(x) = 1 - x^2$, $a = 1$, $n = 5$

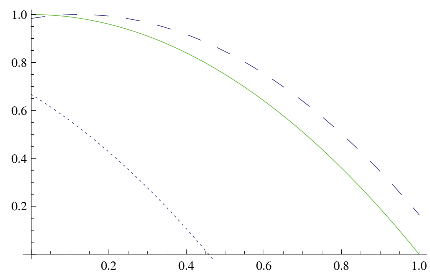

Figure 4-B: $f(x) = 1 - x^2$, $a = 1$, $n = 1$

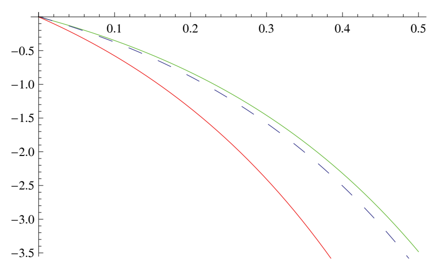

Figure 5-A: $f(x) = 1 - e^{3x}$, $a = 1$, $n = 1$

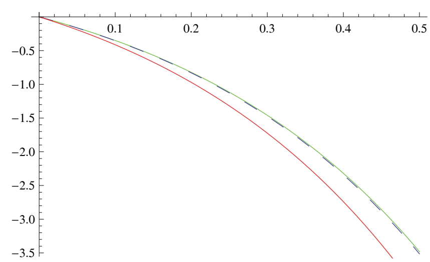

Figure 5-B: $f(x) = 1 - e^{3x}$, $a = 1$, $n = 3$

Modified Moment-type operators: In Figure (3-A)-Figure (4-B), dotted, dashed and colored curves belong to $T_n(f;.)$, $T_n^{\circ}(f;.)$ and original function f, respectively.

$T_n^{\circ}(f;x)$ $Vs.$ $W_n^{\circ}(f;x)$: In Figure (5-A)-Figure (5-B), colored (red), dashed and colored (green) curves belong to $W_n^{\circ}(f;.)$, $T_n^{\circ}(f;.)$ and original function f, respectively.

9. Final comments

The exponential approximation is a rising trend of recent years. In this context, we carried out this study on an operator sequence that is relatively easy to calculate with well-known Gauss-Weierstrass operators. In order to overcome tough calculations, especially in Lemma 2, and to sketch graphs, we used CAS MATHEMATICA. Indeed, we must state that the calculations for the modified Gauss-Weierstrass operators take quite long time, especially for polynomial functions. On the other hand, it takes relatively short time to calculate exponential functions for the same operators. It is also effective that the kernel of the operator sequence is an exponential-type function.

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