Journal of Mathematical Inequalities
Volume 15, Number 3 (2021), 1155–1171 doi:10.7153/jmi-2021-15-77

A CHAIN OF NUMERICAL RADIUS INEQUALITIES IN COMPLEX HILBERT SPACE

MOHAMMED AL-DOLAT*, AYAT DAGHER AND MARWAN ALQURAN

(*Communicated by M. Krnić*)

Abstract. In this paper, we implement the improvement of numerical radius inequalities that were produced by Alomari MW. [Refinements of some numerical radius inequalities for Hilbert space operators. Linear and Multilinear Algebra. 2019 Jun 4:1-6] and devise a new upper bound for 2×2 operator matrices on complex Hilbert space with many examples which show that our bound is sharper than the existing bounds proved by Bani-Domi W, Kittaneh F. [Norm and numerical radius inequalities for Hilbert space operators. Linear and Multilinear Algebra. 2020 Jul 28:1-2], Al-Dolat M, Jaradat I, Al-Husban B. A novel numerical radius upper bounds for 2×2 operator matrices. Linear and Multilinear Algebra. 2020 Apr 23:1-2], Shebrawi K. [Numerical radius inequalities for certain 2×2 operator matrices II. Linear Algebra and its Applications. 2017 Jun 15; 523:1-2] and Hirzallah O, Kittaneh F, Shebrawi K. [Numerical radius inequalities for 2×2 operator matrices. Studia Mathematica. 2012; 210:99-115].

1. Introduction

Let $(H,\langle .,.\rangle)$ be a complex Hilbert space and let $B(H)$ be the Banach algebra of all bounded linear operators from H to H with identity I. For $T\in B(H)$, let

$$w(T)=\sup_{||x||=1}|\langle Tx,x\rangle|,$$

$$r(T)=\sup\{|\lambda|:\lambda\in\sigma(T)\},$$

$$||T||=\sup_{||x||=1}\langle Tx,Tx\rangle^{\frac{1}{2}},$$

denote the numerical radius, the spectral radius and the usual operator norm respectively.

It is known that the numerical radius and the usual operator norm are equivalent norms on $B(H)$ such that

$$\frac{1}{2}||T||\leqslant w(T)\leqslant ||T||, \tag{1.1}$$

for all $T\in B(H)$.

Mathematics subject classification (2020): 47A12, 47A30, 47A63, 47B33.
Keywords and phrases: Numerical radius, spectral radius, operator matrix.
* Corresponding author.

Paper JMI-15-77

In [2], Kittaneh provided a refinement to the upper bound of the inequality (1.1) by showing that

$$w(T) \leqslant \frac{1}{2}\||T|+|T^*|\| \leqslant \frac{1}{2}\left(||T||+||T^2||^{1/2}\right), \tag{1.2}$$

for all $T \in B(H)$.

Another improvement for the inequality (1.1) was given by the same author as follows:

$$\frac{1}{4}\|T^*T+TT^*\| \leqslant w^2(T) \leqslant \frac{1}{2}\|T^*T+TT^*\|, \tag{1.3}$$

for every $T \in B(H)$.

Precisely, the Numerical radius is not submultiplicative that is $w(AB) \leqslant w(A)w(B)$ for all $A,B \in B(H)$ is not true in general, so many authors are interested to estimate lower and upper bounds for $w(AB)$ where $A,B \in B(H)$. For example it is known that

$$w(AB) \leqslant 4w(A)w(B);$$

and if A,B commute, then

$$w(AB) \leqslant 2w(A)w(B);$$

also, if A,B are normal, then

$$w(AB) \leqslant w(A)w(B).$$

Recently, in [3] the author gave a new upper bound for the numerical radius of product of operators, he proved that for $A,B \in B(H)$ such that $|A|B = B^*|A|$ and for nonnegative continuous functions f and g on $[0,\infty)$ satisfying $f(t)g(t) = t,\ (t \geqslant 0)$,

$$w(AB) \leqslant \frac{1}{2}r(B)\left\|f^2(|A|)+g^2(|A^*|)\right\|, \tag{1.4}$$

where $|A| = (A^*A)^{\frac{1}{2}}$ denotes the absolute value of A.
Also, he proved if $p \geqslant 1, \alpha \geqslant \beta > 1$, with $\frac{1}{\alpha}+\frac{1}{\beta} = 1$ and $\beta p \geqslant 2$, then

$$w^p(AB) \leqslant r^p(B)\left\|\frac{1}{\alpha}f^{\alpha p}(|A|)+\frac{1}{\beta}g^{\beta p}(|A^*|)\right\|, \tag{1.5}$$

and if $|A^2|B^2 = (B^2)^*|A^2|$, then

$$w^{2p}(AB) \leqslant \frac{1}{2}\left(||AB||^{2p}+r^p(B^2)\left\|\frac{1}{\alpha}f^{\alpha p}(|A^2|)+\frac{1}{\beta}g^{\beta p}(|(A^2)^*|)\right\|\right). \tag{1.6}$$

On the other hand, many authors are interested to estimate the numerical radius for the operator of matrices. In 2020, Al-Dolat, Jaradat and Al-Husban in [16] showed that if $A,B,C,D \in B(H)$, then

$$w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant \frac{1}{2}\left(w^2(A)+2w(D)+\sqrt{t^2w(A)+||B||^2}+\sqrt{(1-t)^2w^2(A)+||C||^2}\right),$$

for all $t \in [0,1]$.

In 2020, Bani-Domi and Kittaneh proved in [15] if $A,B,C,D \in B(H)$, then

$$w^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant \max\{w^2(A), w^2(D)\} + w^2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) + w\left(\begin{bmatrix} 0 & BD \\ CA & 0 \end{bmatrix}\right) + \frac{1}{2}\max\{\lambda, \mu\},$$

where $\lambda = \left\| |A|^2 + |B^*|^2 \right\|$ and $\mu = \left\| |D|^2 + |C^*|^2 \right\|$.

Let $a,b \geqslant 0$. Then we have

- The Power-Mean inequality:

$$a^{\alpha} b^{1-\alpha} \leqslant \alpha a + (1-\alpha) b \leqslant (\alpha a^p + (1-\alpha) b^p)^{\frac{1}{p}}, \tag{1.7}$$

for all $\alpha \in [0,1]$ and $p \geqslant 1$.

- Kittaneh and Manasrah [1] gave a refinement for (1.7) as follows:

$$a^{\alpha} b^{1-\alpha} \leqslant \alpha a + (1-\alpha) b - r_0(\sqrt{a} - \sqrt{b})^2, \tag{1.8}$$

for all $\alpha \in [0,1]$ where $r_0 = \min\{\alpha, 1-\alpha\}$.

- The authors in [4] presented a generalization for (1.8) as follows:

$$(a^{\alpha} b^{1-\alpha})^k \leqslant (\alpha a + (1-\alpha) b)^k - r_0^k (a^{\frac{k}{2}} - b^{\frac{k}{2}})^2, \tag{1.9}$$

for all $k \in \mathbb{N}$ and $\alpha \in [0,1]$ where $r_0 = \min\{\alpha, 1-\alpha\}$.

- Recently, Choi [5] improved the Power-Mean inequality as follows:

$$(a^{\alpha} b^{1-\alpha})^k \leqslant (\alpha a + (1-\alpha) b)^k - (2r_0)^k \left(\left(\frac{a+b}{2}\right)^k - (ab)^{\frac{k}{2}}\right), \tag{1.10}$$

for all $k \in \mathbb{N}$ and $\alpha \in [0,1]$ where $r_0 = \min\{\alpha, 1-\alpha\}$.

- The Power-Young inequality:

$$ab \leqslant \frac{a^{\alpha}}{\alpha} + \frac{b^{\beta}}{\beta} \leqslant \left(\frac{a^{p\alpha}}{\alpha} + \frac{b^{p\beta}}{\beta}\right)^{1/p}, \tag{1.11}$$

for all $\alpha, \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and $p \geqslant 1$.

In this paper, we present some generalizations and refinements for the numerical radius inequalities. Further, new upper bounds for the numerical radius of 2×2 operator matrices are given.

2. The main results

The aim of this section is to establish a generalizations and refinements for the numerical radius inequalities. To do this, we need the following sequence of lemmas. The first lemma is a result of the spectral Theorem together with Jensen´s inequality (see[7]).

LEMMA 2.1. *Let $T \in B(H)$ be a positive operator and let $x \in H$ be any vector. Then*

a. $\langle Tx,x\rangle^s \leqslant ||x||^{2s-2}\langle T^s x,x\rangle$ *for* $s \geqslant 1$;

b. $\langle T^s x,x\rangle \leqslant ||x||^{2-2s}\langle Tx,x\rangle^s$ *for* $0 < s \leqslant 1$.

The second lemma give an upper bound for the spectral radius which was obtained by Kittaneh [6].

LEMMA 2.2. *Let $A,B \in B(H)$. Then*

$$r(AB) \leqslant \frac{1}{4}\left(||AB||+||BA||+\sqrt{(||AB||-||BA||)^2+4\min\{||A||||BAB||,||B||||ABA||\}}\right).$$

In particular,

$$r(A) \leqslant \frac{1}{2}(||A||+||A^2||^{\frac{1}{2}}).$$

The next lemma is a consequence of the spectral Theorem [7].

LEMMA 2.3. *Let $A,B \in B(H)$ such that $|A|B = B^*|A|$. If f,g are nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t)=t$, where $t \geqslant 0$, then*

$$|\langle ABx,y\rangle| \leqslant r(B)\,\|f(|A|)x\|\,\|g(|A^*|)y\|,$$

for every vectors $x,y \in H$.

Our first main result is the following improvement of (1.4).

THEOREM 2.4. *Let $A_i,B_i \in B(H)$ $(i=1,2,\dots,n)$ such that $|A_i|B_i = B_i^*|A_i|$ and let f and g be nonnegative continuous functions on $[0,\infty)$ such that $f(t)g(t)=t$ for all $t\in[0,\infty)$. Then for every $k\in\mathbb{N}$ and $p,q\geqslant k$,*

$$w^p\left(\sum_{i=1}^n A_iB_i\right) \leqslant \frac{n^{p-k/q}}{2^{k/q}}\left(\max_{1\leqslant i\leqslant n} r^p(B_i)\right)\left\|\sum_{i=1}^n\left(f^{2pq/k}(|A_i|)+g^{2pq/k}(|A_i^*|)\right)\right\|^{k/q} - \inf_{||x||=1}\phi(x) \tag{2.1}$$

where $\phi(x)=\frac{n^{p-1}}{2^k}\left(\max_{1\leqslant i\leqslant n} r^p(B_i)\right)\sum_{i=1}^n\left(\left\langle f^{2p/k}(|A_i|)x,x\right\rangle^{k/2}-\left\langle g^{2p/k}(|A_i^*|)x,x\right\rangle^{k/2}\right)^2.$

Proof. Let $x \in H$ be any unit vector. Then

$$\begin{aligned}
&\left|\left\langle \sum_{i=1}^{n} A_i B_i x, x\right\rangle\right|^p \\
&\leqslant \left(\sum_{i=1}^{n} |\langle A_i B_i x, x\rangle|\right)^p \\
&\leqslant n^{p-1} \sum_{i=1}^{n} |\langle A_i B_i x, x\rangle|^p \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by convexity of } t^p\text{)} \\
&\leqslant n^{p-1} \sum_{i=1}^{n} \left(r(B_i)\, \|f(|A_i|)x\|\, \|g(|A_i^*|)x\|\right)^p \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 2.3)} \\
&\leqslant n^{p-1} \left(\max_{1\leqslant i\leqslant n} r^p(B_i)\right) \sum_{i=1}^{n} \left(\left\langle f^2(|A_i|)x, x\right\rangle^{p/2} \left\langle g^2(|A_i^*|)x, x\right\rangle^{p/2}\right) \\
&\leqslant n^{p-1} \left(\max_{1\leqslant i\leqslant n} r^p(B_i)\right) \sum_{i=1}^{n} \left(\left\langle f^{2p/k}(|A_i|)x, x\right\rangle^{1/2} \left\langle g^{2p/k}(|A_i^*|)x, x\right\rangle^{1/2}\right)^k \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 2.1)} \\
&\leqslant n^{p-1} \left(\max_{1\leqslant i\leqslant n} r^p(B_i)\right) \sum_{i=1}^{n} \left[\left(\frac{\left\langle \left\langle f^{2p/k}(|A_i|)x, x\right\rangle^q + \left\langle g^{2p/k}(|A_i^*|)x, x\right\rangle^q \right\rangle}{2}\right)^{k/q}\right. \\
&\quad \left. - \frac{1}{2^k} \left(\left\langle f^{2p/k}(|A_i|)x, x\right\rangle^{k/2} - \left\langle g^{2p/k}(|A_i^*|)x, x\right\rangle^{k/2}\right)^2\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by inequalities (1.9) and (1.7))} \\
&\leqslant n^{p-1} \left(\max_{1\leqslant i\leqslant n} r^p(B_i)\right) \sum_{i=1}^{n} \left[\left(\frac{\left\langle \left\langle f^{2pq/k}(|A_i|)x, x\right\rangle + \left\langle g^{2pq/k}(|A_i^*|)x, x\right\rangle \right\rangle}{2}\right)^{k/q}\right. \\
&\quad \left. - \frac{1}{2^k} \left(\left\langle f^{2p/k}(|A_i|)x, x\right\rangle^{k/2} - \left\langle g^{2p/k}(|A_i^*|)x, x\right\rangle^{k/2}\right)^2\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 2.1)} \\
&\leqslant \frac{n^{p-k/q}}{2^{k/q}} \left(\max_{1\leqslant i\leqslant n} r^p(B_i)\right) \left\langle \left(\sum_{i=1}^{n} \left\langle f^{2pq/k}(|A_i|)x, x\right\rangle + \left\langle g^{2pq/k}(|A_i^*|)x, x\right\rangle\right) x, x\right\rangle^{k/q} \\
&\quad - \frac{n^{p-1}}{2^k} \left(\max_{1\leqslant i\leqslant n} r^p(B_i)\right) \sum_{i=1}^{n} \left(\left\langle f^{2p/k}(|A_i|)x, x\right\rangle^{k/2} - \left\langle g^{2p/k}(|A_i^*|)x, x\right\rangle^{k/2}\right)^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by concavity of } t^{\frac{k}{q}}\text{).}
\end{aligned}$$

Thus,

$$w^p\left(\sum_{i=1}^n A_iB_i\right) = \sup\left\{\left|\left\langle \sum_{i=1}^n A_iB_ix,x\right\rangle\right|^p : x\in H, ||x||=1\right\}$$
$$\leqslant \frac{n^{p-k/q}}{2^{k/q}}\left(\max_{1\leqslant i\leqslant n} r^p(B_i)\right)\left\|\sum_{i=1}^n \left(f^{2pq/k}(|A_i|)+g^{2pq/k}(|A_i^*|)\right)\right\|^{k/q}$$
$$-\inf_{||x||=1}\phi(x). \quad \square$$

Choosing $n=1$ in Theorem 2.4 then using Lemma 2.2 we obtain the following corollary.

COROLLARY 2.5. *Let $A,B\in B(H)$ such that $|A|B=B^*|A|$ and let f and g be nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t)=t$ for $t\geqslant 0$. Then for $k\in\mathbb{N}$ and $p,q\geqslant k$,*

$$w^p(AB)\leqslant \frac{r^p(B)}{2^{k/q}}\left\|f^{2pq/k}(|A|)+g^{2pq/k}(|A^*|)\right\| - \inf_{||x||=1}\phi(x)$$
$$\leqslant \frac{1}{2^{p+k/q}}(||B||+||B^2||^{1/2})^p\left\|f^{2pq/k}(|A|)+g^{2pq/k}(|A^*|)\right\| - \inf_{||x||=1}\phi(x),$$

where $\phi(x)=\frac{r^p(B)}{2^k}\left(\left\langle f^{2p/k}(|A|)x,x\right\rangle^{k/2}-\left\langle g^{2p/k}(|A^*|)x,x\right\rangle^{k/2}\right)^2$.

The next result follows from Corollary 2.5 by setting $p=q=k=1$ and $f(t)=t^\alpha$, $g(t)=t^{1-\alpha}$ for $\alpha\in[0,1]$.

COROLLARY 2.6. *Let $A,B\in B(H)$ such that $|A|B=B^*|A|$. Then for $\alpha\in[0,1]$,*

$$w(AB)\leqslant \frac{r(B)}{2}\left\||A|^{2\alpha}+|A^*|^{2(1-\alpha)}\right\| - \inf_{||x||=1}\phi(x)$$
$$\leqslant \frac{1}{4}(||B||+||B^2||^{1/2})\left\||A|^{2\alpha}+|A^*|^{2(1-\alpha)}\right\| - \inf_{||x||=1}\phi(x),$$

where $\phi(x)=\frac{r(B)}{2}\left(\left\langle |A|^{2\alpha}x,x\right\rangle^{1/2}-\left\langle |A^*|^{2(1-\alpha)}x,x\right\rangle^{1/2}\right)^2$.

The next result is a simple form follows from Corollary 2.6 by letting $\alpha=1/2$.

COROLLARY 2.7. *Let $A,B\in B(H)$ such that $|A|B=B^*|A|$. Then*

$$w(AB)\leqslant \frac{r(B)}{2}|||A|+|A^*||| - \inf_{||x||=1}\phi(x)$$
$$\leqslant \frac{1}{4}(||B||+||B^2||^{1/2})\,|||A|+|A^*||| - \inf_{||x||=1}\phi(x),$$

where $\phi(x)=\frac{r(B)}{2}\left(\langle |A|x,x\rangle^{1/2}-\langle |A^*|x,x\rangle^{1/2}\right)^2$.

The next lemma is a result of Shebrawi [8] that will be used in the proof of Corollary 2.9.

LEMMA 2.8. *Let $A,B \in B(H)$ and let $t \in [0,1]$. Then*

$$\|A+B\| \leqslant \max(||A||,||B||) + \frac{1}{2}\left(\left\||A|^t|B|^{1-t}\right\| + \left\||A^*|^t|B^*|^{1-t}\right\|\right).$$

Using Corollary 2.7, Lemma 2.8 with $t = \frac{1}{2}$ and the fact $\left\||A|^{1/2}|B|^{1/2}\right\| \leqslant \|AB^*\|^{1/2}$ we obtain the following corollary.

COROLLARY 2.9. *Let $A,B \in B(H)$ such that $|A|B = B^*|A|$. Then*

$$w(AB) \leqslant \frac{1}{4}\left(||B|| + ||B^2||^{1/2}\right)\left(||A|| + ||A^2||^{1/2}\right) - \inf_{||x||=1} \phi(x),$$

where $\phi(x) = \frac{r(B)}{2}\left(\langle |A|x,x\rangle^{1/2} - \langle |A^*|x,x\rangle^{1/2}\right)^2$.

The following theorem gives a generalization for (1.5) which can be stated as follows.

THEOREM 2.10. *Let $A_i,B_i \in B(H)$ $(i = 1,2,\ldots,n)$ such that $|A_i|B_i = B_i^*|A_i|$ and let f, g be nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t$, $(t \geqslant 0)$. Then for $\alpha \geqslant \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $s \geqslant 1$ and $p \geqslant \max\{1, \frac{2}{s\beta}\}$,*

$$\begin{aligned} w^p\left(\sum_{i=1}^{n} A_iB_i\right) &\leqslant n^{p-1/s}\left(\max_{1\leqslant i\leqslant n} r^p(B_i)\right)\left\|\sum_{i=1}^{n}\left(\frac{1}{\alpha}f^{ps\alpha}(|A_i|) + \frac{1}{\beta}g^{ps\beta}(|A_i^*|)\right)\right\|^{1/s} \\ &\leqslant \frac{n^{p-1/s}}{2^p}\sqrt[s]{\max\left\{\frac{1}{\alpha},\frac{1}{\beta}\right\}}\left(\max_{1\leqslant i\leqslant n}(||B_i|| + ||B_i^2||)^p\right) \\ &\quad\times\left\|\sum_{i=1}^{n}\left(f^{ps\alpha}(|A_i|) + g^{ps\beta}(|A_i^*|)\right)\right\|^{1/s}. \end{aligned}$$

Proof. Let $x \in H$ be any unit vector. Then, we have

$$\begin{aligned} &\left|\left\langle \sum_{i=1}^{n} A_iB_ix,x\right\rangle\right|^p \\ &\leqslant n^{p-1}\left(\max_{1\leqslant i\leqslant n} r^p(B_i)\right)\sum_{i=1}^{n}\left(\left\langle f^2(|A_i|)x,x\right\rangle^{ps/2}\left\langle g^2(|A_i^*|)x,x\right\rangle^{ps/2}\right)^{1/s} \\ &\leqslant n^{p-1}\left(\max_{1\leqslant i\leqslant n} r^p(B_i)\right)\sum_{i=1}^{n}\left(\frac{1}{\alpha}\left\langle f^2(|A_i|)x,x\right\rangle^{ps\alpha/2} + \frac{1}{\beta}\left\langle g^2(|A_i^*|)x,x\right\rangle^{ps\beta/2}\right)^{1/s} \end{aligned}$$

(by inequality 1.11)

$$\leqslant n^{p-1}\left(\max_{1\leqslant i\leqslant n} r^p(B_i)\right)\sum_{i=1}^{n}\left(\frac{1}{\alpha}\langle f^{ps\alpha}(|A_i|)x,x\rangle+\frac{1}{\beta}\left\langle g^{ps\beta}(|A_i^*|)x,x\right\rangle\right)^{1/s}$$

(by Lemma 2.1)

$$=n^{p-1}\left(\max_{1\leqslant i\leqslant n} r^p(B_i)\right)\sum_{i=1}^{n}\left\langle\left(\frac{1}{\alpha}f^{ps\alpha}(|A_i|)+\frac{1}{\beta}g^{ps\beta}(|A_i^*|)\right)x,x\right\rangle^{1/s}$$

$$\leqslant n^{p-1/s}\left(\max_{1\leqslant i\leqslant n} r^p(B_i)\right)\left\langle\left(\sum_{i=1}^{n}\frac{1}{\alpha}f^{ps\alpha}(|A_i|)+\frac{1}{\beta}g^{ps\beta}(|A_i^*|)\right)x,x\right\rangle^{1/s}$$

(by concavity of $t^{1/s}$).

Now, the first bound of Theorem 2.10 is obtained by taking the supremum over all unit vectors $x\in H$. We obtain the second bound by applying Lemma 2.2 on the first inequality. □

As a direct consequence of Theorem 2.10 we get the following result which can be considered as a generalization for the first bound of the inequality (1.2)

COROLLARY 2.11. *Let $A\in B(H)$. Then for all $p,s\geqslant 1$,*

$$w^p(A)\leqslant\left\|\frac{|A|^{ps}+|A^*|^{ps}}{2}\right\|^{1/s}.$$

Setting $p=s=1$ in Corollary 2.11 we get the first bound in the inequality (1.2). On the other hand the next result is obtained by letting $n=1$ in Theorem 2.10.

COROLLARY 2.12. *Let $A,B\in B(H)$ such that $|A|B=B^*|A|$. If f,g are nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t)=t$ $(t\geqslant 0)$. Then*

$$w^p(AB)\leqslant r^p(B)\left\|\frac{1}{\alpha}f^{ps\alpha}(|A|)+\frac{1}{\beta}g^{ps\beta}(|A^*|)\right\|^{\frac{1}{s}},$$

where $\alpha\geqslant\beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$, $s\geqslant 1$ and $p\geqslant\max\{1,\frac{2}{s\beta}\}$.

A general refinement for the second bound of (1.3) will be given in the following theorem.

THEOREM 2.13. *Let $A_i,B_i\in B(H)$ $(i=1,2,\dots,n)$ such that $|A_i|B_i=B_i^*|A_i|$ and let f, g be nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t)=t$, $(t\geqslant 0)$. Then for $\alpha\in[0,1]$, $q\geqslant 1$, $k\in\mathbb{N}$ and $p\geqslant k$,*

$$w^p\left(\sum_{i=1}^{n}A_iB_i\right)\leqslant n^{p-1}\left(\max_{1\leqslant i\leqslant n} r^p(B_i)\right)\sum_{i=1}^{n}\left(\left\|\alpha f^{2pq/\alpha k}(|A_i|)+(1-\alpha)g^{2pq/(1-\alpha)k}(|A_i^*|)\right\|^{\frac{k}{q}}-\inf_{||x||=1}\phi(x)\right)^{1/2},$$

where

$$\phi(x) = (2\min\{\alpha, 1-\alpha\})^k \times \left(\left\langle \frac{f^{2p/\alpha k}(|A_i|) + g^{2p/(1-\alpha)k}(|A_i^*|)}{2} x, x \right\rangle^k - \left(\left\langle f^{2p/\alpha k}(|A_i|)x, x \right\rangle \left\langle g^{2p/(1-\alpha)k}(|A_i^*|)x, x \right\rangle \right)^{k/2} \right).$$

Proof. Let $x \in H$ be a unit vector. Then, we have

$$\begin{aligned}
&\left| \left\langle \sum_{i=1}^n A_i B_i x, x \right\rangle \right|^p \\
&\leqslant n^{p-1} \left(\max_{1\leqslant i\leqslant n} r^p(B_i) \right) \sum_{i=1}^n \left\langle f^2(|A_i|)x, x \right\rangle^{p/2} \left\langle g^2(|A_i^*|)x, x \right\rangle^{p/2} \\
&\leqslant n^{p-1} \left(\max_{1\leqslant i\leqslant n} r^p(B_i) \right) \sum_{i=1}^n \left[\left(\left\langle f^{2/\alpha}(|A_i|)x, x \right\rangle^{p\alpha/k} \left\langle g^{2/(1-\alpha)}(|A_i^*|)x, x \right\rangle^{p(1-\alpha)/k} \right)^k \right]^{1/2} \\
&\qquad \text{(by Lemma 2.1)} \\
&\leqslant n^{p-1} \left(\max_{1\leqslant i\leqslant n} r^p(B_i) \right) \sum_{i=1}^n \left[\left(\left\langle f^{2p/\alpha k}(|A_i|)x, x \right\rangle^{\alpha} \left\langle g^{2p/(1-\alpha)k}(|A_i^*|)x, x \right\rangle^{1-\alpha} \right)^k \right]^{1/2} \\
&\qquad \text{(by Lemma 2.1)} \\
&\leqslant n^{p-1} \left(\max_{1\leqslant i\leqslant n} r^p(B_i) \right) \sum_{i=1}^n \left[\left(\alpha \left\langle f^{2p/\alpha k}(|A_i|)x, x \right\rangle^q + (1-\alpha) \left\langle g^{2p/(1-\alpha)k}(|A_i^*|)x, x \right\rangle^q \right)^{k/q} \right. \\
&\quad - (2\min\{\alpha, 1-\alpha\})^k \left(\left\langle \frac{f^{2p/\alpha k}(|A_i|) + g^{2p/(1-\alpha)k}(|A_i^*|)}{2} x, x \right\rangle^k \right. \\
&\quad \left. \left. - \left(\left\langle f^{2p/\alpha k}(|A_i|)x, x \right\rangle \left\langle g^{2p/(1-\alpha)k}(|A_i^*|)x, x \right\rangle \right)^{k/2} \right) \right]^{1/2} \quad \text{(by (1.10) and (1.7))} \\
&\leqslant n^{p-1} \left(\max_{1\leqslant i\leqslant n} r^p(B_i) \right) \sum_{i=1}^n \left[\left\langle \left(\alpha f^{2pq/\alpha k}(|A_i|) + (1-\alpha) g^{2pq/(1-\alpha)k}(|A_i^*|) \right) x, x \right\rangle^{k/q} \right. \\
&\quad - (2\min\{\alpha, 1-\alpha\})^k \left(\left\langle \frac{f^{2p/\alpha k}(|A_i|) + g^{2p/(1-\alpha)k}(|A_i^*|)}{2} x, x \right\rangle^k \right. \\
&\quad \left. \left. - \left(\left\langle f^{2p/\alpha k}(|A_i|)x, x \right\rangle \left\langle g^{2p/(1-\alpha)k}(|A_i^*|)x, x \right\rangle \right)^{k/2} \right) \right]^{1/2} \quad \text{(by Lemma 2.1)}.
\end{aligned}$$

The desired bound is obtained By taking the supremum over all unit vectors $x \in H$. □

Choosing $\alpha = \frac{1}{2}$, $n = p = q = k = 1$, $B = I$ and $f(t) = g(t) = t^{1/2}$ in Theorem 2.13 we get the following corollary.

COROLLARY 2.14. *Let $A \in B(H)$. Then*

$$w^2(A) \leqslant \frac{1}{2}\left\| |A|^2 + |A^*|^2 \right\| - \inf_{\|x\|=1} \phi(x),$$

where

$$\phi(x) = \left\langle \frac{|A|^2 + |A^*|^2}{2} x, x \right\rangle - \left(\left\langle |A|^2 x, x \right\rangle \left\langle |A^*|^2 x, x \right\rangle \right)^{1/2}.$$

The next lemma is a result of Dragomir [9] that will be used in the proof of Theorem 2.16.

LEMMA 2.15. *Let $x, y, e \in H$ such that $\|x\| = 1$. Then*

$$|\langle x, e\rangle \langle e, y\rangle| \leqslant \frac{1}{2}\left(|\langle x, y\rangle| + \|x\|\|y\| \right).$$

Let $A, B \in B(H)$ and let $u \in H$ be unit vector. Then for $e = u$, $x = ABu$ and $y = B^*A^*u$ in Lemma 2.15 we have

$$|\langle ABu, u\rangle|^2 \leqslant \frac{1}{2}\left(\left| \left\langle (AB)^2 u, u \right\rangle \right| + \|ABu\|\|B^*A^*u\| \right). \tag{2.2}$$

The next result provides a generalization for the inequality (1.6).

THEOREM 2.16. *Let $A_i, B_i \in B(H)$ $(i = 1, 2, \ldots, n)$ such that $A_iB_i = B_iA_i$ and $|A_i^2|B_i^2 = (B_i^2)^*|A_i^2|$ and let f, g be nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$, $(t \geqslant 0)$. Then for $\alpha \geqslant \beta > 1$ with $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $s \geqslant 1$ and $p \geqslant$* $\max\{1, \frac{2}{s\beta}\}$,

$$\begin{aligned} w^{2p}\left(\sum_{i=1}^{n} A_iB_i \right) &\leqslant \frac{n^{2p-1}}{2} \sum_{i=1}^{n} \|A_iB_i\|^{2p} + \frac{n^{2p-1/s}}{2} \left(\max_{1\leqslant i\leqslant n} r^p(B_i^2) \right) \\ &\quad \times \left\| \sum_{i=1}^{n} \left(\frac{1}{\alpha} f^{ps\alpha}(|A_i^2|) + \frac{1}{\beta} g^{ps\beta}(|(A_i^2)^*|) \right) \right\|^{1/s} \\ &\leqslant \frac{n^{2p-1}}{2} \left(\sum_{i=1}^{n} \|A_iB_i\| + \frac{\left(\|B_i^2\| + \|B_i^4\|^{1/2} \right)}{n^{-1+1/s}} \right. \\ &\quad \left. \times \left\| \sum_{i=1}^{n} \left(\frac{1}{\alpha} f^{ps\alpha}(|A_i^2|) + \frac{1}{\beta} g^{ps\beta}(|(A_i^2)^*|) \right) \right\|^{1/s} \right). \end{aligned}$$

Proof. For any unit vector $x \in H$ we have

$$\left| \left\langle \sum_{i=1}^{n} A_iB_i x, x \right\rangle \right|^{2p} \leqslant n^{2p-1} \sum_{i=1}^{n} |\langle A_iB_i x, x\rangle|^{2p} \quad \text{(by convexity of } t^{2p}\text{)}$$

$$\leqslant n^{2p-1} \sum_{i=1}^{n} \left(\frac{\left| \left\langle (A_iB_i)^2 x, x \right\rangle \right| + \|A_iB_i x\|\|B_i^*A_i^*\|}{2} \right)^p \quad \text{(by (2.2))}$$

$$\leqslant n^{2p-1}\sum_{i=1}^{n}\left(\frac{\left|\left\langle (A_iB_i)^2x,x\right\rangle\right|^p+||A_iB_ix||^p||B_i^*A_i^*||^p}{2}\right) \quad \text{(by (1.7))}$$

$$\leqslant \frac{n^{2p-1}}{2}\left(\sum_{i=1}^{n}||A_iB_i||^{2p}+\left(\max_{1\leqslant i\leqslant n} r^p(B_i^2)\right)\sum_{i=1}^{n}\left\langle f^2(|A_i^2|)x,x\right\rangle^{p/2}\left\langle g^2(|(A_i^2)^*|)x,x\right\rangle^{p/2}\right)$$

$$\text{(by Lemma 2.3))}$$

$$\leqslant \frac{n^{2p-1}}{2}\left(\sum_{i=1}^{n}||A_iB_i||^{2p}+\frac{(\max_{1\leqslant i\leqslant n} r^p(B_i^2))}{n^{-1+1/s}}\left\langle\left(\sum_{i=1}^{n}\frac{1}{\alpha}f^{ps\alpha}(|A_i^2|)+\frac{1}{\beta}g^{ps\beta}(|(A_i^2)^*|)\right)x,x\right\rangle^{1/s}\right).$$

The last inequality above is obtained by follows the same steps of Theorem 2.10. The proof is finish by taking the supremum over all unit vectors $x\in H$. □

The next result follows from Theorem 2.16 by letting $n=1$ and $B=I$.

COROLLARY 2.17. *Let $A\in B(H)$, and let f, g be nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t)=t$, $(t\geqslant 0)$. Then for $\alpha\geqslant\beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$ and $p\geqslant\max\{\frac{2}{s\beta},1\}$,*

$$w^{2p}(A)\leqslant\frac{1}{2}\left(||A||^{2p}+\left\|\frac{1}{\alpha}f^{ps\alpha}(|A^2|)+\frac{1}{\beta}g^{ps\beta}(|(A^2)^*|)\right\|^{1/s}\right).$$

The final result in this section is the following refinement of [[10], Theorem 3.3].

THEOREM 2.18. *Let $A_i,B_i,X_i\in B(H)$, $(i=1,2,\dots,n)$ such that A_i,B_i positive for each $i=1,2,\dots,n$. Then for $\alpha\in[0,1]$, $k\in\mathbb{N}$, $q\geqslant k$ and $p\geqslant 2k$,*

$$w^p\left(\sum_{i=1}^{n}A_i^{\alpha}X_iB_i^{1-\alpha}\right)\leqslant\left(\max_{1\leqslant i\leqslant n}||X_i||^p\right)\min\{\lambda,\mu\},$$

where

$$\lambda=n^{p-q/k}\left\|\sum_{i=1}^{n}\left(\alpha A_i^{pq/k}+(1-\alpha)B_i^{pq/k}\right)\right\|^{k/q}-\inf_{||x||=1}\phi(x),$$

and

$$\mu=n^{p-q/k}\left\|\sum_{i=1}^{n}\left(\alpha A_i^{pq/k}+(1-\alpha)B_i^{pq/k}\right)\right\|^{k/q}-\inf_{||x||=1}\varphi(x),$$

where

$$\phi(x)=(2r_0)^kn^{p-1}\sum_{i=1}^{n}\left(\left\langle\frac{A_i^{p/k}+B_i^{p/k}}{2}x,x\right\rangle^k-\left(\left\langle A_i^{p/k}x,x\right\rangle\left\langle B_i^{p/k}x,x\right\rangle\right)^{k/2}\right),$$

and

$$\varphi(x) = n^{p-1} r_0^k \sum_{i=1}^{n} \left(\left\langle A_i^{p/k} x, x \right\rangle^{k/2} - \left\langle B_i^{p/k} x, x \right\rangle^{k/2} \right)^2,$$

where $r_0 = \min\{\alpha, 1-\alpha\}$.

Proof. Let $x \in H$ be any unit vector. Then by the Cauchy-Schwartz inequality, Lemma 2.3, Lemma 2.1 and the inequality (1.10) we have

$$\begin{aligned}
&\left| \left\langle \sum_{i=1}^{n} A_i^{\alpha} X_i B_i^{1-\alpha} x, x \right\rangle \right|^p \\
&\leqslant n^{p-1} \sum_{i=1}^{n} \left| \left\langle A_i^{\alpha} X_i B_i^{1-\alpha} x, x \right\rangle \right|^p \\
&= n^{p-1} \sum_{i=1}^{n} \left| \left\langle X_i B_i^{1-\alpha} x, A_i^{\alpha} x \right\rangle \right|^p \\
&\leqslant n^{p-1} \sum_{i=1}^{n} (||X_i||^p ||B_i^{1-\alpha} x||^p ||A_i^{\alpha} x||^p) \\
&\leqslant n^{p-1} \left(\max_{1 \leqslant i \leqslant n} ||X_i||^p \right) \sum_{i=1}^{n} \left(\langle A_i^{2\alpha} x, x \rangle^{p/2k} \left\langle B_i^{2(1-\alpha)} x, x \right\rangle^{p/2k} \right)^k \\
&\leqslant n^{p-1} \left(\max_{1 \leqslant i \leqslant n} ||X_i||^p \right) \sum_{i=1}^{n} \left(\left\langle A_i^{p/k} x, x \right\rangle^{\alpha} \left\langle B_i^{p/k} x, x \right\rangle^{(1-\alpha)} \right)^k \\
&\leqslant \left(\max_{1 \leqslant i \leqslant n} ||X_i||^p \right) \left[n^{p-q/k} \left\langle \left(\sum_{i=1}^{n} \left(\alpha A_i^{pq/k} + (1-\alpha) B_i^{pq/k} \right) \right) x, x \right\rangle^{k/q} \right. \\
&\quad \left. - (2r_0)^k n^{p-1} \sum_{i=1}^{n} \left(\left\langle \frac{A_i^{p/k} + B_i^{p/k}}{2} x, x \right\rangle^k - \left(\left\langle A_i^{p/k} x, x \right\rangle \left\langle B_i^{p/k} x, x \right\rangle \right)^{k/2} \right) \right]
\end{aligned}$$

The last inequality above is obtained by follows the same technique of Theorem 2.4 together with the inequalities (1.10) and (1.7). Taking the supremum over all unit vectors $x \in H$, we get the first bound. Finally, by using (1.9), (1.7) and the same method that gave the first bound we get the second bound. □

COROLLARY 2.19. *Let* $A, B, X \in B(H)$ *such that* A *and* B *are positive. Then for* $\alpha \in [0,1]$, $k \in \mathbb{N}$ *and* $p \geqslant 2k$,

$$w^p(A^{\alpha} X B^{1-\alpha}) \leqslant ||X||^p \min\{\lambda, \mu\},$$

where

$$\lambda = \|\alpha A^p + (1-\alpha) B^p\| - \inf_{||x||=1} \phi(x)$$

and

$$\mu = \|\alpha A^p + (1-\alpha)B^p\| - \inf_{||x||=1} \varphi(x),$$

where

$$\phi(x) = (2r_0)^k \left(\left\langle \frac{A^{p/k} + B^{p/k}}{2} x, x \right\rangle - \left(\left\langle A^{p/k} x, x \right\rangle \left\langle B^{p/k} x, x \right\rangle \right)^{k/2} \right),$$

and

$$\varphi(x) = {r_0}^k \left(\left\langle A^{p/k} x, x \right\rangle^{k/2} - \left\langle B^{p/k} x, x \right\rangle^{k/2} \right)^2,$$

where $r_0 = \min\{\alpha, 1-\alpha\}$.

3. New upper bounds for 2×2 operator matrices

In this section we will give new upper bounds for 2×2 operator matrices. To do this we need some facts about the spectral radius and the numerical radius. The first fact gives some basic properties for the spectral radius.

LEMMA 3.1. *Let $A, B, C, D \in B(H)$. The following statements hold*

a. If $AB = BA$, then $r(A+B) \leqslant r(A) + r(B)$ and $r(AB) \leqslant r(A)r(B)$.

b. $r(A^n) = r^n(A)$ for all $n \in \mathbb{N}$.

c. $r\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant r\left(\begin{bmatrix} ||A|| & ||B|| \\ ||C|| & ||D|| \end{bmatrix}\right)$.

d. $r\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = \sqrt{r(BC)}$.

e. If A is normal, then $r(A) = w(A) = ||A||$.

The second fact gives a useful form for the numerical radius (see [11]).

LEMMA 3.2. *Let $T \in B(H)$. Then $w(T) = \max_{\theta \in \mathbb{R}} ||\mathrm{Re}(e^{i\theta} T)||$.*

Also, we need the following fact see [12] and [13] respectively.

LEMMA 3.3. *Let $A, B, C, D \in B(H)$. Then*

a. $w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \geqslant \max\left\{ w(A), w(B), w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right\}$.

b. $w\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = \frac{1}{2} \max_{\theta \in R} \left\| e^{i\theta} B + e^{-i\theta} C^* \right\|$.

Our first estimate can be stated as follows

THEOREM 3.4. *Let $A,B,C,D \in B(H)$. Then*

$$w^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant \frac{1}{4}\left(w(A)+\sqrt{w^2(A)+4w^2(E)}\right)^2 + \frac{1}{2}w^2(D)+\frac{1}{2}w(D)\sqrt{w^2(A)+4w^2(E)},$$

where $E=\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$.

Proof. Let $T=\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and let $X=e^{i\theta}A+e^{-i\theta}A^*,\ Z=e^{i\theta}D+e^{-i\theta}D^*$ and $Y=e^{i\theta}B+e^{-i\theta}C^*$ where $\theta\in\mathbb{R}$. Then

$$\begin{aligned}
(2w(T))^2 &= \max_{\theta\in\mathbb{R}}\left\|2\mathrm{Re}(e^{i\theta}T)\right\|^2 \\
&= \max_{\theta\in\mathbb{R}}\left\|e^{i\theta}T+e^{-i\theta}T^*\right\|^2 \\
&= \max_{\theta\in\mathbb{R}}\left\|(e^{i\theta}T+e^{-i\theta}T^*)^2\right\| \\
&= \max_{\theta\in\mathbb{R}}\left\|TT^*+T^*T+2\mathrm{Re}(e^{2i\theta}T^2)\right\| \\
&= \max_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} X^2+YY^* & XY+YZ \\ Y^*X+ZY^* & Z^2+Y^*Y \end{bmatrix}\right\| \\
&\leqslant \max_{\theta\in\mathbb{R}}\left(\left\|\begin{bmatrix} X^2+YY^* & XY \\ Y^*X & Y^*Y \end{bmatrix}\right\|+\left\|\begin{bmatrix} 0 & YZ \\ ZY^* & Z^2 \end{bmatrix}\right\|\right) \\
&= \max_{\theta\in R}\left(r\left(\begin{bmatrix} X & Y \\ Y^* & 0 \end{bmatrix}^2\right)+r\left(\begin{bmatrix} 0 & YZ \\ ZY^* & Z^2 \end{bmatrix}\right)\right) \quad \text{(by Lemma 3.1(e))} \\
&= \max_{\theta\in\mathbb{R}}\left(r^2\left(\begin{bmatrix} X & Y \\ Y^* & 0 \end{bmatrix}\right)+r\left(\begin{bmatrix} 0 & YZ \\ (YZ)^* & Z^2 \end{bmatrix}\right)\right) \quad \text{(by Lemma 3.1(b))} \\
&= \max_{\theta\in\mathbb{R}}\left(r^2\left(\begin{bmatrix} ||X|| & ||Y|| \\ ||Y|| & 0 \end{bmatrix}\right)+r\left(\begin{bmatrix} 0 & ||YZ|| \\ ||YZ|| & ||Z^2|| \end{bmatrix}\right)\right) \quad \text{(by Lemma 3.1(c))} \\
&= \max_{\theta\in\mathbb{R}}\left(\frac{1}{2}\left(||X||+\sqrt{||X||^2+4||Y||^2}\right)^2+\frac{1}{2}\left(||Z^2||+\sqrt{||Z^2||^2+4||YZ||^2}\right)\right) \\
&= \left(w(A)+\sqrt{w^2(A)+4w^2(E)}\right)^2+2w^2(D)+2w(D)\sqrt{w^2(D)+4w^2(E)}.
\end{aligned}$$

Hence

$$w^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant \frac{1}{4}\left(w(A)+\sqrt{w^2(A)+4w^2(E)}\right)^2 + \frac{1}{2}w^2(D)+\frac{1}{2}w(D)\sqrt{w^2(D)+4w^2(E)}. \quad \square$$

REMARK 3.5. **1.** The inequality in Theorem 3.4 is sharper than the upper bound provided in [[14], Theorem 2.6], to see this take $B=C=D=0$ which implies that the inequality in our theorem becomes equality while in [[14], Theorem 2.6] we obtain $w(A)\leqslant \|A\|$.

2. In [[15], Theorem 2.2] if we choose $A=D=0$ and $B=C=I$, we obtain $w^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$ $\leqslant 1.5$ whereas in Theorem 3.4 we have $w^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)\leqslant 1$.

3. In [[16], Theorem 2.8] take $A=B=C=D=I$ with $t=1$ to obtain $w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)\leqslant$ $\frac{4+\sqrt{2}}{2}\approx 2.7$, while in Theorem 3.4 we get $w\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)\leqslant \sqrt{2+\sqrt{5}}\approx 2.06$.

4. In [[8], Corollary 3.4] if we choose $A=D=\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $B=C=0$, we obtain $w^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)\leqslant 16$ while in Theorem 3.4 we obtain $w^2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)\leqslant 8$.

The second estimate which concerns with certain 2×2 operator matrix will be given in the following theorem

THEOREM 3.6. *Let $X,Y\in B(H)$ and suppose f,g are nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t)=t$ $(t\geqslant 0)$. Then*

$$w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right)\leqslant \frac{1}{2}\left(1+\sqrt{r(|X||Y|)}\right)\max\{\|f^2(|X|)+g^2(|Y^*|)\|,\|f^2(|Y|)+g^2(|X^*|)\|\}.$$

Also,

$$w(X|Y|)\leqslant \frac{1}{2}\left(1+\sqrt{r(|X||Y|)}\right)\max\{\|f^2(|X|)+g^2(|Y^*|)\|,\|f^2(|Y|)+g^2(|X^*|)\|\}.$$
$$w(Y|X|)\leqslant \frac{1}{2}\left(1+\sqrt{r(|X||Y|)}\right)\max\{\|f^2(|X|)+g^2(|Y^*|)\|,\|f^2(|Y|)+g^2(|X^*|)\|\}.$$

Proof. Let $A=\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}$ and $B=\begin{bmatrix} I & |X| \\ |Y| & I \end{bmatrix}$. Then it is easy to see that $|A|B=B^*|A|$ and so by Lemma 2.3 we have

$$\begin{aligned}
|\langle ABx,x\rangle| &\leqslant r(B)\|f(|A|)x\|\|g(|A^*|)x\| \text{ (where } x\in H\oplus H)\\
&\leqslant r(B)\left\langle f^2(|A|)x,x\right\rangle^{1/2}\left\langle g^2(|A^*|)x,x\right\rangle^{1/2}\\
&\leqslant \frac{1}{2}r(B)\left\langle (f^2(|A|)+g^2(|A^*|))x,x\right\rangle.
\end{aligned}$$

Thus,

$$
\begin{aligned}
& w\left(\begin{bmatrix} X|Y| & X \\ Y & Y|X| \end{bmatrix}\right) \\
& = w(AB) = \sup\{|\langle ABx, x\rangle| : x \in H \oplus H, ||x|| = 1\} \\
& \leqslant \frac{1}{2}\left\|f^2(|A|) + g^2(|A^*|)\right\| r\left(\begin{bmatrix} I & |X| \\ |Y| & I \end{bmatrix}\right) \\
& = \frac{1}{2}\left\|\begin{bmatrix} f^2(|Y|) + g^2(|X^*|) & 0 \\ 0 & f^2(|X|) + g^2(|Y^*|) \end{bmatrix}\right\| r\left(\begin{bmatrix} I & |X| \\ |Y| & I \end{bmatrix}\right) \\
& = \frac{1}{2}\max\left\{\left\|f^2(|X|) + g^2(|Y^*|)\right\|, \left\|f^2(|Y|) + g^2(|X^*|)\right\|\right\} r\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & |X| \\ |Y| & 0 \end{bmatrix}\right) \\
& \leqslant \frac{1}{2}\max\left\{\left\|f^2(|X|) + g^2(|Y^*|)\right\|, \left\|f^2(|Y|) + g^2(|X^*|)\right\|\right\} \\
& \quad \times \left(r\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\right) + r\left(\begin{bmatrix} 0 & |X| \\ |Y| & 0 \end{bmatrix}\right)\right) \text{ (by Lemma 3.1(a))} \\
& = \frac{1}{2}\max\left\{\left\|f^2(|X|) + g^2(|Y^*|)\right\|, \left\|f^2(|Y|) + g^2(|X^*|)\right\|\right\} \\
& \quad \times \left(1 + \sqrt{r(|X||Y|)}\right) \text{ (by Lemma 3.1(d)).}
\end{aligned}
$$

Using the above inequality and Lemma 3.3(a) we get our bounds. □

By Theorem 3.6 and Lemma 2.8 we get the following result.

COROLLARY 3.7. *Let $X \in B(H)$. Then*

$$
\begin{aligned}
w(X|X|) & \leqslant \frac{1}{2}(1 + ||X||)\, |||X| + |X^*||| \\
& \leqslant \frac{1}{2}(1 + ||X||)(||X|| + ||X^2||^{1/2}).
\end{aligned}
$$

REFERENCES

[1] F. KITTANEH Y. MANASRAH, *Improved Young and Heinz inequalities for matrices*, J. Math. Appl. 361 (2010) 262–269.

[2] F. KITTANEH, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math. 158 (2003) 11–17.

[3] M. ALOMRI, *Refinement of some numerical radius inequalities for Hilbert space operators*, Linear Multilinear A. (2019), https://doi.org/10.1080/03081087.2019.1624682.

[4] Y. MANASRAH, F. KITTANEH, *A generalization of two refined Young inequalities*, Positivity 19 (2015) 757–768.

[5] D. CHOI, *A generalization of Younge-type inequalities*, Math. Inequal. Appl. 21 (2018) 99–106.

[6] F. KITTANEH, *Spectral radius inequalities for Hilbert space operators*, Proc. Am. Math. Soc. 134 (2) (2005) 385–390.

[7] F. KITTANEH, *Notes on some inequalities for Hilbert space operators*, Publ. Res. I. Math. Sci. 24 (1988) 283–293.

[8] K. SHEBRAWI, *Numerical radius inequalities for certain 2×2 operator matrices II*, Linear Algebra Appl. 523 (2017) 1–12.

[9] S. S. DRAGOMIR, *Some refinements of Schwarz inequality*, Simposional de Math Si Appl. Polytechnical Inst Timisoara, Romania, 1–2, (1985) 13–16.

[10] M. SATTARI, M. S. MOSLEHIAN, T. YAMAZAKI, *Some generalized numerical radius inequalities of Hilbert space operators*, Linear Algebr Appl. 470 (2015) 216–227.

[11] T. YAMAZAKI, *On upper and lower bounds of the numerical radius and an equality coditions*, Studia mathematica 178 (2007) 83–89.

[12] R. BAHTIA, *Matrix analysis*, volume of Graduate texts in mathematics, 1997.

[13] O. HIRZALLAH, F. KITTANEH, K. SHEBRAWI, *Numerical radius inequality for certain 2×2 operator matrices*, Integral Equations Operator Theory 71 (1997) 129–149.

[14] O. HIRZALLAH, F. KITTANEH, K. SHEBRAWI, *Numerical radius inequalities for 2×2 operator matrices*, Studia Mathematica 15 (2012) 99–115.

[15] W. BANI-DOMI, F. KITTANEH, *Norm and numerical radius inequalities for Hilbert space operators*, Linear and Multilinear Algebra (2020), https://doi.org/10.1080/03081087.2020.1798334.

[16] M. AL-DOLAT, I. JARADAT, I. AL-HUSBAN, *Anovel numerical radius upper bound for 2×2 operator matrices*, Linear and Multilinear Algebra (2020), https://doi.org/10.1080/03081087.2020.1756199.

(Received September 23, 2020)

Mohammed Al-Dolat
Department of Mathematics & Statistics
Jordan University of Science and Technology
P.O. Box 3030, Irbid 22110, Jordan
e-mail: mmaldolat@just.edu.jo

Ayat Dagher
Department of Mathematics & Statistics
Jordan University of Science and Technology
P.O. Box 3030, Irbid 22110, Jordan
e-mail: amdagher17@sci.just.edu.jo

Marwan Alquran
Department of Mathematics & Statistics
Jordan University of Science and Technology
P.O. Box 3030, Irbid 22110, Jordan
e-mail: marwan04@just.edu.jo