COMPOSITE CONVEX FUNCTIONS

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Abstract. Convex functions have played a major role in the field of Mathematical inequalities. In this paper, we introduce a new concept related to convexity, which proves better estimates when the function is somehow more convex than another.

In particular, we define what we called g-composite convexity as a generalization of log-convexity. Then we prove that g-composite convex functions have better estimates in certain known inequalities like the Hermite-Hadamard inequality, super additivity of convex functions, the Majorization inequality and some means inequalities.

Strongly related to this, we define the index of convexity as a measure of "how much the function is convex".

Applications including Hilbert space operators, matrices and entropies will be presented in the end.

1. Introduction

A function $f : [a,b] \to \mathbb{R}$ is said to be convex if

$$f(w_1 x_1 + w_2 x_2) \leqslant w_1 f(x_1) + w_2 f(x_2),$$

for all $x_1, x_2 \in [a,b]$ and positive numbers w_1, w_2 satisfying $w_1 + w_2 = 1$. This is generalized by the so called Jensen's inequality in the form

$$f\left(\sum_{i=1}^{n} w_i x_i\right) \leqslant \sum_{i=1}^{n} w_i f(x_i), \quad n \in \mathbb{N}, \tag{1.1}$$

for $x_i \in [a,b]$ and $w_i > 0$ with $\sum_{i=1}^{n} w_i = 1$.

Convex functions have received a considerable attention in the literature due to their applications in many scientific fields, such as Mathematical inequalities, Mathematical analysis and Mathematical physics.

It can be seen that all known properties of convex functions follow from (1.1). Very recently, a new characterization of convex functions was given in [13], where nonlinear upper bounds of convex functions were found. In this context, we recall that the geometric meaning of a convex function is that the function is bounded above by its linear secants.

However, neither the original definition nor the Jensen inequality differentiates between two convex functions. In other words, when f_1 and f_2 are convex functions, all what the definition says is that

$$f_i(w_1x_1 + w_2x_2) \leqslant w_1f_i(x_1) + w_2f_i(x_2).$$

This does not reflect any of the many other properties of f_i. For example, if $f_1(x) = x^2$ and $f_2(x) = x^4$, then both functions are convex. Hence,

$$(w_1x_1 + w_2x_2)^2 \leqslant w_1x_1^2 + w_2x^2$$

and

$$(w_1x_1 + w_2x_2)^4 \leqslant w_1x_1^4 + w_2x^4.$$

The main goal of this article is somehow to look into "how much the convex function is convex?" For example, according to our argument, we will see that $f(x) = x^4$ is "more convex" than $f(x) = x^2$, and then to see that $f(x) = e^x$ is more convex than polynomials!

The idea we present is a simple idea, where we make a concave function operates on the convex function, then to see the result. For example, the function $f(x) = x^2$, $x > 0$ is convex. It is somehow about "how much power do we need to exert to stop convexity of f?" In this case, we know that $\sqrt{f(x)} = x$. The function x being the "least" convex function, we see that we needed a power of $\frac{1}{2}$ to stop convexity of $f(x) = x^2$, informally.

Our main target is to formalize the above paragraph! We will see that our approach generalizes the well known and useful notion of log-convexity, where a positive function f is called log-convex if the function $\log f$ is convex. It is well known that log-convex functions satisfy better bounds than convex functions. We notice here that the function $g(x) = \log x$ is a concave function that acted on f. Having $\log f$ convex made log-convex functions satisfy better results than convex functions.

Our main definition reads as follows.

DEFINITION 1.1. Let $f : J_1 \rightarrow J_2$ be a continuous function on the interval J_1 and let $g : J_2 \rightarrow J_3$ be strictly increasing and concave (resp., convex) on J_2, such that $g \circ f : J_1 \rightarrow J_3$ is convex (resp., concave). Then, f is said to be g-composite convex (resp., g-composite concave).

We should remark that the definition of k-composite convexity was defined similarly in [2], with the difference that the function k was not assumed concave. Assuming concavity of k imposes more strict convexity behavior of f, somehow, and hence, implies better estimates in general. We refer the reader to [2] for a very useful related reference. We also refer the reader to [14] for strongly related concepts.

We observe that, in our definition, we do not impose the condition that f is convex or concave. However, this follows immediately because

$$f = g^{-1}(g \circ f).$$

Indeed, when g is concave and increasing, it follows immediately that g^{-1} is convex and increasing. The assumption that $g \circ f$ is convex implies that f is the composition of an increasing convex function with another convex function, which is convex by elementary properties of convex functions.

We will show that g-composite convex functions satisfy better bounds than convex functions. However, the significance here is that we treat convex functions as g-composite convex functions, for certain g. Once this idea is established, we show Jensen-type and Hermite-Hadamard inequalities, as refinements of the well known inequalities.

As a special case, we will take the power functions $g(x) = x^{\frac{1}{r}}$, $r \geqslant 1$, to introduce the new notion of "the index of a convex function". This new convexity index aims to present a number that, somehow, measures convexity of f. As a consequence of this index, we will be able to present a new property of convex functions that happens to coincide with log-convexity. Namely, we will show that a positive convex function f satisfies

$$(f')^2 \leqslant f f'' \text{ if and only if the index of convexity of } f \text{ is } \infty,$$

as a new property of convex functions relating f, f' and f''. This property in fact is equivalent to f being log-convex. Thus, this will entail the conclusion that the index of convexity of f is ∞ if and only if it is log-convex.

Then we present some applications for Hilbert space operators and entropies. These applications include better majorization bounds, better bounds in the operator-convex super additivity results and the Jensen inner product inequality.

2. Treatment of convex functions inequalities

In this section, we present some applications of g-composite convex functions in the context of the Jensen inequality, the Hermite-Hadamard inequality and some applications to mean inequalities. Also, super additivity of convex functions will be visited. We begin with the following refinement of Jensen's inequality, whose proof is straightforward.

PROPOSITION 2.1. *Let f be a g-composite convex function on the interval J. Then f is convex and*

$$f\left(\sum_{i=1}^{n} w_i x_i\right) \leqslant g^{-1}\left(\sum_{i=1}^{n} w_i (g \circ f)(x_i)\right) \leqslant \sum_{i=1}^{n} w_i f(x_i) \tag{2.1}$$

for any $x_1, \ldots, x_n \in J$ and $0 \leqslant w_1, \ldots, w_n \leqslant 1$ with $\sum_{i=1}^{n} w_i = 1$.

Related to Proposition 2.1, we refer the reader to [7].

Clarifying g-composite convexity, we present some examples.

EXAMPLE 2.1.

(i) If we take $f(x) := -\log x$, $(0 < x \leqslant 1)$ and $g(x) := x^p$, $(x > 0, 0 \leqslant p \leqslant 1)$, then $h(x) = (-\log x)^p$ and we have

$$-\log\left(\sum_{i=1}^n w_i x_i\right) \leqslant \left(\sum_{i=1}^n w_i(-\log x_i)^p\right)^{1/p} \leqslant -\sum_{i=1}^n w_i \log x_i,$$

which implies

$$\log\prod_{i=1}^n x_i^{w_i} \leqslant \log\exp\left(-\sum_{i=1}^n w_i(-\log x_i)^p\right)^{1/p} \leqslant \log\left(\sum_{i=1}^n w_i x_i\right).$$

If we take $p = 1$, then we get

$$\prod_{i=1}^n x_i^{w_i} \leqslant \sum_{i=1}^n w_i x_i.$$

(ii) If we take $f(x) := \exp(x)$ and $g(x) := x^p$, $(0 \leqslant p \leqslant 1)$, then $(g \circ f)(x) = \exp(px)$, and we have

$$\exp\left(\sum_{i=1}^n w_i x_i\right) \leqslant \left(\sum_{i=1}^n w_i\exp(px_i)\right)^{1/p} \leqslant \sum_{i=1}^n w_i\exp(x_i),$$

which improves the inequality given in (i).

(iii) If we take $f(x) := x^p$, $(x > 0, p \leqslant 0)$ and $g(x) := \log x$, $(x > 0)$, then $(g \circ f)(x) = p\log x$, and we have

$$\left(\sum_{i=1}^n w_i x_i\right)^p \leqslant \prod_{i=1}^n x_i^{pw_i} \leqslant \sum_{i=1}^n w_i x_i^p.$$

We make some space in the following example for the celebrated Young's inequality. Recall that if $a, b > 0$ and $0 \leqslant t \leqslant 1$, then Young's inequality states that

$$a^{1-t}b^t \leqslant (1-t)a + tb. \tag{2.2}$$

This inequality has attracted numerous researchers due to its applications in operator theory and functional analysis, in general. In the following, we present refinements of this inequality using our idea about g-composite convexity. Although the first part of the proposition is well known, we present it as an application of g-composite convexity.

PROPOSITION 2.2. *Let $a, b > 0$ and $0 \leqslant t \leqslant 1$.*

- *If $0 \leqslant p \leqslant 1$, then (2.2) can be refined as*

$$a^{1-t}b^t \leqslant \{(1-t)a^p + tb^p\}^{\frac{1}{p}} \leqslant (1-t)a + tb. \tag{2.3}$$

- *We also have for $0 \leqslant p \leqslant 1$,*

$$\sqrt{ab} \leqslant H_t^{1/p}(a^p, b^p) \leqslant H_t(a,b), \tag{2.4}$$

where $H_t(a,b) := \dfrac{a^{1-t}b^t + a^t b^{1-t}}{2}$ is the Heinz mean.

Proof. Let $f(t) = a^{1-t}b^t$ and $g(t) = t^p$, $(0 \leqslant p \leqslant 1)$. Then, g is increasing concave and $g \circ f = a^{p(1-t)}b^{pt}$ is convex, since we have

$$(g \circ f)''(t) = a^{p(1-t)}b^{pt}p^2 (\log a - \log b)^2 \geqslant 0.$$

Applying Proposition 2.1, with $n = 2$, $w_1 = t$, $x_1 = 1$, $w_2 = 1 - t$ and $x_2 = 0$ implies (2.3).

In the similar setting such as $f(t) = a^{1-t}b^t$ and $g(t) = t^p$, $(0 \leqslant p \leqslant 1)$ with $n = 2$, $w_1 = w_2 = \dfrac{1}{2}$, $x_1 = t$ and $x_2 = 1 - t$ in Proposition 2.1, we have (2.4). $\square$

Notice that the inequality (2.3) is the well known power mean inequality. Thus, we have obtained this celebrated inequality as a special case of our general argument. We note that $\lim\limits_{p \to 0} H_t^{1/p}(a^p, b^p) = \sqrt{ab}$ and $H_t^{1/p}(a^p, b^p) = H_t(a,b)$ when $p = 1$.

On the other hand, g-composite convex functions satisfy better super additivity inequalities. Recall that a convex function $f : [0,a] \to \mathbb{R}$ with $f(0) \leqslant 0$, satisfies

$$f(x) + f(y) \leqslant f(x+y), \quad x,y \in [0,a].$$

The following result presents a better bound for g-composite convex functions.

PROPOSITION 2.3. *Let f be a g-composite convex function on the interval $J := [0,a], a > 0$, with $(g \circ f)(0) \leqslant 0$ and $g(0) \geqslant 0$. Then*

$$f(x) + f(y) \leqslant g^{-1}((g \circ f)(x) + (g \circ f)(y)) \leqslant f(x+y),$$

for any $x,y \in J$.

Proof. Since $h = g \circ f$ is a convex function with $(g \circ f)(0) \leqslant 0$, we have for any $x,y \in J$,

$$(g \circ f)(x) + (g \circ f)(y) \leqslant (g \circ f)(x+y).$$

Since g^{-1} is increasing and convex with $g(0) \geqslant 0$, we have $g^{-1}(0) \leqslant 0$ and then have

$$\begin{aligned}
f(x) + f(y) &= g^{-1}((g \circ f)(x)) + g^{-1}((g \circ f)(y)) \\
&\leqslant g^{-1}((g \circ f)(x) + (g \circ f)(y)) \\
&\leqslant g^{-1}((g \circ f)(x+y)) \\
&= f(x+y).
\end{aligned}$$

This completes the proof of the proposition. □

Our next target is improving the Hermite-Hadamard inequality for g-composite convex functions. We observe that g-composite convex functions satisfy better bounds in the Hermite-Hadamrd inequality than mere convex functions.

THEOREM 2.1. *Let f be a g-composite convex function on the interval J. Then for $a < b$ in J,*

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(z)\,dz$$
$$\leqslant \int_a^b g^{-1}\left(\frac{z-a}{b-a}h(a)+\frac{b-z}{b-a}h(b)\right)dz$$
$$\leqslant \frac{f(a)+f(b)}{2},$$

where $h = g \circ f$.

Proof. On account of Proposition 2.1, it follows that

$$f((1-v)x+vy) \leqslant g^{-1}((1-v)(g\circ f)(x)+v(g\circ f)(y))$$
$$\leqslant (1-v)f(x)+vf(y). \tag{2.5}$$

Now, suppose $z \in [a,b]$. If we substitute $x = a$, $y = b$, and $1-v = (b-z)/(b-a)$ in (2.5), we get

$$f(z) \leqslant g^{-1}\left(\frac{b-z}{b-a}h(a)+\frac{z-a}{b-a}h(b)\right) \leqslant \frac{b-z}{b-a}f(a)+\frac{z-a}{b-a}f(b). \tag{2.6}$$

Since $z \in [a,b]$, it follows that $b+a-z \in [a,b]$. Now, applying the inequality (2.6) and changing the variable z to $b+a-z$, we get

$$f(b+a-z) \leqslant g^{-1}\left(\frac{z-a}{b-a}h(a)+\frac{b-z}{b-a}h(b)\right) \leqslant \frac{z-a}{b-a}f(a)+\frac{b-z}{b-a}f(b). \tag{2.7}$$

By adding inequalities (2.6) and (2.7), we infer that

$$f(b+a-z)+f(z) \leqslant g^{-1}\left(\frac{z-a}{b-a}h(a)+\frac{b-z}{b-a}h(b)\right)+\left(\frac{b-z}{b-a}h(a)+\frac{z-a}{b-a}h(b)\right)$$
$$\leqslant \frac{z-a}{b-a}f(a)+\frac{b-z}{b-a}f(b)+\frac{b-z}{b-a}f(a)+\frac{z-a}{b-a}f(b)$$
$$= f(b)+f(a)$$

which, in turn, leads to

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{a+b-z+z}{2}\right)$$

$$\leqslant \frac{f(a+b-z)+f(z)}{2}$$

$$\leqslant \frac{1}{2}\left(g^{-1}\left(\frac{z-a}{b-a}h(a)+\frac{b-z}{b-a}h(b)\right)+g^{-1}\left(\frac{b-z}{b-a}h(a)+\frac{z-a}{b-a}h(b)\right)\right)$$

$$\leqslant \frac{f(a)+f(b)}{2}.$$

(2.8)

Now, the result follows by integrating the inequality (2.8) over $z \in [a,b]$, and using the fact that $\int_a^b f(z)\,dz = \int_a^b f(a+b-z)\,dz$. □

With the same approach, we can provide another refinement of Hermite-Hadamard inequality.

THEOREM 2.2. *Let f be a g-composite convex function on the interval J. Then for $a < b$ in J,*

$$f\left(\frac{a+b}{2}\right) \leqslant \int_0^1 g^{-1}\left(\frac{h((1-v)a+vb)+h((1-v)b+va)}{2}\right)dv$$

$$\leqslant \int_0^1 f((1-v)a+vb)\,dv$$

$$\leqslant \frac{1}{2}\left(\int_0^1 g^{-1}((1-v)h(a)+vh(b))\,dv + \int_0^1 g^{-1}((1-v)h(b)+vh(a))\,dv\right)$$

$$\leqslant \frac{f(a)+f(b)}{2},$$

where $h = g \circ f$.

Proof. The inequality (2.5) implies that

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{(1-v)a+vb+(1-v)b+va}{2}\right)$$

$$\leqslant g^{-1}\left(\frac{h((1-v)a+vb)+h((1-v)b+va)}{2}\right)$$

$$\leqslant \frac{f((1-v)a+vb)+f((1-v)b+va)}{2}$$

$$\leqslant \frac{g^{-1}((1-v)h(a)+vh(b))+g^{-1}((1-v)h(b)+vh(a))}{2}$$

$$\leqslant \frac{(1-v)f(a)+vf(b)+(1-v)f(b)+vf(a)}{2}$$

$$= \frac{f(a)+f(b)}{2}.$$

(2.9)

Now, the result follows by integrating the inequality (2.9) over $v \in [a,b]$. □

3. Index of convexity

In this section, we define the index of convexity as a positive real number that, somehow, measures how convex the function is. According to this definition, we will see that a function with larger index of convexity is more convex. This definition is motivated by our earlier discussion of convexity of $g \circ f$. So, if we select $g(x) = x^{\frac{1}{r}}$, $r \geqslant 1$, we reach the following definition.

DEFINITION 3.1. Let $f : (a,b) \to (0,\infty)$ be a convex function. With f, we associate a set of real numbers called *the set of convex exponents of f* defined by

$$C_{exp}(f) = \{r \geqslant 1 : (f(x))^{\frac{1}{r}} \text{ is convex}\}.$$

The index of convexity of f is then defined by

$$I_{conv}(f) = \sup_r C_{exp}(f).$$

EXAMPLE 3.1. It can be easily seen that the power function $f(x) = x^r$, $r \geqslant 1$ has index of convexity $I_{conv}(f) = r$.

On the other hand, if $f(x) = e^x$, then

$$C_{exp}(f) = [1,\infty) \text{ and } I_{conv}(f) = \infty.$$

Moreover, the function $f(x) = x^{-1}$, defined on $(0,\infty)$ satisfies

$$C_{exp}(f) = [1,\infty) \text{ and } I_{conv}(f) = \infty.$$

Further, the function $f(x) = \tan x$ is convex on $(0, \pi/2)$, with index of convexity 1.

We show some properties of those newly defined concepts.

PROPOSITION 3.1. *Let $f : (a,b) \to (0,\infty)$ be a given convex function. Then C_{exp} is an interval.*

Proof. We first prove that if for some $r > 1$, the function $k_r(x) := (f(x))^{\frac{1}{r}}$ is concave, then so is $k_{r'}$ for any $r' > r$. Indeed, assuming concavity of k_r, we have, for $\alpha, \beta > 0$ with $\alpha + \beta = 1$,

$$\begin{aligned}
k_{r'}(\alpha x + \beta y) &= (k_r(\alpha x + \beta y))^{\frac{r}{r'}} \\
&\geqslant (\alpha k_r(x) + \beta k_r(y))^{\frac{r}{r'}} \text{ (by concavity of } k_r) \\
&\geqslant \alpha (k_r(x))^{\frac{r}{r'}} + \beta (k_r(y))^{\frac{r}{r'}} \text{ (by concavity of } t \mapsto t^{\frac{r}{r'}}) \\
&= \alpha k_{r'}(x) + \beta k_{r'}(y).
\end{aligned}$$

This shows that if $r \notin C_{exp}(f)$, then $r' \notin C_{exp}(f)$ for all $r' > r$.

Now, if $I_{conv}(f) = \infty$, then $C_{exp}(f) = [1, \infty)$. If not, there would be an $r > 1$ such that $r \notin C_{exp}(f)$, which then implies $r' \notin C_{exp}(f)$ for all $r' > r$, which implies that $C_{exp}(f) \subseteq [1, r)$, and hence $I_{conv}(f) \leqslant r$, contradicting the assumption that $I_{conv}(f) = \infty$.

On the other hand, if $I_{conv}(f) < \infty$, then a similar argument implies that $C_{exp}(f) = [1, I_{conv}(f)]$.

Thus, we have shown that for any convex f, either $C_{exp}(f) = [1, I_{conv}(f)]$ or $C_{exp}(f) = [1, \infty)$, which completes the proof. $\square$

We know that a twice differentiable convex function satisfies $f'' \geqslant 0$. In fact, it turns out that the index of convexity can be used to present a new relation between f, f' and f'' for convex functions. More precisely, we have the following.

THEOREM 3.1. *Let $f : (a, b) \to (0, \infty)$ be a twice differentiable convex function. Then*

$$I_{conv}(f) = \sup\left\{ r \geqslant 1 : \left(1 - \frac{1}{r}\right)(f'(x))^2 \leqslant f(x)f''(x), \forall x \in (a, b) \right\}.$$

In particular, $(f'(x))^2 \leqslant f(x)f''(x)$ if and only if $I_{conv}(f) = \infty$.

Proof. Let $k_r(x) = (f(x))^{\frac{1}{r}}$. Convexity of k_r implies positivity of k_r''. Direct calculus computations then imply

$$k_r'' \geqslant 0 \Leftrightarrow \left(1 - \frac{1}{r}\right)(f')^2 \leqslant ff'',$$

which implies the first assertion, by definition of $I_{conv}(f)$. The second assertion follows immediately from the first. $\square$

Therefore, the above theorem presents a necessary and sufficient condition for a convex function to satisfy $(f'(x))^2 \leqslant f(x)f''(x)$; as a new property of convex functions. In fact, the differential inequality $(f'(x))^2 \leqslant f(x)f''(x)$ is equivalent to the fact that f is log-convex. This follows by differentiating $\log f$ twice. Therefore, we reach the following conclusion about functions with index of convexity ∞.

COROLLARY 3.1. *Let $f : (a, b) \to (0, \infty)$ be a twice differentiable convex function. Then the index of convexity of f is ∞ if and only if f is log-convex.*

At this stage, it is interesting to ask about when we can have an equality in both quantities appearing in Theorem 3.1. Namely, when do we have

$$\left(1 - \frac{1}{r}\right)(f')^2 = ff'' \text{ or } (f')^2 = ff''.$$

This is nicely described next. Solving these two ordinary differential equations, we have the following.

PROPOSITION 3.2. *Let f be a twice differentiable function. Then*

- $\left(1 - \frac{1}{r}\right)(f')^2 = ff''$, $r > 1$, *if and only if*

$$f(x) = \left(\frac{c}{r}x + d\right)^r, \quad c, d \in \mathbb{R}.$$

- $(f')^2 = ff''$ *if and only if*

$$f(x) = \alpha e^{\beta x}, \quad \alpha, \beta \in \mathbb{R}.$$

In fact, simple Calculus computations lead to the following property of convex functions having index of convexity $I_{conv}(f) = \infty$, equivalently log-convex functions. This is explained in the next result.

PROPOSITION 3.3. *Let* $f : [a,b] \to (0,\infty)$ *be an increasing convex function satisfying*
$$(f')^2 \leqslant ff''.$$
Then, for certain real numbers α *and* β,

$$f(x) \geqslant \alpha e^{\beta x}.$$

Proof. Observe first that the condition that f is convex follows from the inequality $(f')^2 \leqslant ff''$. So, we may remove this from the statement of the proposition.

Now, rearranging the given inequality, we have for $x \in [a,b]$,

$$\frac{f'}{f} \leqslant \frac{f''}{f'} \Rightarrow \int_a^x \frac{f'}{f} dt \leqslant \int_a^x \frac{f''}{f'} dt.$$

Performing the integrals implies

$$\log \frac{f(x)}{f(a)} \leqslant \log \frac{f'(x)}{f'(a)} \Rightarrow \frac{f(x)}{f(a)} \leqslant \frac{f'(x)}{f'(a)}.$$

The latter inequality implies

$$\frac{f'(x)}{f(x)} \geqslant \frac{f'(a)}{f(a)} \Rightarrow \log \frac{f(x)}{f(a)} \geqslant \frac{f'(a)}{f(a)}(x-a).$$

This implies that
$$f(x) \geqslant f(a) \exp\left(\frac{f'(a)}{f(a)}(x-a)\right),$$
which implies the desired conclusion. $\square$

Combining Theorem 3.1 with Propositions 3.2 and 3.3 implies the following observation.

COROLLARY 3.2. *Let* $f : [a,b] \to (0,\infty)$ *be an increasing convex function. If* $I_{conv}(f) = \infty$ *(equivalently if* f *is log-convex), then* $f(x) \geqslant \alpha e^{\beta x}$, *for some positive real numbers* α *and* β.

At this point, it is worth looking at the function $f(x) = x^{-1}$, $[1,\infty)$. This function satisfies $I_{conv}(f) = \infty$, however it is not increasing! Therefore, it does not satisfy the conclusion of the above corollary.

4. Applications to Hilbert space operators

In this section we study operator inequalities for a composite function of two functions. We remind the reader, first, of some terminologies and notations. Let $\mathscr{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators acting on a Hilbert space $\mathscr{H}$. When $\mathscr{H}$ is finite dimensional, say of dimension n, the algebra $\mathscr{B}(\mathscr{H})$ is identified with the algebra of all complex $n \times n$ matrices, denoted by $\mathscr{M}_n$. A real function f defined on an interval J is said to be operator monotone if $f(A) \geqslant f(B)$ whenever $A, B \in \mathscr{B}(\mathscr{H})$ are self adjoint operators (or Hermitian matrices) such that $A \geqslant B$, with spectra in J. In this context, we write $A \geqslant B$ if $A - B$ is a positive operator. That is, if $\langle Ax, x \rangle \geqslant \langle Bx, x \rangle$ for all vectors $x \in \mathscr{H}$. On the other hand, f will be called an operator convex function if for any pair of self adjoint operators $A, B \in \mathscr{B}(\mathscr{H})$ and any $t \in [0,1]$, we have the convex inequality $f((1-t)A + tB) \leqslant (1-t)f(A) + tf(B)$. Operator concave functions are defined similarly.

Firstly, we consider two continuous positive functions f and g defined on $(0,\infty)$. If f and g are operator monotone functions, then the composite function $g \circ f$ is clearly operator monotone. For slightly different conditions on f and g, we have the following theorem.

THEOREM 4.1. *Let* $f : [0,\infty) \to [0,\infty)$ *be a real-valued continuous function. If* $g : [0,\infty) \to [0,\infty)$ *is increasing operator convex such that* $g \circ f$ *is operator concave, then* f *is operator concave. In particular, if* $A, B \in \mathscr{B}(\mathscr{H})$ *are two positive operators then*

$$f((1-v)A + vB) \geqslant g^{-1}((1-v)(g \circ f)(A) + v(g \circ f)(B))$$
$$\geqslant (1-v)f(A) + vf(B).$$

Proof. It follows from the operator concavity of $g \circ f$ that

$$(g \circ f)((1-v)A + vB) \geqslant (1-v)(g \circ f)(A) + v(g \circ f)(B).$$

On the other hand, it is shown in [11, Proposition 2.3] that if g is an increasing operator convex function on $[0,\infty)$, then g^{-1} is operator monotone on $[0,\infty)$. Thus,

$$f((1-v)A + vB) \geqslant g^{-1}((1-v)(g \circ f)(A) + v(g \circ f)(B))$$
$$\geqslant (1-v)f(A) + vf(B),$$

where the second inequality follows from the fact that a function h is operator monotone on a half-line $[0,\infty)$ if and only if h is operator concave [1, Theorem 2.3]. $\square$

PROPOSITION 4.1. *Let f be g-composite convex and let $A \in \mathscr{B}(\mathscr{H})$ be self adjoint. If $x \in \mathscr{H}$ is a unit vector, then*

$$f(\langle Ax, x \rangle) \leqslant g^{-1}(\langle (g \circ f)(A)x, x \rangle) \leqslant \langle f(A)x, x \rangle.$$

Proof. Since f is g-composite convex, we have

$$(g \circ f)(\langle Ax, x \rangle) \leqslant \langle (g \circ f)(A)x, x \rangle \leqslant g(\langle f(A)x, x \rangle),$$

which implies the desired result, upon applying g^{-1} to the above inequalities. $\square$

Let $\mathscr{M}_n$ denote the C^*-algebra of $n \times n$ complex matrices with identity I and let $\mathscr{H}_n$ be the set of all Hermitian matrices in $\mathscr{M}_n$. We denote by $\mathscr{H}_n(J)$ the set of all Hermitian matrices in $\mathscr{M}_n$ whose spectra are contained in an interval $J \subseteq \mathbb{R}$. The notation $\prec_w$ will be used to denote weak majorization, while $\lambda(A)$ will denote the eigenvalues vector of the Hermitian matrix A, arranged in a decreasing order.

THEOREM 4.2. *Let $A_1, \ldots, A_k \in \mathscr{H}_m(J)$, f be g-composite convex on the real interval J, and let $w_1, \ldots, w_k$ be positive scalars such that $\sum_{i=1}^k w_i = 1$. Then*

$$\lambda\left(f\left(\sum_{i=1}^k w_i A_i\right)\right) \prec_w \lambda\left(g^{-1}\left(\sum_{i=1}^k w_i(g \circ f)(A_i)\right)\right) \prec_w \lambda\left(\left(\sum_{i=1}^k w_i f(A_i)\right)\right).$$

Proof. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\sum_{i=1}^k w_i A_i$ and let $x_1, \ldots, x_n$ be the corresponding orthonormal eigenvectors arranged such that $f(\lambda_1) \geqslant \ldots \geqslant f(\lambda_n)$. Therefore, for $1 \leqslant l \leqslant n$,

$$\sum_{\ell=1}^l \lambda_\ell\left(f\left(\sum_{i=1}^k w_i A_i\right)\right) = \sum_{\ell=1}^l f\left(\left\langle \sum_{i=1}^k w_i A_i x_\ell, x_\ell \right\rangle\right)$$

$$\leqslant \sum_{\ell=1}^l g^{-1}\left(\sum_{i=1}^k w_i(g \circ f)(\langle A_i x_\ell, x_\ell \rangle)\right) \quad \text{(by (2.1))}$$

$$\leqslant \sum_{\ell=1}^l g^{-1}\left(\sum_{i=1}^k w_i(\langle (g \circ f)(A_i)x_\ell, x_\ell \rangle)\right)$$

(since $g \circ f$ is convex and g^{-1} is increasing)

$$= \sum_{\ell=1}^l g^{-1}\left(\left\langle \sum_{i=1}^k w_i(g \circ f)(A_i)x_\ell, x_\ell \right\rangle\right)$$

$$\leqslant \sum_{\ell=1}^l \left\langle g^{-1}\left(\sum_{i=1}^k w_i(g \circ f)(A_i)\right)x_\ell, x_\ell \right\rangle$$

(since g^{-1} is convex)

$$\leqslant \sum_{\ell=1}^l \lambda_\ell\left(g^{-1}\left(\sum_{i=1}^k w_i(g \circ f)(A_i)\right)\right).$$

Therefore,

$$\lambda\left(f\left(\sum_{i=1}^{k}w_iA_i\right)\right) \prec_w \lambda\left(g^{-1}\left(\sum_{i=1}^{k}w_i(g\circ f)(A_i)\right)\right). \tag{4.1}$$

On the other hand, by [8, Remark 2.1 (ii)]

$$\lambda\left(g^{-1}\left(\sum_{i=1}^{k}w_i(g\circ f)(A_i)\right)\right) \prec_w \lambda\left(\left(\sum_{i=1}^{k}w_ig^{-1}((g\circ f)(A_i))\right)\right)$$

$$= \lambda\left(\left(\sum_{i=1}^{k}w_if(A_i)\right)\right). \tag{4.2}$$

Combining (2.1) and (4.2), we infer that

$$\lambda\left(f\left(\sum_{i=1}^{k}w_iA_i\right)\right) \prec_w \lambda\left(g^{-1}\left(\sum_{i=1}^{k}w_i(g\circ f)(A_i)\right)\right)$$

$$\prec_w \lambda\left(\left(\sum_{i=1}^{k}w_if(A_i)\right)\right).$$

This completes the proof of the theorem. □

As a direct consequence of Theorem 4.2, we have the following result:

COROLLARY 4.1. *Let* $A_1,\ldots,A_k \in \mathcal{H}_m(J)$, *and let* $w_1,\ldots,w_k$ *be positive scalars such that* $\sum_{i=1}^{k}w_i = 1$. *Then for any* $r \geqslant 2$,

$$\lambda\left(\left(\sum_{i=1}^{k}w_iA_i\right)^r\right) \prec_w \frac{1}{2}\lambda\left(2\sum_{i=1}^{k}w_i\left(A_i^r+A_i^{\frac{r}{2}}\right)+I-\sqrt{4\sum_{i=1}^{k}w_i\left(A_i^r+A_i^{\frac{r}{2}}\right)+I}\right)$$

$$\prec_w \lambda\left(\sum_{i=1}^{k}w_iA_i^r\right).$$

Proof. Letting $g(x)=x+\sqrt{x}$ on $[0,\infty)$. Then $g'(x)=\frac{1}{2\sqrt{x}}+1\geqslant 0$ and $g''(x)=-\frac{1}{4\sqrt{x^3}}\leqslant 0$. Thus g is increasing and concave. Put $f(x)=x^r$ $(r\geqslant 2)$ on $[0,\infty)$. Therefore, $g(f(x))=x^r+x^{\frac{r}{2}}$ and $g''(f(x))=\frac{r\left((4r-4)x^r+(r-2)x^{\frac{r}{2}}\right)}{4x^2}\geqslant 0$, namely $g(f(x))$ is a convex function. Since $g^{-1}(x)=\frac{2x+1-\sqrt{4x+1}}{2}$, we get the desired result. □

We give an example to clarify the situation in Corollary 4.1.

EXAMPLE 4.1. Letting $k=2$, $A_1=\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}$, $A_2=\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$, $w_1=w_2=1/2$, and $r=2$. A simple calculation shows that

$$\lambda\left(\left(\frac{A_1+A_2}{2}\right)^2\right)=\lambda\left(\begin{bmatrix}4 & 0\\0 & \frac{9}{4}\end{bmatrix}\right)\approx\{4,2.25\},$$

$$\lambda \left(\frac{A_1^2 + A_1 + I - \sqrt{2\left(A_1^2 + A_1\right) + I} + A_2^2 + A_2 + I - \sqrt{2\left(A_2^2 + A_2\right) + I}}{2} \right)$$

$$\approx \lambda \left(\begin{bmatrix} 4.3771 & 0.487 \\ 0.487 & 3.0551 \end{bmatrix} \right) \approx \{4.5371, 2.895\},$$

and

$$\lambda \left(\frac{A_1^2 + A_2^2}{2} \right) = \lambda \left(\begin{bmatrix} 5 & 0.5 \\ 0.5 & 3.5 \end{bmatrix} \right) \approx \{5.1513, 3.3486\},$$

that is, we have

$$\{4, 2.25\} \prec_w \{4.5371, 2.895\} \prec_w \{5.1513, 3.3486\}.$$

Kosem [5] proved that if $k : (0, \infty) \to \mathbb{R}$ is a convex (resp. concave) function with $k(0) = 0$, then

$$\|k(A) + k(B)\| \leqslant (\text{resp.} \geqslant) \|k(A+B)\|,$$

for positive matrices $A, B \in \mathcal{M}_n$. It turns out that g-composite convex functions satisfy better bounds, as follows.

THEOREM 4.3. *Let $A, B \in \mathcal{M}_n$ be positive and let f be a g-composite convex function on the interval $[0, \infty)$, with $(g \circ f)(0) \leqslant 0$ and $g(0) \geqslant 0$. Then*

$$\|f(A) + f(B)\| \leqslant \left\| g^{-1}(g \circ f(A) + g \circ f(B)) \right\| \leqslant \|f(A+B)\|.$$

Proof. If f is a g-composite convex, we get

$$\|g \circ f(A) + g \circ f(B)\| \leqslant \|g \circ f(A+B)\|.$$

Since g is increasing and concave, we infer that

$$\|g(f(A) + f(B))\| \leqslant \|g \circ f(A) + g \circ f(B)\| \leqslant \|g \circ f(A+B)\|.$$

Now, applying g^{-1}, to get

$$\|f(A) + f(B)\| \leqslant \left\| g^{-1}(g \circ f(A) + g \circ f(B)) \right\| \leqslant \|f(A+B)\|.$$

This completes the proof of the theorem. □

5. Some applications to entropies

In this section, we give a new lower bound of quantum relative entropy as an application in this topic. In quantum information theory [9, 10], the quantum entropy (von Neumann entropy) [17] defined by $S(\rho) := -Tr[\rho \log \rho]$ for a density operator ρ, is an important quantity. A density operator is a self adjoint positive operator with unit trace. The quantum relative entropy [16] is also important quantity and it is defined by

$$D(\rho|\sigma) := Tr[\rho(\log \rho - \log \sigma)]$$

for two density operators ρ and σ. It is known the non-negativity of quantum relative entropy, $D(\rho|\sigma) \geqslant 0$. Our lower bound modify this in the following theorem. To show our theorem we give the following lemma.

LEMMA 5.1.

(i) *If f is a g-composite convex, then we have*

$$f(a) + f'(a)(b-a) \leqslant f(a) + \left(g^{-1}\right)'(h(a))(h(b) - h(a)) \leqslant f(b). \quad (5.1)$$

(ii) *If f is a g-composite concave, then we have*

$$f(a) + f'(a)(b-a) \geqslant f(a) + \left(g^{-1}\right)'(h(a))(h(b) - h(a)) \geqslant f(b). \quad (5.2)$$

Proof. Since clearly g^{-1} is increasing convex under the assumptions of lemma, one can check that

$$
\begin{aligned}
f((1-v)a + vb) &= g^{-1} \circ h(a + v(b-a)) \\
&\leqslant g^{-1}(h(a) + v(h(b) - h(a))) \quad \text{(convexity of h and g is increasing)} \\
&\leqslant f(a) + v(f(b) - f(a)) \quad \text{(convexity of g).}
\end{aligned}
$$

Therefore,

$$
\frac{f(a + v(b-a)) - f(a)}{v} \leqslant \frac{g^{-1}(h(a) + v(h(b) - h(a))) - g^{-1}(h(a))}{v}
$$
$$
\leqslant f(b) - f(a)
$$

Now, if $v \to 0$, we get (5.1). (ii) can be proven similarly. □

THEOREM 5.1. *For two density operators ρ and σ, we have*

$$D(\rho|\sigma) \geqslant S(\sigma) - S(\rho) + Tr[\exp(-\rho \log \rho)\exp(\sigma \log \sigma) - I] \geqslant 0. \quad (5.3)$$

Proof. We take a concave function $f(t) := -t \log t$ for $0 < t \leqslant 1$ and an increasing convex function $g(t) := \exp(t)$. Then $h(t) := g \circ f(t) = \exp(-t \log t) = t^{-t}$ is concave on $(0,1]$. Since $g \circ f''(t) = t^{-t}(1 + \log t)^2 - t^{-t-1} \leqslant 0$ for $t \in (0,1]$. To prove $g \circ f''(t) \leqslant 0$, it is sufficient to consider the function $k(t) := t(1 + \log t)^2$ on $0 < t \leqslant 1$. Then we have $k'(t) = (\log t + 1)(\log t + 3)$. We also easily find that $k'(t) \geqslant 0$ for $0 < t < e^{-3}$, $k'(t) \leqslant 0$ for $e^{-3} < t < e^{-1}$ and $k'(t) \geqslant 0$ for $e^{-1} < t \leqslant 1$. Since $k(e^{-3}) = 4e^{-3} \simeq 0.199148 < 1 = k(1)$, the function $k(t)$ take a maximum value 1 when $t = 1$ for $0 < t \leqslant 1$. Thus we have $k(t) \leqslant 1$ so that $t(1 + \log t)^2 \leqslant 1$ which proves $g \circ f''(t) \leqslant 0$. Thus we have the following inequalities by Lemma 5.1(ii)

$$f(x) - f(y) - f'(y)(x-y) \leqslant f(x) - f(y) - \left(g^{-1}\right)'(h(y))(h(x) - h(y)) \leqslant 0 \quad (5.4)$$

We take spectral decompositions $\rho = \sum_i \lambda_i P_i$ and $\sigma = \sum_j \mu_j Q_j$ with $\sum_i P_i = \sum_j Q_j = I$. Then we have the following inequalities:

$$
\begin{aligned}
-Tr[\rho(\log\rho - \log\sigma)] &= Tr[-\rho\log\rho + \sigma\log\sigma - (-\log\sigma - I)(\rho - \sigma)] \\
&= \sum_{i,j} Tr\left[P_i\left\{-\lambda_i\log\lambda_i + \mu_j\log\mu_j - (-\log\mu_j - 1)(\lambda_i - \mu_j)\right\}Q_j\right] \\
&= \sum_{i,j}\left\{-\lambda_i\log\lambda_i + \mu_j\log\mu_j - (-\log\mu_j - 1)(\lambda_i - \mu_j)\right\}Tr[P_iQ_j] \\
&\leqslant \sum_{i,j}\left\{-\lambda_i\log\lambda_i + \mu_j\log\mu_j - (\mu_j^{\mu_j})(\lambda_i^{-\lambda_i} - \mu_j^{-\mu_j})\right\}Tr[P_iQ_j] \\
&= \sum_{i,j}\left\{-\lambda_i\log\lambda_i + \mu_j\log\mu_j - \mu_j^{\mu_j}\lambda_i^{-\lambda_i} + 1\right\}Tr[P_iQ_j] \\
&\leqslant 0.
\end{aligned}
$$

The inequalities above are due to (5.4). Finally we derive

$$
\begin{aligned}
&\sum_{i,j}\left\{-\lambda_i\log\lambda_i + \mu_j\log\mu_j - \mu_j^{\mu_j}\lambda_i^{-\lambda_i} + 1\right\}Tr[P_iQ_j] \\
&= \sum_{i,j}Tr\left[P_i\left\{-\lambda_i\log\lambda_i + \mu_j\log\mu_j - \exp(\mu_j\log\mu_j)\exp(-\lambda_i\log\lambda_i) + 1\right\}Q_j\right] \\
&= Tr[-\rho\log\rho + \sigma\log\sigma - \exp(\sigma\log\sigma)\exp(-\rho\log\rho) + I],
\end{aligned}
$$

since we have $\sum_{i,j}Tr[P_i f(\lambda_i)g(\mu_j)Q_j] = Tr[\sum_i f(\lambda_i)P_i \sum_j g(\mu_j)Q_j] = Tr[f(\rho)g(\sigma)]$. Thus we have

$$
-Tr[\rho(\log\rho - \log\sigma)] \leqslant Tr[-\rho\log\rho + \sigma\log\sigma - \exp(-\rho\log\rho)\exp(\sigma\log\sigma) + I] \leqslant 0,
$$

which implies (5.3). $\square$

REMARK 5.1. The inequalities (5.3) are equivalent to

$$
Tr[(\sigma - \rho)\log\sigma] \geqslant Tr[\exp(-\rho\log\rho)\exp(\sigma\log\sigma) - I] \geqslant S(\rho) - S(\sigma).
$$

If we consider the special case $\rho = \sigma$, then both sides in the above inequalities become to 0, so that equality holds.

REMARK 5.2. From (5.3), we have the lower bound of quantum Jeffrey divergence [3]:

$$
J(\rho|\sigma) := \frac{1}{2}(D(\rho|\sigma) + D(\sigma|\rho))
$$

as

$$
J(\rho|\sigma) \geqslant \frac{1}{2}(Tr[\exp(-\rho\log\rho)\exp(\sigma\log\sigma) + \exp(\rho\log\rho)\exp(-\sigma\log\sigma) - 2I]).
$$

The following examples present the inequalities (5.3).

EXAMPLE 5.1. We take density matrices as

$$\rho := \frac{1}{7}\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}, \quad \sigma := \frac{1}{6}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

From the eigenvalue decomposition for any normal operator A, we generally have

$$A = \sum_{j=1}^{n} a_j |\psi_j\rangle\langle\psi_j|,$$

where $|\psi_j\rangle$ are orthonormal eigenvectors corresponding eigenvalues a_j.

For two density matrices ρ and σ, we have the eigenvalue decomposition as

$$\rho = \frac{6}{7}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\frac{1}{\sqrt{5}}(1 \quad 2) + \frac{1}{7}\frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}\frac{1}{\sqrt{5}}(-2 \quad 1).$$

Then we have

$$-S(\rho) = Tr[\rho\log\rho] = \frac{6}{7}\log\frac{6}{7} + \frac{1}{7}\log\frac{1}{7}.$$

Similarly we have

$$\sigma = \frac{2}{3}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}(1 \quad 1) + \frac{1}{3}\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}(-1 \quad 1).$$

Then we have

$$Tr[\rho\log\sigma] = Tr\left[\frac{1}{7}\begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} -\frac{1}{2}\log\frac{9}{2} & -\frac{1}{2}\log\frac{1}{2} \\ -\frac{1}{2}\log\frac{1}{2} & -\frac{1}{2}\log\frac{9}{2} \end{pmatrix}\right]$$

$$= \frac{1}{7}\left(\log 2 - \frac{5}{2}\log\frac{9}{2}\right) + \frac{1}{7}\left(\log 2 - \log\frac{9}{2}\right)$$

By the numerical computations, we thus have

$$D(\rho|\sigma) = Tr[\rho\log\rho] - Tr[\rho\log\sigma]$$

$$= \frac{6}{7}\log\frac{6}{7} + \frac{1}{7}\log\frac{1}{7} - \frac{1}{7}\left(\log 2 - \frac{5}{2}\log\frac{9}{2}\right) + \frac{1}{7}\left(\log 2 - \log\frac{9}{2}\right)$$

$$\simeq 0.14388.$$

Next we compute

$$S(\sigma) - S(\rho) + Tr[\exp(-\rho\log\rho)\exp(\sigma\log\sigma) - I].$$

The von Neumann entropies $S(\sigma) = -\frac{2}{3}\log\frac{2}{3} - \frac{1}{3}\log\frac{1}{3}$ and $S(\rho) = -\frac{6}{7}\log\frac{6}{7} - \frac{1}{7}\log\frac{1}{7}$ are similarly computaed as above. We also compute

$$\exp(-\rho\log\rho) = \exp(-\rho)\rho$$

$$= \left(\frac{6}{7}\right)^{-\frac{6}{7}}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\frac{1}{\sqrt{5}}(1 \quad 2) + \left(\frac{1}{7}\right)^{-\frac{1}{7}}\frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}\frac{1}{\sqrt{5}}(-2 \quad 1).$$

Similarly we have

$$\exp(\sigma \log \sigma) = \exp(\sigma)\sigma$$

$$= \left(\frac{2}{3}\right)^{\frac{2}{3}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} (1 \quad 1) + \left(\frac{1}{3}\right)^{\frac{1}{3}} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} (-1 \quad 1).$$

Therefore we compute as

$$Tr[\exp(-\rho \log \rho)\exp(\sigma \log \sigma)]$$

$$= Tr\left[\left\{\frac{\left(\frac{6}{7}\right)^{-\frac{6}{7}}}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} + \frac{\left(\frac{1}{7}\right)^{-\frac{1}{7}}}{5}\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\right\}\left\{\frac{\left(\frac{2}{3}\right)^{\frac{2}{3}}}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{\left(\frac{1}{3}\right)^{\frac{1}{3}}}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\right\}\right]$$

$$= \frac{7^{1/7}\left(6 \times 2^{2/3} + 54 \times 3^{1/3} + 9 \times 2^{17/21} \times 3^{1/2} \times 7^{5/7} + 2^{1/7} \times 3^{10/21} \times 7^{5/7}\right)}{60 \times 3^{2/3}}.$$

By the numerical computations, we thus have

$$S(\sigma) - S(\rho) + Tr[\exp(-\rho \log \rho)\exp(\sigma \log \sigma) - I]$$

$$= -\frac{2}{3}\log\frac{2}{3} - \frac{1}{3}\log\frac{1}{3} + \frac{6}{7}\log\frac{6}{7} + \frac{1}{7}\log\frac{1}{7}$$

$$+ \frac{7^{1/7}\left(6 \times 2^{2/3} + 54 \times 3^{1/3} + 9 \times 2^{17/21} \times 3^{1/2} \times 7^{5/7} + 2^{1/7} \times 3^{10/21} \times 7^{5/7}\right)}{60 \times 3^{2/3}} - 2$$

$$\simeq 0.0141518.$$

By the similar way, for the case

$$\rho := \frac{1}{6}\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad \sigma := \frac{1}{7}\begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}$$

we also have

$$D(\rho|\sigma) \simeq 0.174615$$

and

$$S(\sigma) - S(\rho) + Tr[\exp(-\rho \log \rho)\exp(\sigma \log \sigma) - I] \simeq 0.0155788.$$

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