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ON A HILBERT–TYPE INEQUALITY WITH THE KERNEL INVOLVING EXTENDED HARDY OPERATOR

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Abstract. In this paper by defining a extended Hardy operator, a new kernel function including both the homogeneous and the non-homogeneous cases is constructed. Dealing with these cases in a unified way, a Hilbert-type inequality involving the newly constructed kernel is established, and the constant factor is proved to be the best possible. The equivalent Hardy-type inequality is also considered in parallel. Furthermore, by specifying the kernel function, some special and meaningful Hilbert-type inequalities with the constant factors related to the higher derivative of trigonometric functions and special functions are presented at the end of the paper, and these newly obtained inequalities are proved to be the extensions of some classical Hilbert-type inequalities.

1. Introduction

Throughout this paper, it is assumed that p and q satisfy $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ except where specifically noted.

Let $f,g \geqslant 0$, $f \in L^p(\mathbb{R}_+)$, $g \in L^q(\mathbb{R}_+)$, then we have [7]:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \mathrm{d}x\mathrm{d}y < \frac{\pi}{\sin\frac{\pi}{p}} \|f\|_p \|g\|_q, \tag{1.1}$$

$$\int_0^\infty \int_0^\infty \frac{\ln\frac{x}{y}}{x-y} f(x)g(y)\mathrm{d}x\mathrm{d}y < \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^2 \|f\|_p \|g\|_q, \tag{1.2}$$

where the constant factors $\frac{\pi}{\sin\frac{\pi}{p}}$ and $\left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^2$ are the best possible.

Inequality (1.1) is called Hardy-Hilbert inequality, and (1.2) is usually named as Hilbert-type inequality. Although more than 100 years have passed since they were stated at the beginning of the 20th century, numerous researchers are still interested in the refinements, extensions, and analogies of inequalities (1.1) and (1.2). Through

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continuous innovation of kernel functions and parameterization of the new kernel functions, a large number of new Hilbert-type inequalities were established in the past several decades. Considering the discrete form, half-discrete form and high-dimensional generalizations of these inequalities, such type of inequalities have grown into a vast inequality system.

The inequality below is a classical extension of (1.1) that was established by Yang [19] in 1998. That is

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\rho} \mathrm{d}x\mathrm{d}y < B\left(\frac{\rho}{2},\frac{\rho}{2}\right) \|f\|_{2,\mu}\|g\|_{2,\nu}, \tag{1.3}$$

where $\rho > 0,\ \mu(x) = \nu(x) = x^{1-\rho}$, and

$$B(u,v) := \int_0^\infty \frac{x^{u-1}}{(1+x)^{u+v}} \mathrm{d}x = B(v,u) \ \ (u,v>0).$$

In 2004, Yang [20] gave another extension of (1.1) as follows:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\rho + y^\rho} \mathrm{d}x\mathrm{d}y < \frac{\pi}{\rho \sin\frac{\pi}{r}} \|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{1.4}$$

where $\rho > 0,\ \mu(x) = x^{p(1-\frac{\rho}{r})-1},\ \nu(y) = y^{q(1-\frac{\rho}{s})-1},\ \frac{1}{r}+\frac{1}{s} = 1$, and the constant factor is the best possible.

Other extensions of inequality (1.1), as well as some discrete and half-discrete forms, can be found in [9, 10, 11, 12, 21, 22, 23, 24, 25, 26]. Furthermore, some Hilbert-type inequalities with newly constructed kernel functions appeared sporadically in the past several years. For example, in 2012, Yang [27] established the following inequality with a homogeneous kernel involving exponent function:

$$\int_0^\infty \int_0^\infty e^{-\frac{ax}{y}} f(x)g(y)\mathrm{d}x\mathrm{d}y < a^{-\beta}\Gamma(\beta)\|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{1.5}$$

where $a > 0,\ \beta > 0,\ \mu(x) = x^{p(1-\beta)-1}$, and $\nu(y) = y^{q(1+\beta)-1}$. In addition, in 2013, Liu [14] established an inequality with a non-homogeneous kernel involving hyperbolic secant function as follows:

$$\int_0^\infty \int_0^\infty \operatorname{sech}(xy) f(x)g(y)\mathrm{d}x\mathrm{d}y < 2c_0\|f\|_{2,\mu}\|g\|_{2,\mu} \tag{1.6}$$

where $\mu(x) = x^{-3}$ and $c_0 = \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^2} = 0.91596559^+$, which is the Catalan constant. Thereafter, Yang [28] gave a similar inequality involving hyperbolic cosecant function:

$$\int_0^\infty \int_0^\infty \operatorname{csch}(x^\delta y) f(x)g(y)\mathrm{d}x\mathrm{d}y < \frac{\pi^2}{4}\|f\|_{2,\mu}\|g\|_{2,\nu}\ , \tag{1.7}$$

where $\delta \in \{1,-1\},\ \mu(x) = x^{1-4\delta}$, and $\nu(x) = x^{-3}$.

It should be noted that, by the introduction of parameters δ, (1.7) gives a unified treatment of Hilbert-type inequalities involving a homogeneous kernel of degree–0 and a non-homogeneous kernel. This method of dealing with kernel functions can also be found in [8, 16]. Furthermore, some other Hilbert-type inequalities with new kernels can be found in [4, 9, 16, 25, 26, 24].

Besides the above-mentioned inequalities of Hilbert-type, another classical inequality of great significance in analysis should be presented. It was stated by Hardy in 1920, and is usually expressed as follows:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\mathrm{d}t\right)^p \mathrm{d}x < \left(\frac{p}{p-1}\|f\|_p\right)^p, \tag{1.8}$$

where $p > 1$, and f is a non-negative function such that $f \in L^p(\mathbb{R}_+)$. Just like Hilbert's inequality, in the past 100 years, researchers have never stopped studying Hardy's inequality, and numerous results were established [1, 2, 3, 5, 13, 15, 17, 29]. As is well known, Hilbert's inequality (1.1) has an equivalent form which is similar to Hardy's inequality (1.8), that is

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y}\mathrm{d}x\right)^p \mathrm{d}y < \left(\frac{\pi}{\sin\frac{\pi}{p}}\|f\|_p\right)^p. \tag{1.9}$$

Inequality (1.9) is the so-called Hardy-type inequality. Generally, a Hilbert-type inequality has an equivalent inequality of Hardy-type. Therefore, they always appear in pairs in the literature.

It is generally known that Hardy's inequality implies the classical Hardy operator, which reveals the arithmetic mean of a function in integral form. It can be written as follows:

$$(\mathrm{H}f)(u) := \frac{1}{u}\int_0^u f(t)\mathrm{d}t. \tag{1.10}$$

For some extended forms of Hardy operator, we can refer to [1, 3, 6, 13, 15]. Inspired by these literature, a new hardy-type operator is proposed as follows:

$$(\mathrm{T}h)(u) := \frac{1}{\psi(u)-\varphi(u)}\int_{\varphi(u)}^{\psi(u)} h(t)\mathrm{d}t \tag{1.11}$$

where $\varphi(u)$ and $\psi(u)$ are two real valued functions defined on $\mathbb{R}_+$, $\varphi(u) \neq \psi(u)$ for arbitrary $u \in \mathbb{R}_+$, $h(t)$ is a continuous function defined on $\mathbb{R}$ and $h(t) \geqslant 0$. Setting $u = x^{\beta_1}y^{\beta_2}$, $\beta_1\beta_2 \neq 0$ in (1.11), a new kernel function can be constructed to include both the homogeneous and the non-homogeneous cases in a unified manner, and a new Hilbert-type inequality can be established with a best possible constant factor. Detailed definitions and lemmas will be presented in the next section.

2. Definitions and lemmas

DEFINITION 2.1. Let $u > 0$,

$$\Gamma(u) := \int_0^\infty x^{u-1}e^{-x}\mathrm{d}x$$

is the Γ-function. Specially, we have $\Gamma(u)=(u-1)!$ for $u\in\mathrm{N}^{+}$.

LEMMA 2.2. *Let $\beta>0$, $\beta_1\beta_2\neq 0$. Let $\varphi(u)$ and $\psi(u)$ be two real valued functions defined on $\mathbb{R}_+$, and $\varphi(u)\neq\psi(u)$ for arbitrary $u\in\mathbb{R}_+$. Suppose that $h(t)$ is a continuous function defined on $\mathbb{R}$, $h(t)\geqslant 0$, and $(\mathrm{T}h)(u)$ is defined by (1.11), such that*

$$C(h,\varphi,\psi,\beta):=\int_0^\infty u^{\beta-1}(\mathrm{T}h)(u)\mathrm{d}u<\infty.$$

For arbitrary sufficiently small positive number ε, define the functions $f_\varepsilon(x)$ and $g_\varepsilon(x)$ as follows:

$$f_\varepsilon(x):=\begin{cases}x^{\frac{p(\beta\beta_1-1)-\beta_1\varepsilon}{p}}, & x\in\Omega_1\\ 0 & x\in\mathbb{R}_+\setminus\Omega_1\end{cases},\quad g_\varepsilon(y):=\begin{cases}y^{\frac{q(\beta\beta_2-1)+\beta_2\varepsilon}{q}}, & y\in\Omega_2\\ 0, & y\in\mathbb{R}_+\setminus\Omega_2\end{cases}.$$

where $\Omega_1=\left\{x:x>0,\ x^{\frac{\beta_1}{|\beta_1|}}>1\right\}$, and $\Omega_2=\left\{y:y>0,\ y^{\frac{\beta_2}{|\beta_2|}}<1\right\}$. Then

$$\varepsilon\int_0^\infty\int_0^\infty(\mathrm{T}h)(x^{\beta_1}y^{\beta_2})f_\varepsilon(x)g_\varepsilon(y)\mathrm{d}x\mathrm{d}y=\frac{1}{|\beta_1\beta_2|}C(h,\varphi,\psi,\beta)+o(1).\tag{2.1}$$

Proof. Setting $x^{\beta_1}y^{\beta_2}=t$, we obtain

$$\begin{aligned}&\varepsilon\int_0^\infty\int_0^\infty(\mathrm{T}h)(x^{\beta_1}y^{\beta_2})f_\varepsilon(x)g_\varepsilon(y)\mathrm{d}x\mathrm{d}y\\
&=\varepsilon\int_{\Omega_2}y^{\frac{q(\beta\beta_2-1)+\beta_2\varepsilon}{q}}\left(\int_{\Omega_1}(\mathrm{T}h)(x^{\beta_1}y^{\beta_2})x^{\frac{p(\beta\beta_1-1)-\beta_1\varepsilon}{p}}\mathrm{d}x\right)\mathrm{d}y\\
&=\frac{\varepsilon}{|\beta_1|}\int_{\Omega_2}y^{\beta_2\varepsilon-1}\left(\int_{y^{\beta_2}}^\infty(\mathrm{T}h)(t)t^{\beta-\frac{\varepsilon}{p}-1}\mathrm{d}t\right)\mathrm{d}y\\
&=\frac{\varepsilon}{|\beta_1|}\int_{\Omega_2}y^{\beta_2\varepsilon-1}\left(\int_1^\infty(\mathrm{T}h)(t)t^{\beta-\frac{\varepsilon}{p}-1}\mathrm{d}t\right)\mathrm{d}y\\
&\quad+\frac{\varepsilon}{|\beta_1|}\int_{\Omega_2}y^{\beta_2\varepsilon-1}\left(\int_{y^{\beta_2}}^1(\mathrm{T}h)(t)t^{\beta-\frac{\varepsilon}{p}-1}\mathrm{d}t\right)\mathrm{d}y\\
&=\frac{1}{|\beta_1\beta_2|}\int_1^\infty(\mathrm{T}h)(t)t^{\beta-\frac{\varepsilon}{p}-1}\mathrm{d}t+\frac{\varepsilon}{|\beta_1|}\int_{\Omega_2}y^{\beta_2\varepsilon-1}\left(\int_{y^{\beta_2}}^1(\mathrm{T}h)(t)t^{\beta-\frac{\varepsilon}{p}-1}\mathrm{d}t\right)\mathrm{d}y.\end{aligned}\tag{2.2}$$

No matter $\beta_2>0$ or $\beta_2<0$, it follows from Fubini's theorem that

$$\int_{\Omega_2}y^{\beta_2\varepsilon-1}\left(\int_{y^{\beta_2}}^1(\mathrm{T}h)(t)t^{\beta-\frac{\varepsilon}{p}-1}\mathrm{d}t\right)\mathrm{d}y=\frac{1}{|\beta_2|\,\varepsilon}\int_0^1(\mathrm{T}h)(t)t^{\beta+\frac{\varepsilon}{q}-1}\mathrm{d}t.\tag{2.3}$$

Applying (2.3) to (2.2), we can obtain

$$\begin{aligned}&\varepsilon\int_0^\infty\int_0^\infty(\mathrm{T}h)(x^{\beta_1}y^{\beta_2})f_\varepsilon(x)g_\varepsilon(y)\mathrm{d}x\mathrm{d}y\\
&\qquad=\frac{1}{|\beta_1\beta_2|}\left\{\int_1^\infty(\mathrm{T}h)(t)t^{\beta-\frac{\varepsilon}{p}-1}\mathrm{d}t+\int_0^1(\mathrm{T}h)(t)t^{\beta+\frac{\varepsilon}{q}-1}\mathrm{d}t\right\}.\end{aligned}\tag{2.4}$$

Let $\varepsilon \to 0^+$, we arrive at (2.1). Lemma 2.2 is proved. $\square$

LEMMA 2.3. *Let* $\rho,\beta,\gamma>0$, $0<\beta<\gamma-\rho$ *and* $\phi(x)=\cot x$. *Then*

$$\int_0^\infty \frac{t^{\beta-1}-t^{\rho+\beta-1}}{1-t^\gamma}\mathrm{d}t = \frac{\pi}{\gamma}\left\{\phi\left(\frac{\beta\pi}{\gamma}\right)-\phi\left(\frac{(\rho+\beta)\pi}{\gamma}\right)\right\}. \tag{2.5}$$

Proof. Write

$$\int_0^\infty \frac{t^{\beta-1}-t^{\rho+\beta-1}}{1-t^\gamma}\mathrm{d}t = \int_0^1 \frac{t^{\beta-1}-t^{\rho+\beta-1}}{1-t^\gamma}\mathrm{d}t + \int_1^\infty \frac{t^{\beta-1}-t^{\rho+\beta-1}}{1-t^\gamma}\mathrm{d}t := I_1+I_2. \tag{2.6}$$

Expanding $\frac{1}{1-t^\gamma}$ into power series and exchange the order of integral and summation, then

$$I_1=\sum_{k=0}^\infty \int_0^1 \left\{t^{k\gamma+\beta-1}-t^{k\gamma+\rho+\beta-1}\right\} = \sum_{k=0}^\infty\left\{\frac{1}{k\gamma+\beta}-\frac{1}{k\gamma+\rho+\beta}\right\}. \tag{2.7}$$

Similarly, by using variable substitution $t=\frac{1}{u}$, we can get

$$I_2=\int_0^1 \frac{u^{\gamma-\rho-\beta-1}-u^{\gamma-\beta-1}}{1-u^\gamma}\mathrm{d}u = \sum_{k=0}^\infty\left\{\frac{1}{k\gamma+\gamma-\rho-\beta}-\frac{1}{k\gamma+\gamma-\beta}\right\}. \tag{2.8}$$

Applying (2.7) and (2.8) to (2.6), we can get

$$\int_0^\infty \frac{t^{\beta-1}-t^{\rho+\beta-1}}{1-t^\gamma}\mathrm{d}t = \sum_{k=0}^\infty\left\{\frac{1}{k\gamma+\beta}-\frac{1}{k\gamma+\gamma-\beta}+\frac{1}{k\gamma+\gamma-\rho-\beta}-\frac{1}{k\gamma+\rho+\beta}\right\}. \tag{2.9}$$

$\square$

Since $\phi(x)=\cot x$ has the form of rational fraction expansion [18] as follows:

$$\phi(x)=\frac{1}{x}+\sum_{k=1}^\infty\left\{\frac{1}{x+k\pi}+\frac{1}{x-k\pi}\right\}, \tag{2.10}$$

letting $x=\frac{\beta\pi}{\gamma}$, and by careful computation, it follows that

$$\sum_{k=0}^\infty\left\{\frac{1}{k\gamma+\beta}-\frac{1}{k\gamma+\gamma-\beta}\right\}=\frac{\pi}{\gamma}\phi\left(\frac{\beta\pi}{\gamma}\right). \tag{2.11}$$

Similarly, letting $x=\frac{(\rho+\beta)\pi}{\gamma}$, we have

$$\sum_{k=0}^\infty\left\{\frac{1}{k\gamma+\rho+\beta}-\frac{1}{k\gamma+\gamma-\rho-\beta}\right\}=\frac{\pi}{\gamma}\phi\left(\frac{(\rho+\beta)\pi}{\gamma}\right). \tag{2.12}$$

Combining (2.9), (2.11) and (2.12), we have (2.5).

LEMMA 2.4. *Let* $0<\rho<\gamma$, *and* $n\in\mathrm{N}$. *Let* $\sinh(t)=\frac{e^t-e^{-t}}{2}$, $\operatorname{csch}(t)=\frac{2}{e^t-e^{-t}}$, *and* $\Phi(x)=\tan x$. *Then*

$$\int_0^\infty t^{2n}\sinh(\rho t)\operatorname{csch}(\gamma t)\mathrm{d}t=\left(\frac{\pi}{\gamma}\right)^{2n+1}\Phi^{(2n)}\left(\frac{\rho\pi}{2\gamma}\right). \tag{2.13}$$

Proof. Since $\frac{1}{e^{\gamma t}-e^{-\gamma t}}=\sum_{k=0}^\infty e^{-(2k+1)\gamma t}$, we can obtain

$$\int_0^\infty t^{2n}\sinh(\rho t)\operatorname{csch}(\gamma t)\mathrm{d}t=\sum_{k=0}^\infty\int_0^\infty\left\{e^{-(2\gamma k+\gamma-\rho)t}t^{2n}-e^{-(2\gamma k+\gamma+\rho)t}t^{2n}\right\}\mathrm{d}t. \tag{2.14}$$

Setting $(2\gamma k+\gamma-\rho)t=u$, then

$$\int_0^\infty e^{-(2\gamma k+\gamma-\rho)t}t^{2n}\mathrm{d}t=\frac{(2n)!}{(2\gamma k+\gamma-\rho)^{2n+1}}. \tag{2.15}$$

Similarly, we have

$$\int_0^\infty e^{-(2\gamma k+\gamma+\rho)t}t^{2n}\mathrm{d}t=\frac{(2n)!}{(2\gamma k+\gamma+\rho)^{2n+1}}. \tag{2.16}$$

Applying (2.15) and (2.16) to (2.14), we have

$$\int_0^\infty t^{2n}\sinh(\rho t)\operatorname{csch}(\gamma t)\mathrm{d}t=\sum_{k=0}^\infty\left\{\frac{(2n)!}{(2\gamma k+\gamma-\rho)^{2n+1}}-\frac{(2n)!}{(2\gamma k+\gamma+\rho)^{2n+1}}\right\}. \tag{2.17}$$

Take the $2n^{\text{th}}$ derivative of (2.10), then

$$\phi^{(2n)}(x)=(2n)!\left\{\frac{1}{x^{2n+1}}+\sum_{k=1}^\infty\left(\frac{1}{(x+k\pi)^{2n+1}}+\frac{1}{(x-k\pi)^{2n+1}}\right)\right\}. \tag{2.18}$$

Letting $x=\frac{\gamma-\rho}{2\gamma}\pi$ in (2.18), in view of $\phi^{(2n)}\left(\frac{(\gamma-\rho)\pi}{2\gamma}\right)=\Phi^{(2n)}\left(\frac{\rho\pi}{2\gamma}\right)$, we have

$$\Phi^{(2n)}\left(\frac{\rho\pi}{2\gamma}\right)=\frac{(2n)!(2\gamma)^{2n+1}}{\pi^{2n+1}}\sum_{k=0}^\infty\left\{\frac{1}{(2\gamma k+\gamma-\rho)^{2n+1}}-\frac{1}{(2\gamma k+\gamma+\rho)^{2n+1}}\right\}. \tag{2.19}$$

Combining (2.18) and (2.19), we arrive at (2.13), and the proof of Lemma 2.4 is completed. □

LEMMA 2.5. *Let* $\gamma>0$, $n\in\mathrm{N}$, *and* $\phi(x)=\cot x$. *Then*

$$\int_0^\infty t^{2n+1}\operatorname{csch}(\gamma t)\mathrm{d}t=-\left(\frac{\pi}{2\gamma}\right)^{2n+2}\phi^{(2n+1)}\left(\frac{\pi}{2}\right). \tag{2.20}$$

Proof. Similar to the proof of Lemma 2.4, we can obtain

$$\int_0^{\infty} t^{2n+1}\operatorname{csch}(\gamma t)\mathrm{d}t = \frac{2(2n+1)!}{\gamma^{2n+2}}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n+2}}. \tag{2.21}$$

Take the derivative of (2.18). Then

$$\phi^{(2n+1)}(x) = -(2n+1)!\left\{\frac{1}{x^{2n+2}}+\sum_{k=1}^{\infty}\left(\frac{1}{(x+k\pi)^{2n+2}}+\frac{1}{(x-k\pi)^{2n+2}}\right)\right\}. \tag{2.22}$$

Let $x=\frac{\pi}{2}$, then

$$\phi^{(2n+1)}\left(\frac{\pi}{2}\right) = \frac{-(2n+1)!2^{2n+3}}{\pi^{2n+2}}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n+2}}. \tag{2.23}$$

Combining (2.22) and (2.23), we can obtain (2.21), and Lemma 2.5 is proved. □

LEMMA 2.6. *Let* $0<\beta<\gamma$, $n\in \mathrm{N}$, *and* $\phi(x)=\cot x$. *Then*

$$\int_0^{\infty}\frac{t^{\beta-1}(\ln t)^{2n+1}}{t^{\gamma}-1}\mathrm{d}t = -\left(\frac{\pi}{\gamma}\right)^{2n+2}\phi^{(2n+1)}\left(\frac{\beta\pi}{\gamma}\right). \tag{2.24}$$

Proof. Expanding $\frac{1}{1-t^{\gamma}}$ into power series and exchange the order of integral and summation, then

$$\begin{aligned}
&\int_0^{\infty}\frac{t^{\beta-1}(\ln t)^{2n+1}}{t^{\gamma}-1}\mathrm{d}t \\
&=\int_0^{1}\frac{t^{\beta-1}(\ln t)^{2n+1}}{t^{\gamma}-1}\mathrm{d}t+\int_1^{\infty}\frac{t^{\beta-1}(\ln t)^{2n+1}}{t^{\gamma}-1}\mathrm{d}t \\
&=\int_0^{1}\frac{t^{\beta-1}(\ln t)^{2n+1}}{t^{\gamma}-1}\mathrm{d}t+\int_0^{1}\frac{t^{\gamma-\beta-1}(\ln t)^{2n+1}}{t^{\gamma}-1}\mathrm{d}t \\
&=-\sum_{k=0}^{\infty}\left\{\int_0^1 t^{k\gamma+\beta-1}(\ln t)^{2n+1}\mathrm{d}t+\int_0^1 t^{k\gamma+\gamma-\beta-1}(\ln t)^{2n+1}\mathrm{d}t\right\}.
\end{aligned} \tag{2.25}$$

Setting $\ln t=\frac{-u}{k\gamma+\beta}$, then

$$\int_0^1 t^{k\gamma+\beta-1}(\ln t)^{2n+1}\mathrm{d}t = \frac{-1}{(k\gamma+\beta)^{2n+2}}\int_0^{\infty}e^{-u}u^{2n+1}\mathrm{d}u = \frac{-(2n+1)!}{(k\gamma+\beta)^{2n+2}}. \tag{2.26}$$

Similarly, we can obtain

$$\int_0^1 t^{k\gamma+\gamma-\beta-1}(\ln t)^{2n+1}\mathrm{d}t = \frac{-(2n+1)!}{(k\gamma+\gamma-\beta)^{2n+2}}. \tag{2.27}$$

Applying (2.26) and (2.27) to (2.25), we obtain

$$\int_0^\infty \frac{t^{\beta-1}(\ln t)^{2n+1}}{t^\gamma-1}\mathrm{d}t=(2n+1)!\sum_{k=0}^{\infty}\left\{\frac{1}{(k\gamma+\beta)^{2n+2}}+\frac{1}{(k\gamma+\gamma-\beta)^{2n+2}}\right\}. \tag{2.28}$$

Let $x=\frac{\beta\pi}{\gamma}$ in (2.22), and apply the result to (2.28), then we can obtain (2.24). □

LEMMA 2.7. *Let $0<a<b$, and $0<\beta<\gamma$. Then*

$$\int_0^\infty u^{\beta-2}\int_{au}^{bu}\frac{1}{(1+t)^\gamma}\mathrm{d}t\mathrm{d}u=\begin{cases}\frac{\ln a-\ln b}{1-\gamma}, & \beta=1,\\ \frac{b^{1-\beta}-a^{1-\beta}}{1-\beta}B(\beta,\gamma-\beta), & \beta\neq 1.\end{cases} \tag{2.29}$$

Proof. If $\beta=1$, then $\gamma>1$. By Fubini's theorem, we can obtain

$$\int_0^\infty u^{\beta-2}\int_{au}^{bu}\frac{1}{(1+t)^\gamma}\mathrm{d}t\mathrm{d}u=\int_0^\infty\frac{1}{(1+t)^\gamma}\int_{\frac{t}{b}}^{\frac{t}{a}}\frac{1}{u}\mathrm{d}u\mathrm{d}t=\frac{\ln a-\ln b}{1-\gamma}.$$

If $\beta\neq 1$, by Fubini's theorem, we can have

$$\int_0^\infty u^{\beta-2}\int_{au}^{bu}\frac{1}{(1+t)^\gamma}\mathrm{d}t\mathrm{d}u=\int_0^\infty\frac{1}{(1+t)^\gamma}\int_{\frac{t}{b}}^{\frac{t}{a}}u^{\beta-2}\mathrm{d}u\mathrm{d}t=\frac{b^{1-\beta}-a^{1-\beta}}{1-\beta}\int_0^\infty\frac{t^{\beta-1}}{(1+t)^\gamma}\mathrm{d}t$$
$$=\frac{b^{1-\beta}-a^{1-\beta}}{1-\beta}B(\beta,\gamma-\beta).$$

Lemma 2.7 is proved. □

3. Main results

THEOREM 3.1. *Let $\beta>0$, $\beta_1\beta_2\neq 0$. Let $\varphi(u)$ and $\psi(u)$ be two real valued functions defined on $\mathbb{R}_+$, and $\varphi(u)\neq\psi(u)$ for arbitrary $u\in\mathbb{R}_+$. Suppose that $h(t)$ is a continuous function defined on $\mathbb{R}$, $h(t)\geqslant 0$, and $(\mathrm{T}h)(u)$ is defined by (1.11), such that*

$$C(h,\varphi,\psi,\beta):=\int_0^\infty u^{\beta-1}(\mathrm{T}h)(u)\mathrm{d}u<\infty.$$

Let $\mu(x)=x^{p(1-\beta\beta_1)-1}$, $\nu(y)=y^{q(1-\beta\beta_2)-1}$ and $f(x)$, $g(x)\geqslant 0$ with $f(x)\in L^p_\mu(\mathbb{R}_+)$, and $g(x)\in L^q_\nu(\mathbb{R}_+)$. Then

$$\int_0^\infty\int_0^\infty(\mathrm{T}h)(x^{\beta_1}y^{\beta_2})f(x)g(y)\mathrm{d}x\mathrm{d}y<|\beta_1|^{-\frac{1}{q}}|\beta_2|^{-\frac{1}{p}}C(h,\varphi,\psi,\beta)\|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{3.1}$$

where the constant factor $|\beta_1|^{-\frac{1}{q}}|\beta_2|^{-\frac{1}{p}}C(h,\varphi,\psi,\beta)$ is the best possible.

Proof. By Hölder's inequality, we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty (\mathrm{T}h)(x^{\beta_1}y^{\beta_2})f(x)g(y)\mathrm{d}x\mathrm{d}y\\
&=\int_0^\infty\int_0^\infty\left(\left((\mathrm{T}h)(x^{\beta_1}y^{\beta_2})\right)^{\frac{1}{p}}x^{\frac{1-\beta\beta_1}{q}}y^{\frac{\beta\beta_2-1}{p}}f(x)\right)\\
&\quad\times\left(\left((\mathrm{T}h)(x^{\beta_1}y^{\beta_2})\right)^{\frac{1}{q}}y^{\frac{1-\beta\beta_2}{p}}x^{\frac{\beta\beta_1-1}{q}}g(y)\right)\mathrm{d}x\mathrm{d}y\\
&\leqslant\left(\int_0^\infty\int_0^\infty (\mathrm{T}h)(x^{\beta_1}y^{\beta_2})y^{\beta\beta_2-1}x^{\frac{p(1-\beta\beta_1)}{q}}f^p(x)\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}\\
&\quad\times\left(\int_0^\infty\int_0^\infty (\mathrm{T}h)(x^{\beta_1}y^{\beta_2})x^{\beta\beta_1-1}y^{\frac{q(1-\beta\beta_2)}{p}}g^q(y)\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{q}}\\
&=\left(\int_0^\infty\omega(x)x^{\frac{p(1-\beta\beta_1)}{q}}f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_0^\infty\varpi(y)y^{\frac{q(1-\beta\beta_2)}{p}}g^q(y)\mathrm{d}y\right)^{\frac{1}{q}},
\end{aligned}\tag{3.2}$$

where $\omega(x)=\int_0^\infty(\mathrm{T}h)(x^{\beta_1}y^{\beta_2})y^{\beta\beta_2-1}\mathrm{d}y$, and $\varpi(y)=\int_0^\infty(\mathrm{T}h)(x^{\beta_1}y^{\beta_2})x^{\beta\beta_1-1}\mathrm{d}x$.

Setting $x^{\beta_1}y^{\beta_2}=u$, we have

$$\omega(x)=\frac{x^{-\beta\beta_1}}{|\beta_2|}\int_0^\infty u^{\beta-1}(\mathrm{T}h)(u)\mathrm{d}u=\frac{1}{|\beta_2|}C(h,\varphi,\psi,\beta)x^{-\beta\beta_1}. \tag{3.3}$$

Similarly, we obtain

$$\varpi(y)=\frac{y^{-\beta\beta_2}}{|\beta_1|}\int_0^\infty u^{\beta-1}(\mathrm{T}h)(u)\mathrm{d}u=\frac{1}{|\beta_1|}C(h,\varphi,\psi,\beta)y^{-\beta\beta_2}. \tag{3.4}$$

Applying (3.3) and (3.4) to (3.2), then

$$\int_0^\infty\int_0^\infty(\mathrm{T}h)(x^{\beta_1}y^{\beta_2})f(x)g(y)\mathrm{d}x\mathrm{d}y\leqslant|\beta_1|^{-\frac{1}{q}}|\beta_2|^{-\frac{1}{p}}C(h,\varphi,\psi,\beta)\|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{3.5}$$

If (3.5) takes the form of equality, then there must exist two constants A_1 and A_2 that are not both equal to zero, such that

$$A_1(\mathrm{T}h)(x^{\beta_1}y^{\beta_2})y^{\beta\beta_2-1}x^{\frac{p(1-\beta\beta_1)}{q}}f^p(x)=A_2(\mathrm{T}h)(x^{\beta_1}y^{\beta_2})x^{\beta\beta_1-1}y^{\frac{q(1-\beta\beta_2)}{p}}g^q(y),$$

$a.\ e.\ in\ \mathbb{R}_+^2$. That is

$$A_1x^{p(1-\beta\beta_1)}f^p(x)=A_2y^{q(1-\beta\beta_2)}g^q(y),\quad a.\ e.\ in\ \mathbb{R}_+^2.$$

Therefore, there exists a constant A such that

$$A_1x^{p(1-\beta\beta_1)}f^p(x)=A,\quad a.\ e.\ in\ \mathbb{R}_+;$$

and

$$A_2 y^{q(1-\beta\beta_2)} g^q(y) = A, \quad a.\ e.\ \ in\ \ \mathbb{R}_+.$$

Without loss of generality, assuming $A_1 \neq 0$, it follows that $x^{p(1-\beta\beta_1)-1} f^p(x) = \frac{A}{A_1 x}$ a.e. in $\mathbb{R}_+$, and this contradicts the fact $f(x) \in L^p_\mu(\mathbb{R}_+)$. Thus, (3.5) takes the form of strict inequality, and (3.1) is obtained.

It remains to show that the constant factor in (3.1) is the best possible. Assume that there is a positive constant $C < |\ \beta_1\ |^{-\frac{1}{q}} |\ \beta_2\ |^{-\frac{1}{p}} C(h,\varphi,\psi,\beta)$, such that (3.1) still holds for $|\ \beta_1\ |^{-\frac{1}{q}} |\ \beta_2\ |^{-\frac{1}{p}} C(h,\varphi,\psi,\beta)$ being replaced by C. That is

$$\int_0^\infty \int_0^\infty (\mathrm{T}h)(x^{\beta_1} y^{\beta_2}) f(x) g(y) \mathrm{d}x\mathrm{d}y < C\|f\|_{p,\mu}\|g\|_{q,\nu}. \tag{3.6}$$

Replacing f and g in (3.6) by f_ε and g_ε defined in Lemma 2.2 respectively, and by virtue of (2.1), it follows that

$$\begin{aligned}
&\frac{1}{|\ \beta_1\beta_2\ |} C(h,\varphi,\psi,\beta) + o(1)\\
&= \varepsilon \int_0^\infty \int_0^\infty (\mathrm{T}h)(x^{\beta_1} y^{\beta_2}) f_\varepsilon(x) g_\varepsilon(y) \mathrm{d}x\mathrm{d}y\\
&< \varepsilon C \left(\int_{\Omega_1} x^{-\beta_1\varepsilon-1} \mathrm{d}x \right)^{\frac{1}{p}} \left(\int_{\Omega_2} x^{\beta_2\varepsilon-1} \mathrm{d}x \right)^{\frac{1}{q}} = C\,|\ \beta_1\ |^{-\frac{1}{p}} |\ \beta_2\ |^{-\frac{1}{q}}.
\end{aligned}$$

Let $\varepsilon \to 0$, then we can obtain $|\ \beta_1\ |^{-\frac{1}{q}} |\ \beta_2\ |^{-\frac{1}{p}} C(h,\varphi,\psi,\beta) \leqslant C$, and this contradicts the assumption. Thus, the constant factor in (3.1) is the best possible. Theorem 3.1 is proved. $\square$

THEOREM 3.2. *Let $\beta > 0$, $\beta_1\beta_2 \neq 0$. Let $\varphi(u)$ and $\psi(u)$ be two real valued functions defined on $\mathbb{R}_+$, and $\varphi(u) \neq \psi(u)$ for arbitrary $u \in \mathbb{R}_+$. Suppose that $h(t)$ is a continuous function defined on $\mathbb{R}$, $h(t) \geqslant 0$, and $(\mathrm{T}h)(u)$ is defined by (1.11), such that*

$$C(h,\varphi,\psi,\beta) := \int_0^\infty u^{\beta-1} (\mathrm{T}h)(u) \mathrm{d}u < \infty.$$

Let $\mu(x) = x^{p(1-\beta\beta_1)-1}$, $\nu(y) = y^{q(1-\beta\beta_2)-1}$ and $f(x) \geqslant 0$ with $f(x) \in L^p_\mu(\mathbb{R}_+)$. Then

$$\int_0^\infty y^{p\beta\beta_2-1} \left(\int_0^\infty (\mathrm{T}h)(x^{\beta_1} y^{\beta_2}) f(x) \mathrm{d}x \right)^p \mathrm{d}y < \left(|\ \beta_1\ |^{-\frac{1}{q}} |\ \beta_2\ |^{-\frac{1}{p}} C(h,\varphi,\psi,\beta) \|f\|_{p,\mu} \right)^p, \tag{3.7}$$

where the constant factor $\left(|\ \beta_1\ |^{-\frac{1}{q}} |\ \beta_2\ |^{-\frac{1}{p}} C(h,\varphi,\psi,\beta) \right)^p$ is the best possible, and (3.7) is equivalent to (3.1).

Proof. Setting $g(y) := y^{p\beta\beta_2-1}\left(\int_0^\infty (\mathrm{T}h)(x^{\beta_1}y^{\beta_2})f(x)\mathrm{d}x\right)^{p-1}$, by Theorem 3.1, we can obtain

$$\begin{aligned}
0 < (\|g\|_{q,\nu})^{pq} &= \left(\int_0^\infty y^{q(1-\beta\beta_2)-1}g^q(y)\mathrm{d}y\right)^p \\
&= \left(\int_0^\infty y^{p\beta\beta_2-1}\left(\int_0^\infty (\mathrm{T}h)(x^{\beta_1}y^{\beta_2})f(x)\mathrm{d}x\right)^p \mathrm{d}y\right)^p \\
&= \left(\int_0^\infty\int_0^\infty (\mathrm{T}h)(x^{\beta_1}y^{\beta_2})f(x)g(y)\mathrm{d}x\mathrm{d}y\right)^p \\
&\leqslant \left(|\,\beta_1\,|^{-\frac{1}{q}}|\,\beta_2\,|^{-\frac{1}{p}}\,C(h,\varphi,\psi,\beta)\|f\|_{p,\mu}\|g\|_{q,\nu}\right)^p. \qquad (3.8)
\end{aligned}$$

Thus

$$\begin{aligned}
0 &< \int_0^\infty y^{p\beta\beta_1-1}\left(\int_0^\infty (\mathrm{T}h)(x^{\beta_1}y^{\beta_2})f(x)\mathrm{d}x\right)^p \mathrm{d}y \\
&= (\|g\|_{q,\nu})^q \leqslant \left(|\,\beta_1\,|^{-\frac{1}{q}}|\,\beta_2\,|^{-\frac{1}{p}}\,C(h,\varphi,\psi,\beta)\|f\|_{p,\mu}\right)^p. \qquad (3.9)
\end{aligned}$$

Since $f(x)\in L^p_\mu(\mathbb{R}_+)$, it follows from (3.9) that $g(x)\in L^q_\nu(\mathbb{R}_+)$. By using Theorem 3.1 once more, both (3.8) and (3.9) take the form of strict inequalities, and therefore (3.7) is proved.

On the other hand, if (3.7) is valid, by Hölder's inequality, we obtain

$$\begin{aligned}
&\int_0^\infty\int_0^\infty (\mathrm{T}h)(x^{\beta_1}y^{\beta_2})f(x)g(y)\mathrm{d}x\mathrm{d}y \\
&= \int_0^\infty \left(y^{-(1-\beta\beta_2-\frac{1}{q})}\int_0^\infty (\mathrm{T}h)(x^{\beta_1}y^{\beta_2})f(x)\mathrm{d}x\right)\left(y^{1-\beta\beta_2-\frac{1}{q}}g(y)\right)\mathrm{d}y \\
&\leqslant \left(\int_0^\infty y^{p\beta\beta_2-1}\left(\int_0^\infty (\mathrm{T}h)(x^{\beta_1}y^{\beta_2})f(x)\mathrm{d}x\right)^p \mathrm{d}y\right)^{\frac{1}{p}}\|g\|_{q,\nu}. \qquad (3.10)
\end{aligned}$$

Applying (3.7) to (3.10), we can get (3.1). Therefore, (3.1) is equivalent to (3.7). From the equivalence of (3.1) and (3.7), the constant factor $\left(|\,\beta_1\,|^{-\frac{1}{q}}|\,\beta_2\,|^{-\frac{1}{p}}\,C(h,\varphi,\psi,\beta)\right)^p$ in (3.7) is the best possible obviously, and therefore the proof of Theorem 3.2 is completed. □

4. Applications

In Theorem 3.1, let $h(t)=t^{\frac{\rho}{\gamma}-1}$, $\psi(x)=t^\gamma$, $\gamma>\rho>0$, and $\varphi(x)=1$. By using Lemma 2.3, we obtain the following corollary.

COROLLARY 4.1. *Let* $\rho,\beta,\gamma>0$, $0<\beta<\gamma-\rho$, *and* $\beta_1\beta_2\neq 0$. $\phi(x)=\cot x$, $\mu(x)=x^{p(1-\beta\beta_1)-1}$, *and* $\nu(y)=y^{q(1-\beta\beta_2)-1}$. *Suppose that* $f(x),\ g(x)\geqslant 0$, *such that*

$f(x) \in L^p_\mu(\mathbb{R}_+)$, *and* $g(x) \in L^q_\nu(\mathbb{R}_+)$. *Then*

$$\int_0^\infty \int_0^\infty \frac{(x^{\beta_1} y^{\beta_2})^\rho - 1}{(x^{\beta_1} y^{\beta_2})^\gamma - 1} f(x)g(y)\mathrm{d}x\mathrm{d}y \\ <| \beta_1 |^{-\frac{1}{q}} | \beta_2 |^{-\frac{1}{p}} \frac{\pi}{\gamma} \left\{ \phi\left(\frac{\beta\pi}{\gamma}\right) - \phi\left(\frac{(\rho+\beta)\pi}{\gamma}\right) \right\} \|f\|_{p,\mu} \|g\|_{q,\nu}. \tag{4.1}$$

In corollary 4.1, let $\gamma = 2\rho,\ \beta_1 = \beta_2 = 1$, then $0 < \beta < \rho$, and (4.1) reduces to

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{1+(xy)^\rho} \mathrm{d}x\mathrm{d}y < \frac{\pi}{\rho \sin\left(\frac{\beta\pi}{\rho}\right)} \|f\|_{p,\mu} \|g\|_{q,\nu}, \tag{4.2}$$

where $\mu(x) = x^{p(1-\beta)-1},\ \nu(y) = y^{q(1-\beta)-1}$.

In corollary 4.1, Let $\gamma = 2\rho,\ \beta_1 = 1,\ \beta_2 = -1$, then $0 < \beta < \rho$. By replacing $g(y)y^\rho$ with $g(y)$, (4.1) reduces to

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\rho + y^\rho} \mathrm{d}x\mathrm{d}y < \frac{\pi}{\rho \sin\left(\frac{\beta\pi}{\rho}\right)} \|f\|_{p,\mu} \|g\|_{q,\nu}, \tag{4.3}$$

where $\mu(x) = x^{p(1-\beta)-1},\ \nu(y) = y^{q(1+\beta-\rho)-1}$. Let $\beta = \frac{\rho}{r},\ \frac{1}{r} + \frac{1}{s} = 1$ in (4.3), we can obtain (1.4).

In corollary 4.1, let $\gamma = 3\rho,\ \beta_1 = 1,\ \beta_2 = -1$, then $0 < \beta < 2\rho$, and

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^{2\rho} + (xy)^\rho + y^{2\rho}} \mathrm{d}x\mathrm{d}y < \frac{\pi}{3\rho} \left\{ \phi\left(\frac{\beta\pi}{3\rho}\right) - \phi\left(\frac{(\rho+\beta)\pi}{3\rho}\right) \right\} \|f\|_{p,\mu} \|g\|_{q,\nu}, \tag{4.4}$$

where $\mu(x) = x^{p(1-\beta)-1},\ \nu(y) = y^{q(1+\beta-2\rho)-1}$. Let $\beta = \rho$ in (4.4), then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^{2\rho} + (xy)^\rho + y^{2\rho}} \mathrm{d}x\mathrm{d}y < \frac{2\sqrt{3}\pi}{9\rho} \|f\|_{p,\mu} \|g\|_{q,\nu}. \tag{4.5}$$

In Theorem 3.1, let $h(t) = t^{\frac{\rho}{\gamma}-1},\ \psi(t) = e^{\gamma t},\ \varphi(t) = e^{-\gamma t},\ 0 < \rho < \gamma$, and $\beta = 2n+1$. By using Lemma 2.4, we obtain corollary 4.2.

COROLLARY 4.2. *Let* $0 < \rho < \gamma,\ n \in \mathrm{N},\ \sinh(t) = \frac{e^t - e^{-t}}{2},\ \operatorname{csch}(t) = \frac{2}{e^t - e^{-t}}$, $\Phi(x) = \tan x,\ \mu(x) = x^{p(1-(2n+1)\beta_1)-1}$, *and* $\nu(y) = y^{q(1-(2n+1)\beta_2)-1}$. *Suppose that* $f(x),\ g(x) \geqslant 0$, *such that* $f(x) \in L^p_\mu(\mathbb{R}_+)$, *and* $g(x) \in L^q_\nu(\mathbb{R}_+)$. *Then*

$$\int_0^\infty \int_0^\infty \sinh\left(\rho x^{\beta_1} y^{\beta_2}\right) \operatorname{csch}\left(\gamma x^{\beta_1} y^{\beta_2}\right) f(x)g(y)\mathrm{d}x\mathrm{d}y \\ <| \beta_1 |^{-\frac{1}{q}} | \beta_2 |^{-\frac{1}{p}} \left(\frac{\pi}{2\gamma}\right)^{2n+1} \Phi^{(2n)}\left(\frac{\rho\pi}{2\gamma}\right) \|f\|_{p,\mu} \|g\|_{q,\nu}. \tag{4.6}$$

Let $\gamma = 2\rho,\ \beta_1 = \beta_2 = 1$ in (4.6), then

$$\int_0^\infty \int_0^\infty \operatorname{sech}(\rho xy) f(x)g(y)\mathrm{d}x\mathrm{d}y < \frac{1}{2^{4n+1}}\left(\frac{\pi}{\rho}\right)^{2n+1} \phi^{(2n)}\left(\frac{\pi}{4}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}, \quad (4.7)$$

where $\mu(x) = x^{-(2pn+1)},\ \nu(y) = y^{-(2qn+1)}$. Let $\rho = 1,\ n = 0$ in (4.7), we can get a Hilbert-type inequality similar to (1.6) with the best constant factor $\frac{\pi}{2}$.

In Theorem 3.1, let $h(t) = \frac{1}{t},\ \psi(t) = e^{\gamma t},\ \varphi(t) = e^{-\gamma t},\ \gamma > 0$, and $\beta = 2n+1$. By using Lemma 2.5, and replacing $f(x)x^{\beta_1}$ with $f(x)$, $g(y)y^{\beta_2}$ with $g(y)$, we can obtain the following corollary.

COROLLARY 4.3. *Let $\gamma > 0,\ \beta_1\beta_2 \neq 0$, and $n \in \mathrm{N}$. Let $\mu(x) = x^{p(1-(2n+2)\beta_1)-1}$, and $\nu(y) = y^{q(1-(2n+2)\beta_2)-1}$. Suppose that $f(x),\ g(x) \geqslant 0$, such that $f(x) \in L^p_\mu(\mathbb{R}_+)$, and $g(x) \in L^q_\nu(\mathbb{R}_+)$. Then*

$$\int_0^\infty \int_0^\infty \operatorname{csch}\left(\gamma x^{\beta_1} y^{\beta_2}\right) f(x)g(y)\mathrm{d}x\mathrm{d}y$$
$$< -\mid \beta_1 \mid^{-\frac{1}{q}} \mid \beta_2 \mid^{-\frac{1}{p}} \left(\frac{\pi}{2\gamma}\right)^{2n+2} \phi^{(2n+1)}\left(\frac{\pi}{2}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}. \quad (4.8)$$

Let $\beta_1 = \delta,\ \delta \in \{1,-1\},\ \beta_2 = 1,\ \gamma = 1,\ n = 0$, and $p = q = 2$, then we can obtain (1.7).

In Theorem 3.1, let $h(t) = \frac{(\ln t)^{2n}}{t},\ n \in \mathrm{N},\ \psi(t) = t^\gamma$, and $\varphi(t) = 1$. By using Lemma 2.6, we obtain corollary 4.4.

COROLLARY 4.4. *Let $0 < \beta < \gamma,\ \beta_1\beta_2 \neq 0,\ n \in \mathrm{N},\ \mu(x) = x^{p(1-\beta\beta_1)-1}$, and $\nu(y) = y^{q(1-\beta\beta_2)-1}$. Suppose that $f(x),\ g(x) \geqslant 0$, such that $f(x) \in L^p_\mu(\mathbb{R}_+)$, and $g(x) \in L^q_\nu(\mathbb{R}_+)$. Then*

$$\int_0^\infty \int_0^\infty \frac{\left(\ln\left(x^{\beta_1} y^{\beta_2}\right)\right)^{2n+1}}{\left(x^{\beta_1} y^{\beta_2}\right)^\gamma - 1} f(x)g(y)\mathrm{d}x\mathrm{d}y$$
$$< -\mid \beta_1 \mid^{-\frac{1}{q}} \mid \beta_2 \mid^{-\frac{1}{p}} \left(\frac{\pi}{\gamma}\right)^{2n+2} \phi^{(2n+1)}\left(\frac{\beta\pi}{\gamma}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}. \quad (4.9)$$

Let $\beta_1 = 1,\ \beta_2 = -1$ in (4.9), and replace $g(y)y^\gamma$ with $g(y)$, then

$$\int_0^\infty \int_0^\infty \frac{\left(\ln\frac{x}{y}\right)^{2n+1}}{x^\gamma - y^\gamma} f(x)g(y)\mathrm{d}x\mathrm{d}y < -\left(\frac{\pi}{\gamma}\right)^{2n+2} \phi^{(2n+1)}\left(\frac{\beta\pi}{\gamma}\right)\|f\|_{p,\mu}\|g\|_{q,\nu}. \quad (4.10)$$

where $\mu(x) = x^{p(1-\beta)-1},\ \nu(y) = y^{q(1+\beta-\gamma)-1}$. Obviously, (4.10) is an extension of (1.2). In fact, setting $n = 0,\ \gamma = 1$, and $\beta = \frac{1}{q}$ in (4.10), we can obtain (1.2).

In Theorem 3.1, let $h(t) = \frac{1}{\sqrt{1+t}},\ \psi(t) = bt,\ \varphi(t) = at$, and $b > a > 0$. By using Lemma 2.7, then we can get the following corollary.

COROLLARY 4.5. *Let $0<a<b,\ 0<\beta<\frac{1}{2}$, and $\beta_1\beta_2\neq 0$. Let $\mu(x)=x^{p(1-\beta\beta_1)-1}$, and $\nu(y)=y^{q(1-\beta\beta_2)-1}$. Suppose that $f(x),\ g(x)\geqslant 0$, such that $f(x)\in L^p_\mu(\mathbb{R}_+)$, and $g(x)\in L^q_\nu(\mathbb{R}_+)$. Then*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)\,\mathrm{d}x\mathrm{d}y}{\sqrt{1+ax^{\beta_1}y^{\beta_2}}+\sqrt{1+bx^{\beta_1}y^{\beta_2}}} <|\beta_1|^{-\frac{1}{q}}|\beta_2|^{-\frac{1}{p}}C(a,b,\beta)\|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{4.11}$$

where $C(a,b,\beta):=\frac{a^{1-\beta}-b^{1-\beta}}{2(a-b)(1-\beta)}B\left(\beta,\frac{1}{2}-\beta\right)$.

Let $\beta_1=1,\ \beta_2=-1$ in (4.11), and replace $\sqrt{y}g(y)$ by $g(y)$, then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{\sqrt{ax+y}+\sqrt{bx+y}}\mathrm{d}x\mathrm{d}y<C(a,b,\beta)\|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{4.12}$$

where $\mu(x)=x^{p(1-\beta)-1},\ \mu(y)=y^{q(\frac{1}{2}+\beta)-1}$.

In Theorem 3.1, let $h(t)=\frac{1}{(1+t)^2},\ \psi(t)=bt,\ \varphi(t)=at$, and $b>a>0$. Then we can also get the corollary as follows.

COROLLARY 4.6. *Let $0<a<b,\ 0<\beta<2$, and $\beta_1\beta_2\neq 0$. Let $\mu(x)=x^{p(1-\beta\beta_1)-1}$, and $\nu(y)=y^{q(1-\beta\beta_2)-1}$. Suppose that $f(x),\ g(x)\geqslant 0$, such that $f(x)\in L^p_\mu(\mathbb{R}_+)$, and $g(x)\in L^q_\nu(\mathbb{R}_+)$. Then*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)\mathrm{d}x\mathrm{d}y}{\left(1+ax^{\beta_1}y^{\beta_2}\right)\left(1+bx^{\beta_1}y^{\beta_2}\right)} <|\beta_1|^{-\frac{1}{q}}|\beta_2|^{-\frac{1}{p}}C^*(a,b,\beta)\|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{4.13}$$

where $C^*(a,b,\beta)=\frac{a^{1-\beta}-b^{1-\beta}}{(a-b)(1-\beta)}B(\beta,2-\beta)$ for $\beta\neq 1$; $C^*(a,b,\beta)=\frac{\ln a-\ln b}{a-b}$ for $\beta=1$.

Let $\beta_1=1,\ \beta_2=-1$ in (4.13), and replace $\sqrt{y}g(y)$ with $g(y)$, then we can obtain

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(ax+y)(bx+y)}\mathrm{d}x\mathrm{d}y<C^*(a,b,\beta)\|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{4.14}$$

where $\mu(x)=x^{p(1-\beta)-1},\ \mu(y)=y^{q(\frac{1}{2}+\beta)-1}$.

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