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ON THE ARITHMETIC–GEOMETRIC MEAN INEQUALITY

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Abstract. In this article, we present a new treatment of the arithmetic-geometric mean inequality and its siblings, the Heinz and the Young inequalities. New refinements via calculus computations and convex analysis are presented and a new Heinz-type inequality is presented for any symmetric operator mean.

1. Introduction

The simple inequality

$$\sqrt{ab} \leqslant \frac{a+b}{2}, \quad a,b>0$$

is known in the literature as the arithmetic-geometric mean (AM-GM) inequality. Though simple, this inequality has received a considerable attention due to its applications in mathematical inequalities. A Multivariate version of the AM-GM inequality states that

$$\prod_{i=1}^{n} x_i^{w_i} \leqslant \sum_{i=1}^{n} w_i x_i; \quad x_i>0,\ w_i>0,\ \sum_{i=1}^{n} w_i = 1. \tag{1.1}$$

When $n=2$, (1.1) reduces to the so called Young's inequality, stating that

$$a^{1-t}b^t \leqslant (1-t)a+tb, \quad 0 \leqslant t \leqslant 1,\ a,b>0. \tag{1.2}$$

Applying Young's inequality twice implies the celebrated Heinz inequality

$$\sqrt{ab} \leqslant \frac{a^{1-t}b^t + a^t b^{1-t}}{2} \leqslant \frac{a+b}{2}, \ (0 \leqslant t \leqslant 1). \tag{1.3}$$

A simple proof of (1.3) is obtained noting convexity of the function $t \mapsto a^{1-t}b^t + a^t b^{1-t}$ and its symmetry about $t=\frac{1}{2}$. It is customary to use the notations

$$a\sharp_t b = a^{1-t}b^t,\ a\nabla_t b = (1-t)a+tb,\ a!_t b = ((1-t)a^{-1}+tb^{-1})^{-1},\ a,b>0,\ 0\leqslant t\leqslant 1,$$

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to denote the geometric, arithmetic and harmonic means of the scalars a,b. When $t=\frac{1}{2}$, we usually drop it from the notation.

Heinz inequality (1.3) has received a considerable attention in the literature due to its application in Matrix theory. We refer the reader to [4, 9, 11] for some treatments of Heinz inequality, and their applications in Matrix theory.

In [5], a refinement of (1.2) was found in the following form

$$a^{1-t}b^t+\min\{t,1-t\}\left(\sqrt{a}-\sqrt{b}\right)^2 \leqslant (1-t)a+tb, \tag{1.4}$$

while a reverse was shown in [6] as follows

$$a^{1-t}b^t+\max\{t,1-t\}\left(\sqrt{a}-\sqrt{b}\right)^2 \geqslant (1-t)a+tb. \tag{1.5}$$

We refer the reader to [1, 2, 3, 7, 8] for recent references treating refinements and reverses of (1.2).

A common disadvantage among these refinements and reverses is the fact that when $t=\frac{1}{2}$, both (1.4) and (1.5) become trivial identities. Our first target in this article is to present a non-trivial refinement and reverse of Young's inequality, when $t=\frac{1}{2}$. That is, we provide a refinement and reverse of the simple inequality $2\sqrt{ab}\leqslant a+b$. A matrix version then will be shown.

Once we show this, we move to the related Heinz inequality, where a simple application towards the Cauchy-Schwartz inequality is give first.

When dealing with operator/matrix versions, we recall some terminologies. Let $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded linear operators acting on a complex Hilbert space $\mathscr{H}$, with inner product $\langle\cdot,\cdot\rangle$. An operator mean σ_f in the sense of Kubo-Ando is defined by a positive operator monotone function f on the half interval $(0,\infty)$ with $f(1)=1$ as

$$A\sigma B=A^{\frac{1}{2}}f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}},$$

where $A,B\in\mathscr{B}(\mathscr{H})$ are positive operators. Here, f is referred to as the representing function of σ. We recall here that an operator $A\in\mathscr{B}(\mathscr{H})$ is said to be positive when $\langle Ax,x\rangle>0$, for all non-zero vectors $x\in\mathscr{H}$.

The most three common operator means are the harmonic, geometric and arithmetic means, respectively defined for $0\leqslant t\leqslant 1$ by

$$A!_tB=((1-t)A^{-1}+tB^{-1})^{-1},\ A\sharp_tB=A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^tA^{\frac{1}{2}},\ A\nabla_tB=(1-t)A+tB.$$

When $t\notin[0,1]$, we still use the same notations, although they do not represents operator means. Also, for these values of t, $A!_tB$ is not always defined.

The organization of this paper will be as follows. First, we present a new convex approach that implies some AM-GM related inequalities and refinements, then we discuss the AM-GM inequality with its operator versions, where new refinements are shown. In the end, we discuss related Heinz and Cauchy inequalities.

2. Main results

2.1. A new convex analysis approach

We begin our main results with a convex argument that implies certain refinements of the Young and weighted power mean inequalities. While this paper is focused on young and Heinz-type inequalities, the convex inequality we obtain presents a new treatment of convex functions. We refer the reader to [10] for general treatment of convex functions related to this approach.

THEOREM 2.1. *Let f be an increasing function on $[0,\infty)$ with $f(0)=0$ such that $f\left(\sqrt{t}\right)$ is convex. If $r=\min\{t,1-t\}$ and $0\leqslant t\leqslant 1$, then*

$$\begin{aligned}
&f((1-t)a+tb)\\
&\leqslant f((1-t)a+tb)+f\left(\sqrt{t(1-t)}|a-b|\right)+2r\left(\frac{f(a)+f(b)}{2}-f\left(\sqrt{\frac{a^2+b^2}{2}}\right)\right)\\
&\leqslant f\left(\sqrt{(1-t)a^2+tb^2}\right)+2r\left(\frac{f(a)+f(b)}{2}-f\left(\sqrt{\frac{a^2+b^2}{2}}\right)\right)\\
&\leqslant (1-t)f(a)+tf(b).
\end{aligned}$$

Proof. Some ideas in our proof are similar to the ones used in [8, 10]. Assume that $0\leqslant t\leqslant 1$. We have

$$\begin{aligned}
&(1-t)a^2+tb^2-((1-t)a+tb)^2-(1-t)^2(a-b)^2\\
&=(1-t)(2t-1)a^2+(1-t)(2t-1)b^2-2(1-t)(2t-1)ab\\
&=(1-t)(2t-1)\left(a^2+b^2-2ab\right)\\
&=(1-t)(2t-1)(a-b)^2.
\end{aligned}$$

That is,

$$(1-t)a^2+tb^2-((1-t)a+tb)^2-(1-t)^2(a-b)^2=(1-t)(2t-1)(a-b)^2,$$

and

$$(1-t)a^2+tb^2=t(1-t)(a-b)^2+((1-t)a+tb)^2.$$

Assume that $0\leqslant t\leqslant 1/2$. Let $g(t)=f\left(\sqrt{t}\right)$, $t\in[0,\infty)$. Then g is an increasing convex function on $[0,\infty)$. This implies

$$\begin{aligned}
&(1-t)g\left(a^2\right)+tg\left(b^2\right)-2r\left(\frac{g\left(a^2\right)+g\left(b^2\right)}{2}-g\left(\frac{a^2+b^2}{2}\right)\right)\\
&=(1-2t)g\left(a^2\right)+2tg\left(\frac{a^2+b^2}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\geqslant g\left((1-2t)a^2+2t\frac{a^2+b^2}{2}\right)\\
&= g\left((1-t)a^2+tb^2\right)\\
&= g\left(t(1-t)(a-b)^2+((1-t)a+tb)^2\right)\\
&\geqslant g\left(t(1-t)(a-b)^2\right)+g\left(((1-t)a+tb)^2\right).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&g\left(((1-t)a+tb)^2\right)\\
&\leqslant g\left(((1-t)a+tb)^2\right)+g\left(t(1-t)(a-b)^2\right)+2r\left(\frac{g\left(a^2\right)+g\left(b^2\right)}{2}-g\left(\frac{a^2+b^2}{2}\right)\right)\\
&\leqslant g\left((1-t)a^2+tb^2\right)+2r\left(\frac{g\left(a^2\right)+g\left(b^2\right)}{2}-g\left(\frac{a^2+b^2}{2}\right)\right)\\
&\leqslant (1-t)g\left(a^2\right)+tg\left(b^2\right),
\end{aligned}
$$

where $r=\min\{t,1-t\}$. The above inequality is also valid for $1/2 \leqslant t \leqslant 1$. Thus,

$$
\begin{aligned}
&f((1-t)a+tb)\\
&\leqslant f((1-t)a+tb)+f\left(\sqrt{t(1-t)}|a-b|\right)+2r\left(\frac{f(a)+f(b)}{2}-f\left(\sqrt{\frac{a^2+b^2}{2}}\right)\right)\\
&\leqslant f\left(\sqrt{(1-t)a^2+tb^2}\right)+2r\left(\frac{f(a)+f(b)}{2}-f\left(\sqrt{\frac{a^2+b^2}{2}}\right)\right)\\
&\leqslant (1-t)f(a)+tf(b),
\end{aligned}
$$

where $r=\min\{t,1-t\}$ and $0\leqslant t\leqslant 1$. This completes the proof. □

REMARK 2.1. Let the assumptions of Theorem 2.1 be valid. We observe that

$$
\begin{aligned}
0 &\leqslant f\left(\frac{|a-b|}{2}\right)+\frac{f(a)+f(b)}{2}-f\left(\sqrt{\frac{a^2+b^2}{2}}\right)\\
&\leqslant \frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right).
\end{aligned}
\tag{2.1}
$$

Inequality (2.1) says that

$$
f\left(\frac{a+b}{2}\right)\leqslant f\left(\sqrt{\frac{a^2+b^2}{2}}\right)-f\left(\frac{|a-b|}{2}\right)\leqslant \frac{f(a)+f(b)}{2}.
$$

COROLLARY 2.1. *Let $a,b \geqslant 0$ and $0 \leqslant t \leqslant 1$. Then, for $r \geqslant 2$,*

$$\begin{aligned} a^{1-t}b^t &\leqslant \left((1-t)a^{\frac{1}{r}}+tb^{\frac{1}{r}}\right)^r \\ &\leqslant \left((1-t)a^{\frac{2}{r}}+tb^{\frac{2}{r}}\right)^{\frac{r}{2}} - (t(1-t))^{\frac{r}{2}}\left|a^{\frac{1}{r}}-b^{\frac{1}{r}}\right|^r \\ &\leqslant (1-t)a+tb. \end{aligned}$$

In particular, when $t=\frac{1}{2}$, we obtain

$$\sqrt{ab} \leqslant \left(\frac{a^{\frac{1}{r}}+b^{\frac{1}{r}}}{2}\right)^r \leqslant \left(\frac{a^{\frac{2}{r}}+b^{\frac{2}{r}}}{2}\right)^{\frac{r}{2}} - \left(\frac{\left|a^{\frac{1}{r}}-b^{\frac{1}{r}}\right|}{2}\right)^r \leqslant \frac{a+b}{2}.$$

Proof. Applying Theorem 2.1, we have, for $0 \leqslant t \leqslant 1$

$$\begin{aligned} f((1-t)a+tb) &\leqslant f\left(\sqrt{(1-t)a^2+tb^2}\right) - f\left(\sqrt{t(1-t)}|a-b|\right) \\ &\leqslant (1-t)f(a)+tf(b). \end{aligned} \tag{2.2}$$

On the other hand, when $a,b>0$ and $0 \leqslant t \leqslant 1$, Young's inequality states that $a^{1-t}b^t \leqslant (1-t)a+tb$. Since f is increasing, we obtain

$$\begin{aligned} f\left(a^{1-t}b^t\right) &\leqslant f((1-t)a+tb) \\ &\leqslant f\left(\sqrt{(1-t)a^2+tb^2}\right) - f\left(\sqrt{t(1-t)}|a-b|\right) \\ &\leqslant (1-t)f(a)+tf(b). \end{aligned}$$

When $f(t)=t^r, r \geqslant 2$, this implies

$$\begin{aligned} \left(a^{1-t}b^t\right)^r &\leqslant ((1-t)a+tb)^r \\ &\leqslant \left(\sqrt{(1-t)a^2+tb^2}\right)^r - \left(\sqrt{t(1-t)}|a-b|\right)^r \\ &\leqslant (1-t)a^r+tb^r. \end{aligned}$$

Replacing a with $a^{1/r}$ and b with $b^{1/r}$, we obtain the desired result. □

REMARK 2.2. In this remark, we discuss the behavior of the middle term in Corollary 2.1. Namely, we compute

$$\lim_{r\to\infty}\left\{\left((1-t)a^{\frac{2}{r}}+tb^{\frac{2}{r}}\right)^{\frac{r}{2}} - (t(1-t))^{\frac{r}{2}}\left|a^{\frac{1}{r}}-b^{\frac{1}{r}}\right|^r\right\}.$$

In fact, direct Calculus computations, with the aid of L'hopital's rule imply

$$\lim_{r\to\infty}\left((1-t)a^{\frac{2}{r}}+tb^{\frac{2}{r}}\right)^{\frac{r}{2}} = a^{1-t}b^t.$$

On the other hand, since $a^{\frac{1}{r}} - b^{\frac{1}{r}} \to 0$, we have

$$\lim_{r\to\infty}\left\{(t(1-t))^{\frac{r}{2}}\left|a^{\frac{1}{r}}-b^{\frac{1}{r}}\right|^{r}\right\}=0,\ 0\leqslant t\leqslant 1.$$

Consequently,

$$\lim_{r\to\infty}\left\{\left((1-t)a^{\frac{2}{r}}+tb^{\frac{2}{r}}\right)^{\frac{r}{2}}-(t(1-t))^{\frac{r}{2}}\left|a^{\frac{1}{r}}-b^{\frac{1}{r}}\right|^{r}\right\}=a^{1-t}b^{t},\ 0\leqslant t\leqslant 1.$$

Notice that Corollary 2.1 implies

$$a^{1-t}b^{t}+(t(1-t))^{\frac{r}{2}}\left|a^{\frac{1}{r}}-b^{\frac{1}{r}}\right|^{r}\leqslant\left((1-t)a^{\frac{2}{r}}+tb^{\frac{2}{r}}\right)^{\frac{r}{2}},\ 0\leqslant t\leqslant 1,\ r\geqslant 2.$$

Convexity of the function $x\mapsto x^{\frac{r}{2}}$, when $r\geqslant 2$, implies

$$a^{1-t}b^{t}+(t(1-t))^{\frac{r}{2}}\left|a^{\frac{1}{r}}-b^{\frac{1}{r}}\right|^{r}\leqslant\left((1-t)a^{\frac{2}{r}}+tb^{\frac{2}{r}}\right)^{\frac{r}{2}}\leqslant(1-t)a+tb. \tag{2.3}$$

When $r\geqslant 2,\ 0<\frac{2}{r}\leqslant 1$. So, letting $p=\frac{2}{r}$ in (2.3) implies the following refinement of the well known weighted power mean inequality.

COROLLARY 2.2. *Let $a,b>0,\ 0\leqslant t\leqslant 1$ and $0\leqslant p\leqslant 1$. Then*

$$a^{1-t}b^{t}+(t(1-t))^{\frac{1}{p}}\left(a^{\frac{p}{2}}-b^{\frac{p}{2}}\right)^{\frac{2}{p}}\leqslant((1-t)a^{p}+tb^{p})^{\frac{1}{p}}\leqslant(1-t)a+tb.$$

In particular, when $t=\frac{1}{2}$, we obtain

$$\sqrt{ab}+\left(\frac{a^{\frac{p}{2}}-b^{\frac{p}{2}}}{2}\right)^{\frac{2}{p}}\leqslant\left(\frac{a^{p}+b^{p}}{2}\right)^{\frac{1}{p}}\leqslant\frac{a+b}{2}.$$

REMARK 2.3. The weighted power mean interpolates between the weighted arithmetic and the weighted geometric means. In fact for any $a,b\geqslant 0$

$$a^{1-t}b^{t}\leqslant\left((1-t)a^{\frac{1}{p}}+tb^{\frac{1}{p}}\right)^{p}\leqslant(1-t)a+tb,\ \ (0\leqslant t\leqslant 1,\ p\geqslant 1). \tag{2.4}$$

Thus, Corollary 2.2 provides a refinement of the first inequality in (2.4), while Corollary 2.1 improves the second inequality in (2.4).

2.2. The AM-GM inequality

In this subsection, we present new non trivial refinement and reverse of the simple inequality $\sqrt{ab}\leqslant\frac{a+b}{2}$, or $a\sharp b\leqslant a\nabla b$. It is worth noting that this inequality has not been refined or reversed in the literature, although the Young inequality (1.2) has been extensively studied. Also, we should compare the results presented in this new subsection with the results of the previous subsection.

THEOREM 2.2. *Let* $a,b>0$.

1. *If* $0\leqslant p\leqslant \frac{1}{2}$, *then* $\sqrt{ab}+2\left(\frac{|a^p-b^p|}{2}\right)^{\frac{1}{p}}\leqslant \frac{a+b}{2}$.

2. *If* $\frac{1}{2}\leqslant p\leqslant 1$, *then* $\sqrt{ab}+2\left(\frac{|a^p-b^p|}{2}\right)^{\frac{1}{p}}\geqslant \frac{a+b}{2}$.

The equality in (1) and (2) holds if and only if $p=1/2$ *or* $a=b$.

Proof. Without loss of generality, we may assume $a>b$. Let

$$f(p)=\left(\frac{a^p-b^p}{2}\right)^{\frac{1}{p}}.$$

Then

$$\log f(p)=\frac{\log(a^p-b^p)-\log 2}{p}:=g(p).$$

Direct calculations show that $g'(p)=\frac{h(p)}{p^2}$, where

$$h(p)=\log 2+\frac{p(a^p\log a-b^p\log b)}{a^p-b^p}-\log(a^p-b^p).$$

Now

$$h'(p)=-\frac{a^pb^pp(\log a-\log b)^2}{(a^p-b^p)^2}<0.$$

So, h is decreasing on $(0,\infty)$. In particular, when $0\leqslant p\leqslant 1$, we have

$$h(p)\geqslant h(1)=\frac{a\log 2-b\log 2+a\log a+(-a+b)\log(a-b)-b\log b}{a-b}.$$

Now, fix a and define

$$k(b)=a\log 2-b\log 2+a\log a+(-a+b)\log(a-b)-b\log b,\ b\leqslant a.$$

Then

$$k'(b)=\log(a-b)-\log(2b).$$

When $b=\frac{a}{3}$, $k'(b)=0$. Further, $k'(b)>0$ when $0<b<\frac{a}{3}$ and $k'(b)<0$ when $\frac{a}{3}<b<a$. Since $\lim\limits_{x\downarrow 0}x\log x=0$, we have

$$k(b)\geqslant \min\{k(0),k(a)\}=\min\{a\log 2,0\}=0,\ 0<b<a.$$

This shows that $k(b)>0$, hence

$$h(p)\geqslant h(1)\geqslant 0\Rightarrow h'(p)<0\Rightarrow h(p)\geqslant h(1),\ 0\leqslant p\leqslant 1.$$

That is, when $0\leqslant p\leqslant 1$,

$$h(p)\geqslant h(1)\Rightarrow h(p)\geqslant 0\Rightarrow g'(p)\geqslant 0.$$

This shows that f is increasing on $[0,1]$. This leads to:

1. When $0 \leqslant p \leqslant \frac{1}{2}$, $f(p) \leqslant f(1/2)$, which implies
$$\sqrt{ab}+2\left(\frac{a^p-b^p}{2}\right)^{\frac{1}{p}} \leqslant \frac{a+b}{2}.$$
2. When $\frac{1}{2} \leqslant p \leqslant 1$, $f(p) \geqslant f(1/2)$, which implies
$$\sqrt{ab}+2\left(\frac{a^p-b^p}{2}\right)^{\frac{1}{p}} \geqslant \frac{a+b}{2}.$$

This completes the proof. $\square$

The case $p=1/4$ in Theorem 2.2 reduces to the following inequality
$$\sqrt{ab}+\left[F_{1/4}(a,b)-H_{1/4}(a,b)\right] \leqslant \frac{a+b}{2},$$
where $H_v(a,b)=\frac{a^{1-v}b^v+a^vb^{1-v}}{2}$ and $F_v(a,b)=(1-v)\sqrt{ab}+v\frac{a+b}{2}$ are the Heinz mean and the Heron mean, respectively.

As for the operator inequalities for Theorem 2.2, we have the following.

COROLLARY 2.3. *Let $A,B \in \mathscr{B}(\mathscr{H})$ be positive operators such that $A>B$.*

(1) If $0 \leqslant p \leqslant \frac{1}{2}$, then
$$A\sharp B+2^{1-\frac{1}{p}}A\sharp_{\frac{1}{p}}(A-A\sharp_p B) \leqslant A\nabla B.$$
(2) If $\frac{1}{2} \leqslant p \leqslant 1$, then
$$A\sharp B+2^{1-\frac{1}{p}}A\sharp_{\frac{1}{p}}(A-A\sharp_p B) \geqslant A\nabla B.$$

Proof. For the first inequality, we have
$$\sqrt{t}+2\left(\frac{1-t^p}{2}\right)^{\frac{1}{p}} \leqslant \frac{1+t}{2}, \quad (0<t<1).$$
Applying functional calculus with $t:=A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ (then $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}<I$ from the assumption $A>B$) and multiplying $A^{\frac{1}{2}}$ from both sides, we get
$$A\sharp B+2^{1-\frac{1}{p}}A^{\frac{1}{2}}\left(I-\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p\right)^{\frac{1}{p}}A^{\frac{1}{2}} \leqslant \frac{A+B}{2}.$$
This implies the first inequality, since we have
$$A^{\frac{1}{2}}\left(I-\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^p\right)^{\frac{1}{p}}A^{\frac{1}{2}}=A^{\frac{1}{2}}\left\{A^{-\frac{1}{2}}(A-A\sharp_p B)A^{-\frac{1}{2}}\right\}^{\frac{1}{p}}A^{\frac{1}{2}}=A\sharp_{\frac{1}{p}}(A-A\sharp_p B).$$
Note that the assumption $A>B>0$ assures $A-A\sharp_p B>0$. A Similar argument shows the second inequality. $\square$

2.3. The Heinz inequality

In this subsection, we tickle the Heinz inequality and related applications. The first application is the following refinement and reverse of the Cauchy-Schwartz inequality.

THEOREM 2.3. *Let* a_i, b_i $(1 \leqslant i \leqslant n)$ *be positive numbers.*
(I) If $0 \leqslant v \leqslant 1$*, then*

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leqslant \frac{1}{4}\left[\left(\frac{\sum_{i=1}^n b_i^2}{\sum_{i=1}^n a_i^2}\right)^{\frac{1-2v}{2}} \sum_{i=1}^n \left(a_i^{1-v} b_i^v\right)^2 + \left(\frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n b_i^2}\right)^{\frac{1-2v}{2}} \sum_{i=1}^n \left(a_i^v b_i^{1-v}\right)^2\right]^2$$
$$\leqslant \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right).$$

(II) If $v \geqslant 1$ *or* $v \leqslant 0$*, then*

$$\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right) \leqslant \frac{1}{4}\left[\left(\frac{\sum_{i=1}^n b_i^2}{\sum_{i=1}^n a_i^2}\right)^{\frac{1-2v}{2}} \sum_{i=1}^n a_i^{2(1-v)} b_i^{2v} + \left(\frac{\sum_{i=1}^n a_i^2}{\sum_{i=1}^n b_i^2}\right)^{\frac{1-2v}{2}} \sum_{i=1}^n a_i^{2v} b_i^{2(1-v)}\right]^2.$$

Proof. Assume first that $\sum_{i=1}^n a_i^2 = \sum_{i=1}^n b_i^2 = 1$. For $v \in \mathbb{R}$, define the function

$$f(v) = \frac{1}{2}\sum_{i=1}^n \left(a_i^{2v} b_i^{2(1-v)} + a_i^{2(1-v)} b_i^{2v}\right).$$

Each summand is convex, hence f is convex. Further, f is symmetric about $v = \frac{1}{2}$. This means that for $0 \leqslant v \leqslant 1$, one has $f\left(\frac{1}{2}\right) \leqslant f(v) \leqslant f(1)$, which implies

$$\sum_{i=1}^n a_i b_i \leqslant \frac{1}{2}\sum_{i=1}^n \left(a_i^{2v} b_i^{2(1-v)} + a_i^{2(1-v)} b_i^{2v}\right) \leqslant 1.$$

This proves the first result (I) when $\sum_{i=1}^n a_i^2 = \sum_{i=1}^n b_i^2 = 1$. Now for the general case, replace a_i and b_i by $\frac{a_i}{\left(\sum_{i=1}^n a_i^2\right)^{1/2}}$ and $\frac{b_i}{\left(\sum_{i=1}^n b_i^2\right)^{1/2}}$, respectively, to obtain the desired inequalities.

The inequality in (II) follows from the fact that $f(1) \leqslant f(v)$ when $v \geqslant 1$ or $v \leqslant 0$. $\square$

For the rest of this subsection, we discuss the Heinz inequality in a more general setting, motivated by the observation that (1.3) can be looked at as

$$a\sharp b \leqslant \frac{a\sharp_t b + a\sharp_{1-t} b}{2} \leqslant a\nabla b,\ 0 \leqslant t \leqslant 1,\ a,b > 0$$

where $\sharp$ and ∇ stand for the geometric and arithmetic means, respectively, defined as

$$a\sharp_t b = a^{1-t} b^t,\ a\nabla_t b = (1-t)a + tb.$$

When $t=\frac{1}{2}$, it is customary to write $a\sharp b$ and $a\nabla b$. To introduce the new study, we need to recall some terminologies. The operator Heinz version of (1.3) asserts that when $A,B\in\mathscr{B}(\mathscr{H})$ are positive,

$$A\sharp B\leqslant (A\sharp_t B)\nabla(A\sharp_{1-t}B)\leqslant A\nabla B,\ 0\leqslant t\leqslant 1. \tag{2.5}$$

Our motivation begins with (2.5), where we look for possible natural extensions of this inequality for other operator means. For example, one may consider the similar inequality

$$A!B\leqslant (A!_v B)\sharp(B!_v A)\leqslant A\sharp B. \tag{2.6}$$

The scalar case of this inequality can be shown easily as follows.

PROPOSITION 2.1. *Let $a,b>0$ and $v\in[0,1]$. Then*

$$a!b\leqslant (a!_v b)\sharp(b!_v a)\leqslant a\sharp b. \tag{2.7}$$

Proof. The inequalities (2.7) are equivalent to

$$\frac{2t}{1+t}\leqslant\sqrt{\left(1-v+\frac{v}{t}\right)^{-1}\left(\frac{1-v}{t}+v\right)^{-1}}\leqslant\sqrt{t},\quad (t>0). \tag{2.8}$$

The second inequality in (2.8) is equivalent to

$$\left(1-v+\frac{v}{t}\right)\left(\frac{1-v}{t}+v\right)\geqslant\frac{1}{t},\quad (t>0).$$

Multiplying $t^2>0$ to the both sides in the above inequality, we have

$$\{(1-v)t+v\}\{(1-v)+vt\}\geqslant t,\quad (t>0)$$

which is equivalent to

$$v(1-v)(t-1)^2\geqslant 0$$

by simple calculations. Thus the second inequality in (2.8) is true.

Similarly, the first inequality in (2.8) is equivalent to

$$\frac{(1+t)^2}{4t^2}\geqslant\left((1-v)+\frac{v}{t}\right)\left(\frac{1-v}{t}+v\right),\quad (t>0)$$

which is equivalent to

$$(2v-1)^2(t-1)^2\geqslant 0$$

by simple calculations. Thus the first inequality in (2.8) is also true. □

Now we present the generalized operator version of the Heinz-type inequalities (2.5) and (2.6).

THEOREM 2.4. *Assume that σ_p and τ_p are interpolational paths for the symmetric operator means σ and τ, respectively, with $\sigma \leqslant \tau$. Then*

$$A\sigma B \leqslant (A\sigma_p B)\,\tau\,(A\sigma_{1-p}B) \leqslant A\tau B, \ \ (0 \leqslant p \leqslant 1).$$

Proof. We have

$$\begin{aligned} A\sigma B &= \left(A\sigma_{\frac{1-t}{2}}B\right)\sigma\left(A\sigma_{\frac{1+t}{2}}B\right) \leqslant \left(A\sigma_{\frac{1-t}{2}}B\right)\tau\left(A\sigma_{\frac{1+t}{2}}B\right) \\ &\leqslant \left(A\tau_{\frac{1-t}{2}}B\right)\tau\left(A\tau_{\frac{1+t}{2}}B\right) = A\tau B, \end{aligned}$$

where the second inequality follows from the following fact

$$\sigma \leqslant \tau \ \Rightarrow \sigma_p \leqslant \tau_p \quad (0 \leqslant p \leqslant 1).$$

Interchange $(1-t)/2$ and p where $0 \leqslant p \leqslant 1/2$, to get

$$A\sigma B \leqslant (A\sigma_p B)\,\tau\,(A\sigma_{1-p}B) \leqslant A\tau B$$

as desired, for $\leqslant p \leqslant 1/2$.

For $1/2 \leqslant p \leqslant 1$, we replace p by $1-p$, and the proof is complete. $\square$

Related to our study, though in a different scope, we note that the Heinz inequality is not valid if we replace the symmetric means with weighted means. In the following observation, we present conditions on a, b, and v ensuring that this inequality remains valid.

THEOREM 2.5. *Let $a,b>0$ and $v\in[0,1]$. If we have the condition (i) $b \geqslant a$ and $v\in[0,1/2]$ or (ii) $b \leqslant a$ and $v \in [1/2,1]$, then the first inequality of (2.9) holds. If we also have the condition (iii) $b \geqslant a$ and $v\in[1/2,1]$ or (iv) $b\leqslant a$ and $v\in[0,1/2]$, then the second inequality of (2.9) holds.*

$$a^{1-v}b^v \leqslant (1-v)a^{1-v}b^v + va^v b^{1-v} \leqslant (1-v)a + vb. \tag{2.9}$$

Proof. The inequalities (2.9) are equivalent to the following inequalities

$$t^v \leqslant (1-v)t^v + vt^{1-v} \leqslant (1-v) + vt, \quad (t>0). \tag{2.10}$$

Since $(1-v)t^v + vt^{1-v} - t^v = vt^v(t^{1-2v}-1)$, we have the first inequality of (2.10) holds when (i) $t \geqslant 1$ and $v \in [0,1/2]$, or (ii) $0<t\leqslant 1$ and $v\in[1/2,1]$. To prove the second inequality in (2.10) under the condition (i) $t\geqslant 1$ and $v\in[0,1/2]$, or (ii) $0<t\leqslant 1$ and $v\in[1/2,1]$, we set

$$f_v(t) := (1-v) + vt - (1-v)t^v - vt^{1-v}$$

for (iii) $t \geqslant 1$ and $v\in[1/2,1]$. Then we have $f_v'(t) = v - v(1-v)(t^{-v}+t^{v-1})$ and $f_v''(t) = v(1-v)t^{-v-2}\left(vt + (1-v)t^{2v}\right) \geqslant 0$. Thus we have $f_v'(t) \geqslant f_v'(1) = v(2v-1) \geqslant 0$. Thus we have $f_v(t) \geqslant f_v(1) = 0$. We set again

$$f_v(t) := (1-v) + vt - (1-v)t^v - vt^{1-v}$$

for (iv) $0<t\leqslant 1$ and $v\in[0,1/2]$. By similar way, we have $f_v'(t) \leqslant f_v'(1) = v(2v-1) \leqslant 0$ and then $f_v(t) \geqslant f_v(1) = 0$. $\square$

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