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SINGULAR VALUE AND NORM INEQUALITIES OF DAVIDSON–POWER TYPE

WASIM AUDEH

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Abstract. Let A,B,X and Y be $n\times n$ complex matrices such that A and B are positive semidefinite, then

$$\|AX+YB\| \leqslant \frac{1}{4}\left(\|W_1\|+\|W_2\|+W_4\right),$$

where

$$W_1 = A+A^{1/2}\left|X^*\right|^2 A^{1/2},$$
$$W_2 = B+B^{1/2}\left|Y\right|^2 B^{1/2},$$
$$W_3 = A^{1/2}XB^{1/2}+A^{1/2}YB^{1/2}$$

and

$$W_4 = \sqrt{(\|W_1\|-\|W_2\|)^2+4\|W_3\|^2}.$$

Multiple results are given in this paper.

1. Introduction

Let $\mathbb{M}_n$ denote the algebra of all $n\times n$ complex matrices. For $A\in\mathbb{M}_n$, the singular values of A are the eigenvalues of $|A| = (A^*A)^{1/2}$ which are denoted by $s_1(A)\geqslant s_2(A)\geqslant\ldots\geqslant s_n(A)$, they satisfy $s_j(A)=s_j(A^*)=s_j(|A|)$ for $j=1,2,\ldots,n$. The spectral norm $\|.\|$ is defined as $\|A\|=s_1(A)$ and the Schatten p-norms $\|.\|_p$ are defined as $\|A\|_p=\left(\sum\limits_{j=1}^{n} s_j^p(A)\right)^{1/p}$ for $1\leqslant p\leqslant\infty$. The symbol $|||.|||$ will denote any unitarily invariant norm which are norms on $\mathbb{M}_n$ satisfying $|||UAV|||=|||A|||$ for all $A\in\mathbb{M}_n$ and all unitary matrices $U,\ V\in\mathbb{M}_n$, (see, e.g., [6] or [11]). It is pointed out in [9] that if $A,B\in\mathbb{M}_n$ are positive semidefinite, then

$$||A+B|| \leqslant \max\{\|A\|,\|B\|\}+\|AB\|^{1/2}. \tag{1.1}$$

Kittaneh [13] gave a refinement of inequality (1.1) so that

$$||A+B|| \leqslant \max\{\|A\|,\|B\|\}+\left\|A^{1/2}B^{1/2}\right\|. \tag{1.2}$$

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Kittaneh [14] provided an improvement of inequality (1.2) as follows

$$||A+B|| \leqslant \frac{1}{2}\left(||A||+||B||+\sqrt{(||A||-||B||)^2+4\left|\left|A^{1/2}B^{1/2}\right|\right|^2}\right). \tag{1.3}$$

It is shown in [4] that if $A,B \in \mathbb{M}_n$ are positive semidefinite, then

$$s_j(A+B) \leqslant s_j\left(\left(A+\left|B^{1/2}A^{1/2}\right|\right)\oplus\left(B+\left|A^{1/2}B^{1/2}\right|\right)\right) \tag{1.4}$$

for $j=1,2,\ldots,2n$. Norm inequalities versions of inequality (1.4) are listed below

$$||A+B|| \leqslant \max\left\{\left\|A+\left|B^{1/2}A^{1/2}\right|\right\|,\left\|B+\left|A^{1/2}B^{1/2}\right|\right\|\right\} \tag{1.5}$$

and

$$\|A+B\|_p \leqslant \left(\left\|A+\left|B^{1/2}A^{1/2}\right|\right\|_p^p+\left\|B+\left|A^{1/2}B^{1/2}\right|\right\|_p^p\right)^{1/p} \tag{1.6}$$

for $1 \leqslant p \leqslant \infty$. Zhan in [15] showed that if $A,B \in \mathbb{M}_n$ where A,B are positive semidefinite, then

$$s_j(A-B) \leqslant s_j(A\oplus B) \tag{1.7}$$

for $j=1,2,\ldots,2n$. A generalization of inequality (1.4) is given in [2], so that

$$s_j(AX+XB) \leqslant s_j(C\oplus D) \tag{1.8}$$

for $j=1,2,\ldots,2n$, where

$$C = C_1+|C_2|,$$

$$C_1 = \frac{1}{2}A+\frac{1}{2}A^{1/2}\left|X^*\right|^2A^{1/2},$$

$$C_2 = B^{1/2}X^*A^{1/2},$$

$$D = D_1+|D_2|,$$

$$D_1 = \frac{1}{2}B+\frac{1}{2}B^{1/2}\left|X\right|^2B^{1/2},$$

and

$$D_2 = A^{1/2}XB^{1/2}.$$

Audeh, in the same paper, showed that

$$s_j(AX-XB) \leqslant s_j(M\oplus N) \tag{1.9}$$

for $j=1,2,\ldots,2n$, where

$$M = \frac{1}{2}A+\frac{1}{2}A^{1/2}\left|X^*\right|^2A^{1/2}$$

and

$$N = \frac{1}{2}B+\frac{1}{2}B^{1/2}\left|X\right|^2B^{1/2}.$$

Readers interested in singular value inequalities should return to [1], [3], [4], [5] and [10]. We present a considerable generalizations of the inequalities (1.3), (1.4), (1.5), (1.6), (1.8) and (1.9).

2. Main results

We need the following lemmas. The first lemma is proved in [7], the second lemma is obtained in [6], the third lemma is provided in [8] and the fourth lemma is given in [12].

LEMMA 2.1. *Let* $A,B \in \mathbb{M}_n$. *Then*

$$s_j(AB^*) \leqslant \frac{1}{2} s_j(A^*A + B^*B)$$

for $j = 1,2,\ldots,n$.

LEMMA 2.2. *Let* $A \in \mathbb{M}_n$ *be positive semidefinite and let* f *be a non-negative increaing function on* $[0,\infty)$. *Then*

$$s_j(f(A)) = f(s_j(A))$$

for $j = 1,2,\ldots,n$.

LEMMA 2.3. *Let* $A,B \in \mathbb{M}_n$ *be normal and let* f *be a nonnegative concave function on* $[0,\infty)$. *Then*

$$|||f(|A+B|)||| \leqslant |||f(|A|) + f(|B|)|||.$$

LEMMA 2.4. *Let* $A,B,C,D \in \mathbb{M}_n(\mathbb{C})$. *Then*

$$\left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\| \leqslant \left\| \begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix} \right\|.$$

All functions in this study are continuous, the symbols A and B denote for positive semidefinite matrices. A considerable generalization of inequality (1.3) will now be presented .

THEOREM 2.5. *Let* $A,B,X,Y \in \mathbb{M}_n(\mathbb{C})$. *Then*

$$\|AX + YB\| \leqslant \frac{1}{4}\left(\|W_1\| + \|W_2\| + W_4\right), \tag{2.1}$$

where

$$W_1 = A + A^{1/2}|X^*|^2 A^{1/2},$$

$$W_2 = B + B^{1/2}|Y|^2 B^{1/2},$$

$$W_3 = A^{1/2}XB^{1/2} + A^{1/2}YB^{1/2}$$

and

$$W_4 = \sqrt{(\|W_1\| - \|W_2\|)^2 + 4\|W_3\|^2}.$$

Proof. Let $S = \begin{bmatrix} A^{1/2} & YB^{1/2} \\ 0 & 0 \end{bmatrix}$ and $T^* = \begin{bmatrix} A^{1/2}X & 0 \\ B^{1/2} & 0 \end{bmatrix}$. Then

$$\begin{aligned}
\|AX+YB\| &= \|ST^*\| \\
&\leqslant \frac{1}{2}\|S^*S+T^*T\| \\
&= \frac{1}{2}\left\| \begin{bmatrix} A & A^{1/2}YB^{1/2} \\ B^{1/2}Y^*A^{1/2} & B^{1/2}|Y|^2B^{1/2} \end{bmatrix} + \begin{bmatrix} A^{1/2}|X^*|^2A^{1/2} & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & B \end{bmatrix} \right\| \\
&= \frac{1}{2}\left\| \begin{bmatrix} W_1 & W_3 \\ W_3^* & W_2 \end{bmatrix} \right\| \\
&\leqslant \frac{1}{2}\left\| \begin{bmatrix} \|W_1\| & \|W_3\| \\ \|W_3^*\| & \|W_2\| \end{bmatrix} \right\| \\
&\quad \text{(by Lemma 2.4)} \\
&= \frac{1}{4}\left[\|W_1\|+\|W_2\|+\sqrt{(\|W_1\|-\|W_2\|)^2+4\|W_3\|^2}\right].
\end{aligned}$$

Thus we have proven our claim. □

REMARK 2.6. Inequality (1.3) will be obtained by letting $X = Y = I$ in inequality (2.1).

A generalization of inequality (1.9) will now be presented.

THEOREM 2.7. *Let $A,B,X,Y \in \mathbb{M}_n$. Then*

$$s_j(AX-YB) \leqslant s_j(K \oplus L) \tag{2.2}$$

for $j = 1,2,\ldots,2n$, where

$$K = K_1 + |K_2|,$$

$$K_1 = \frac{1}{2}A + \frac{1}{2}A^{1/2}|X^*|^2A^{1/2},$$

$$K_2 = \frac{1}{2}B^{1/2}Y^*A^{1/2} - \frac{1}{2}B^{1/2}X^*A^{1/2},$$

$$L = L_1 + |L_2|,$$

$$L_1 = \frac{1}{2}B + \frac{1}{2}B^{1/2}|Y|^2B^{1/2}$$

and

$$L_2 = \frac{1}{2}A^{1/2}YB^{1/2} - \frac{1}{2}A^{1/2}XB^{1/2}.$$

Proof. Let

$$S = \begin{bmatrix} A^{1/2} & YB^{1/2} \\ 0 & 0 \end{bmatrix},$$

$$R^* = \begin{bmatrix} A^{1/2}X & 0 \\ -B^{1/2} & 0 \end{bmatrix},$$

$$M = \begin{bmatrix} A & A^{1/2}YB^{1/2} \\ B^{1/2}Y^*A^{1/2} & B^{1/2}|Y|^2B^{1/2} \end{bmatrix}$$

and

$$N = \begin{bmatrix} A^{1/2}|X^*|^2A^{1/2} & -A^{1/2}XB^{1/2} \\ -B^{1/2}X^*A^{1/2} & B \end{bmatrix}.$$

Thus,

$$\begin{aligned}
s_j(AX - YB) &= s_j(SR^*) \\
&\leqslant \frac{1}{2}s_j(S^*S + R^*R) \text{ (by Lemma 2.1)} \\
&= s_j\left(\frac{1}{2}M + \frac{1}{2}N\right) \\
&= s_j\left(\begin{bmatrix} K_1 & L_2 \\ K_2 & L_1 \end{bmatrix}\right) \\
&= s_j\left(\begin{bmatrix} K_1 & 0 \\ 0 & L_1 \end{bmatrix} + \begin{bmatrix} 0 & L_2 \\ K_2 & 0 \end{bmatrix}\right) \\
&\leqslant s_j\left(\left|\begin{bmatrix} K_1 & 0 \\ 0 & L_1 \end{bmatrix}\right| + \left|\begin{bmatrix} 0 & L_2 \\ K_2 & 0 \end{bmatrix}\right|\right) \\
&= s_j\left(\begin{bmatrix} K_1 & 0 \\ 0 & L_1 \end{bmatrix} + \begin{bmatrix} |K_2| & 0 \\ 0 & |L_2| \end{bmatrix}\right) \\
&= s_j\left(\begin{bmatrix} K_1 + |K_2| & 0 \\ 0 & L_1 + |L_2| \end{bmatrix}\right) \\
&= s_j\left(\begin{bmatrix} K & 0 \\ 0 & L \end{bmatrix}\right) = s_j(K \oplus L).
\end{aligned}$$

Thus we have proven our claim. □

REMARK 2.8. Inequality (1.7) can be obtained by letting $X = Y = I$ in inequality (2.2).

REMARK 2.9. Inequality (1.9) can be given by letting $Y = X$ in inequality (2.2).

As an application of Theorem 2.7, we present the following result which is a generalization of inequalities (1.4) and (1.8).

COROLLARY 2.10. *Let $A,B,X,Y \in \mathbb{M}_n$. Then*

$$s_j(AX+YB) \leqslant s_j(C \oplus D) \tag{2.3}$$

for $j = 1,2,\dots,2n$, where

$$C = C_1 + |C_2|,$$

$$C_1 = \frac{1}{2}A + \frac{1}{2}A^{1/2}|X^*|^2 A^{1/2},$$

$$C_2 = \frac{1}{2}B^{1/2}X^*A^{1/2} + \frac{1}{2}B^{1/2}Y^*A^{1/2},$$

$$D = D_1 + |D_2|,$$

$$D_1 = \frac{1}{2}B + \frac{1}{2}B^{1/2}|Y|^2 B^{1/2}$$

and

$$D_2 = \frac{1}{2}A^{1/2}XB^{1/2} + \frac{1}{2}A^{1/2}YB^{1/2}.$$

Proof. Letting $Y = -Y$, $K_2 = -C_2$ and $L_2 = -D_2$ in Theorem 2.7, we give inequality (2.3). □

REMARK 2.11. Inequality (1.8) can be obtained by letting $X = Y$ in inequality (2.3).

REMARK 2.12. Inequality (1.4) can be given by letting $X = Y = I$ in inequality (2.3).

COROLLARY 2.13. *Let $A,B,X,Y \in \mathbb{M}_n$. Then*

$$\|AX+YB\| \leqslant \max\{\|C\|,\|D\|\}, \tag{2.4}$$

where C and D are given in Corollary 2.10.

Proof. Inequality (2.4) is a direct consequence of inequality (2.3) by applying the spectral norm. □

REMARK 2.14. Inequality (1.5) can be obtained by letting $X = Y = I$ in inequality (2.4).

COROLLARY 2.15. *Let $A,B,X,Y \in \mathbb{M}_n$. Then*

$$\|AX+YB\|_p \leqslant \left(\|C\|_p^p + \|D\|_p^p\right)^{1/p}, \tag{2.5}$$

where C and D are given in Corollary 2.10.

Proof. Apply the Schatten p-norms on inequality (2.3), we give inequality (2.5). □

REMARK 2.16. Inequality (1.6) can be obtained by letting $X = Y = I$ in inequality (2.5) .

A generalization of the generalized anticommutator will now be given.

THEOREM 2.17. *Let $A, B, X, Y \in \mathbb{M}_n$ and let f be a nonnegative increasing concave function on $[0, \infty)$. Then*

$$|||f(|(AX+YB)\oplus 0|)||| \leqslant |||Z \oplus W|||, \tag{2.6}$$

where

$$Z = f(K_1) + f(|K_2|),$$

$$K_1 = \frac{1}{2}A + \frac{1}{2}A^{1/2}|X^*|^2 A^{1/2},$$

$$K_2 = \frac{1}{2}B^{1/2}X^*A^{1/2} + \frac{1}{2}B^{1/2}Y^*A^{1/2},$$

$$W = f(L_1) + f(|L_2|),$$

$$L_1 = \frac{1}{2}B + \frac{1}{2}B^{1/2}|Y|^2 B^{1/2}$$

and

$$L_2 = \frac{1}{2}A^{1/2}XB^{1/2} + \frac{1}{2}A^{1/2}YB^{1/2}.$$

Proof. Let

$$S = \begin{bmatrix} A^{1/2} & YB^{1/2} \\ 0 & 0 \end{bmatrix},$$

$$T = \begin{bmatrix} X^*A^{1/2} & B^{1/2} \\ 0 & 0 \end{bmatrix},$$

$$E = \begin{bmatrix} A & A^{1/2}YB^{1/2} \\ B^{1/2}Y^*A^{1/2} & B^{1/2}|Y|^2 B^{1/2} \end{bmatrix}$$

and

$$F = \begin{bmatrix} A^{1/2}|X^*|^2 A^{1/2} & A^{1/2}XB^{1/2} \\ B^{1/2}X^*A^{1/2} & B \end{bmatrix}.$$

Then, we have

$$\begin{aligned}
s_j(f(|(AX+YB)\oplus 0|)) &= s_j(f(|ST^*|)) \\
&= f(s_j(ST^*)) \\
&\leqslant f\left(\frac{1}{2}s_j(S^*S+T^*T)\right) \text{ (by Lemma 2.1)} \\
&= f\left(\frac{1}{2}s_j(E+F)\right) \\
&= s_j\left(f\left(\left|\frac{1}{2}E+\frac{1}{2}F\right|\right)\right) \text{ (by Lemma 2.2)} \\
&= s_j\left(f\left(\left|\begin{bmatrix} K_1 & L_2 \\ K_2 & L_1 \end{bmatrix}\right|\right)\right).
\end{aligned}$$

This implies that,

$$\begin{aligned}
|||f(|(AX+YB)\oplus 0|)||| &\leqslant \left|\left|\left|f\left(\left|\begin{bmatrix} K_1 & L_2 \\ K_2 & L_1 \end{bmatrix}\right|\right)\right|\right|\right| \\
&= \left|\left|\left|f\left(\left|\begin{bmatrix} K_1 & 0 \\ 0 & L_1 \end{bmatrix}+\begin{bmatrix} 0 & L_2 \\ K_2 & 0 \end{bmatrix}\right|\right)\right|\right|\right| \\
&\leqslant \left|\left|\left|f\left(\left|\begin{bmatrix} K_1 & 0 \\ 0 & L_1 \end{bmatrix}\right|\right)+f\left(\left|\begin{bmatrix} 0 & L_2 \\ K_2 & 0 \end{bmatrix}\right|\right)\right|\right|\right|, \\
&\quad \text{(by Lemma 2.3)}, \\
&\leqslant \left|\left|\left|f\left(\begin{bmatrix} K_1 & 0 \\ 0 & L_1 \end{bmatrix}\right)+f\left(\begin{bmatrix} |K_2| & 0 \\ 0 & |L_2| \end{bmatrix}\right)\right|\right|\right| \\
&= \left|\left|\left|\begin{bmatrix} f(K_1) & 0 \\ 0 & f(L_1) \end{bmatrix}+\begin{bmatrix} f(|K_2|) & 0 \\ 0 & f(|L_2|) \end{bmatrix}\right|\right|\right| \\
&= \left|\left|\left|\begin{bmatrix} Z & 0 \\ 0 & W \end{bmatrix}\right|\right|\right| \\
&= |||Z\oplus W|||,
\end{aligned}$$

which is precisely inequality (2.6). □

COROLLARY 2.18. *Let $A,B,X,Y \in \mathbb{M}_n$. Then*

$$\|AX+YB\| \leqslant \max\{\|K_1+|K_2|\|,\|L_1+|L_2|\|\}, \tag{2.7}$$

where K_1,K_2,L_1 and L_2 are given in Theorem 2.17.

Proof. Inequality (2.7) is a direct consequence of Theorem 2.17 by considering $\|.\|$ and letting $f(t)=t$. □

REMARK 2.19. Inequality (1.5) can be obtained by letting $X=Y=I$ in Corollary 2.18.

COROLLARY 2.20. *Let $A,B,X,Y \in \mathbb{M}_n$. Then for $1 \leqslant p \leqslant \infty$,*

$$\|AX+YB\|_p \leqslant \left(\|K_1+|K_2|\|_p^p+\|L_1+|L_2|\|_p^p\right)^{1/p}, \tag{2.8}$$

where K_1, K_2, L_1 and L_2 are given in Theorem 2.17.

Proof. Inequality (2.8) is a direct consequence of Theorem 2.17 by considering $\|.\|_p$ and letting $f(t)=t$. □

REMARK 2.21. Inequality (1.6) can be obtained by letting $X=Y=I$ in Corollary 2.20.

Some applications of Theorem 2.17 will now be given.

COROLLARY 2.22. *Let $A,B,X,Y \in \mathbb{M}_n$. Then*

$$|||\log(|(AX+YB)|+I)||| \leqslant |||M \oplus N|||, \tag{2.9}$$

where

$$M = (\log(K_1+I)+\log(|K_2|+I))$$

and

$$N = (\log(L_1+I)+\log(|L_2|+I)).$$

K_1, K_2, L_1 and L_2 are given in Theorem 2.17.

Proof. Inequality (2.9) is a direct consequence of Theorem 2.17 by letting $f(t)=\log(t+1)$. □

COROLLARY 2.23. *Let $A,B,X,Y \in \mathbb{M}_n$. Then, for $r \in (0,1]$, we have*

$$||||(AX+YB)|^r||| \leqslant |||P \oplus Q|||, \tag{2.10}$$

where

$$P = (K_1^r+|K_2|^r) \text{ and } Q = (L_1^r+|L_2|^r).$$

K_1, K_2, L_1 and L_2 are given in Theorem 2.17.

Proof. Inequality (2.10) is a direct consequence of Theorem 2.17 by letting $f(t)=t^r$ and $r \in (0,1]$. □

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Wasim Audeh
Department of Mathematics
Petra University
Amman, Jordan
e-mail: waudeh@uop.edu.jo