SINGULAR VALUE AND NORM
INEQUALITIES OF DAVIDSON–POWER TYPE

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Abstract. Let \( A, B, X \) and \( Y \) be \( n \times n \) complex matrices such that \( A \) and \( B \) are positive semidefinite, then
\[
\|AX + YB\| \leq \frac{1}{4}(\|W_1\| + \|W_2\| + W_4),
\]
where
\[
W_1 = A + A^{1/2}X^*^2A^{1/2},
\]
\[
W_2 = B + B^{1/2}Y^2B^{1/2},
\]
\[
W_3 = A^{1/2}XB^{1/2} + A^{1/2}YB^{1/2}
\]
and
\[
W_4 = \sqrt{(\|W_1\| - \|W_2\|)^2 + 4\|W_3\|^2}.
\]

Multiple results are given in this paper.

1. Introduction

Let \( \mathbb{M}_n \) denote the algebra of all \( n \times n \) complex matrices. For \( A \in \mathbb{M}_n \), the singular values of \( A \) are the eigenvalues of \( |A| = (A^*A)^{1/2} \) which are denoted by \( s_1(A) \geq s_2(A) \geq \ldots \geq s_n(A) \), they satisfy \( s_j(A) = s_j(A^*) = s_j(|A|) \) for \( j = 1, 2, \ldots, n \). The spectral norm \( \|\cdot\| \) is defined as \( \|A\| = s_1(A) \) and the Schatten \( p \)-norms \( \|\cdot\|_p \) are defined as \( \|A\|_p = \left( \sum_{j=1}^n s_j^p(A) \right)^{1/p} \) for \( 1 \leq p \leq \infty \). The symbol \( |||\cdot||| \) will denote any unitarily invariant norm which are norms on \( \mathbb{M}_n \) satisfying \( |||UAV||| = |||A||| \) for all \( A \in \mathbb{M}_n \) and all unitary matrices \( U, V \in \mathbb{M}_n \), (see, e.g., [6] or [11]). It is pointed out in [9] that if \( A, B \in \mathbb{M}_n \) are positive semidefinite, then
\[
\|A + B\| \leq \max \{\|A\|, \|B\|\} + \|AB\|^{1/2}.
\]
Kittaneh [13] gave a refinement of inequality (1.1) so that
\[
\|A + B\| \leq \max \{\|A\|, \|B\|\} + \left\|A^{1/2}B^{1/2}\right\|^{1/2}.
\]

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Kittaneh [14] provided an improvement of inequality (1.2) as follows
\[ \|A + B\| \leq \frac{1}{2} \left( \|A\| + \|B\| + \sqrt{\left(\|A\| - \|B\|\right)^2 + 4 \|A^{1/2}B^{1/2}\|^2} \right). \] (1.3)

It is shown in [4] that if \( A, B \in \mathbb{M}_n \) are positive semidefinite, then
\[ s_j(A + B) \leq s_j \left( \left( A + \left| B^{1/2}A^{1/2} \right| \right) \oplus \left( B + \left| A^{1/2}B^{1/2} \right| \right) \right) \] (1.4)
for \( j = 1, 2, \ldots, 2n \). Norm inequalities versions of inequality (1.4) are listed below
\[ \|A + B\| \leq \max \left\{ \left\| A + \left| B^{1/2}A^{1/2} \right| \right\|, \left\| B + \left| A^{1/2}B^{1/2} \right| \right\| \right\} \] (1.5)
and
\[ \|A + B\|_p \leq \left( \left\| A + \left| B^{1/2}A^{1/2} \right| \right\|_p^p + \left\| B + \left| A^{1/2}B^{1/2} \right| \right\|_p^p \right)^{1/p} \] (1.6)
for \( 1 \leq p \leq \infty \). Zhan in [15] showed that if \( A, B \in \mathbb{M}_n \) where \( A, B \) are positive semidefinite, then
\[ s_j(A - B) \leq s_j(A \oplus B) \] (1.7)
for \( j = 1, 2, \ldots, 2n \). A generalization of inequality (1.4) is given in [2], so that
\[ s_j(AX + XB) \leq s_j(C \oplus D) \] (1.8)
for \( j = 1, 2, \ldots, 2n \), where
\[
C = C_1 + |C_2|, \\
C_1 = \frac{1}{2} A + \frac{1}{2} A^{1/2} |X^*|^2 A^{1/2}, \\
C_2 = B^{1/2}X^*A^{1/2}, \\
D = D_1 + |D_2|, \\
D_1 = \frac{1}{2} B + \frac{1}{2} B^{1/2} |X|^2 B^{1/2}, \\
\text{and} \\
D_2 = A^{1/2}XB^{1/2}.
\]

Audeh, in the same paper, showed that
\[ s_j(AX - XB) \leq s_j(M \oplus N) \] (1.9)
for \( j = 1, 2, \ldots, 2n \), where
\[
M = \frac{1}{2} A + \frac{1}{2} A^{1/2} |X^*|^2 A^{1/2} \\
\text{and} \\
N = \frac{1}{2} B + \frac{1}{2} B^{1/2} |X|^2 B^{1/2}.
\]

Readers interested in singular value inequalities should return to [1], [3], [4], [5] and [10]. We present a considerable generalizations of the inequalities (1.3), (1.4), (1.5), (1.6), (1.8) and (1.9).
2. Main results

We need the following lemmas. The first lemma is proved in [7], the second lemma is obtained in [6], the third lemma is provided in [8] and the fourth lemma is given in [12].

**Lemma 2.1.** Let $A, B \in \mathbb{M}_n$. Then

$$s_j(AB^*) \leq \frac{1}{2} s_j(A^*A + B^*B)$$

for $j = 1, 2, \ldots, n$.

**Lemma 2.2.** Let $A \in \mathbb{M}_n$ be positive semidefinite and let $f$ be a non-negative increasing function on $[0, \infty)$. Then

$$s_j(f(A)) = f(s_j(A))$$

for $j = 1, 2, \ldots, n$.

**Lemma 2.3.** Let $A, B \in \mathbb{M}_n$ be normal and let $f$ be a nonnegative concave function on $[0, \infty)$. Then

$$\|\|f(|A + B|)\|\| \leq \|\|f(|A|) + f(|B|)\|\|.$$

**Lemma 2.4.** Let $A, B, C, D \in \mathbb{M}_n(\mathbb{C})$. Then

$$\|\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]\| \leq \|\| A \|\| \| B \|\|.$$

All functions in this study are continuous, the symbols $A$ and $B$ denote for positive semidefinite matrices. A considerable generalization of inequality (1.3) will now be presented.

**Theorem 2.5.** Let $A, B, X, Y \in \mathbb{M}_n(\mathbb{C})$. Then

$$\|AX + YB\| \leq \frac{1}{4} (\|W_1\| + \|W_2\| + W_4),$$

where

$$W_1 = A + A^{1/2}|X^*|^2A^{1/2},$$

$$W_2 = B + B^{1/2}|Y|^2B^{1/2},$$

$$W_3 = A^{1/2}XB^{1/2} + A^{1/2}YB^{1/2}$$

and

$$W_4 = \sqrt{\|W_1\|^2 - \|W_2\|^2 + 4\|W_3\|^2}.$$
Proof. Let \( S = \begin{bmatrix} A^{1/2} & YB^{1/2} \\ 0 & 0 \end{bmatrix} \) and \( T^* = \begin{bmatrix} A^{1/2}X & 0 \\ B^{1/2} & 0 \end{bmatrix} \). Then

\[
\|AX + YB\| = \|ST^*\| \\
\leq \frac{1}{2} \|S^*S + T^*T\| \\
= \frac{1}{2} \left\| \begin{bmatrix} A & A^{1/2}YB^{1/2} \\ B^{1/2} & Y^2B^{1/2} \end{bmatrix} \right\| \\
= \frac{1}{2} \left\| \begin{bmatrix} W_1 & W_3 \\ W_3^* & W_2 \end{bmatrix} \right\| \\
\leq \frac{1}{2} \left\| \begin{bmatrix} \|W_1\| & \|W_3\| \\ \|W_3^*\| & \|W_2\| \end{bmatrix} \right\| \\
= \frac{1}{4} \left( \|W_1\| + \|W_2\| + \sqrt{\|W_1\| - \|W_2\|}^2 + 4\|W_3\|^2 \right).
\]

Thus we have proven our claim. \( \square \)

Remark 2.6. Inequality (1.3) will be obtained by letting \( X = Y = I \) in inequality (2.1).

A generalization of inequality (1.9) will now be presented.

Theorem 2.7. Let \( A, B, X, Y \in \mathbb{M}_n \). Then

\[
s_j(AX - YB) \leq s_j(K \oplus L)
\] (2.2)

for \( j = 1, 2, \ldots, 2n \), where

\[
K = K_1 + |K_2|, \\
K_1 = \frac{1}{2}A + \frac{1}{2}A^{1/2}|X^*|^2A^{1/2}, \\
K_2 = \frac{1}{2}B^{1/2}Y^*A^{1/2} - \frac{1}{2}B^{1/2}X^*A^{1/2}, \\
L = L_1 + |L_2|, \\
L_1 = \frac{1}{2}B + \frac{1}{2}B^{1/2}|Y|^2B^{1/2}
\]

and

\[
L_2 = \frac{1}{2}A^{1/2}YB^{1/2} - \frac{1}{2}A^{1/2}XB^{1/2}.
\]
Proof. Let
\[ S = \begin{bmatrix} A^{1/2} & YB^{1/2} \\ 0 & 0 \end{bmatrix}, \]
\[ R^* = \begin{bmatrix} A^{1/2}X & 0 \\ -B^{1/2} & 0 \end{bmatrix}, \]
\[ M = \begin{bmatrix} A & A^{1/2}YB^{1/2} \\ B^{1/2}Y^*A^{1/2} & B^{1/2}Y^*B^{1/2} \end{bmatrix} \]
and
\[ N = \begin{bmatrix} A^{1/2}|X^*|^2 & A^{1/2} \\ -B^{1/2}X^*A^{1/2} & B \end{bmatrix}. \]
Thus,
\[ sj(AX - YB) = sj(SR^*) \]
\[ \leq \frac{1}{2} sj(S^*S + R^*R) \text{ (by Lemma 2.1)} \]
\[ = sj\left( \frac{1}{2}M + \frac{1}{2}N \right) \]
\[ = sj\left( \begin{bmatrix} K_1 & L_2 \\ K_2 & L_1 \end{bmatrix} \right) \]
\[ = sj\left( \begin{bmatrix} K_1 & 0 \\ 0 & L_1 \end{bmatrix} + \begin{bmatrix} 0 & L_2 \\ K_2 & 0 \end{bmatrix} \right) \]
\[ \leq sj\left( \begin{bmatrix} K_1 & 0 \\ 0 & L_1 \end{bmatrix} + \begin{bmatrix} 0 & L_2 \\ K_2 & 0 \end{bmatrix} \right) \]
\[ = sj\left( \begin{bmatrix} K_1 & 0 \\ 0 & L_1 \end{bmatrix} + \begin{bmatrix} |K_2| & 0 \\ 0 & |L_2| \end{bmatrix} \right) \]
\[ = sj\left( \begin{bmatrix} K_1 + |K_2| & 0 \\ 0 & L_1 + |L_2| \end{bmatrix} \right) \]
\[ = sj\left( \begin{bmatrix} K & 0 \\ 0 & L \end{bmatrix} \right) = sj(K \oplus L). \]
Thus we have proven our claim. \( \square \)

Remark 2.8. Inequality (1.7) can be obtained by letting \( X = Y = I \) in inequality (2.2).

Remark 2.9. Inequality (1.9) can be given by letting \( Y = X \) in inequality (2.2).

As an application of Theorem 2.7, we present the following result which is a generalization of inequalities (1.4) and (1.8).
Corollary 2.10. Let $A, B, X, Y \in \mathbb{M}_n$. Then

$$s_j(AX + YB) \leq s_j(C \oplus D) \quad (2.3)$$

for $j = 1, 2, \ldots, 2n$, where

$$C = C_1 + |C_2|,$$

$$C_1 = \frac{1}{2}A + \frac{1}{2}A^{1/2}|X^*|A^{1/2},$$

$$C_2 = \frac{1}{2}B^{1/2}X^*A^{1/2} + \frac{1}{2}B^{1/2}Y^*A^{1/2},$$

$$D = D_1 + |D_2|,$$

$$D_1 = \frac{1}{2}B + \frac{1}{2}B^{1/2}|Y|B^{1/2},$$

and

$$D_2 = \frac{1}{2}A^{1/2}XB^{1/2} + \frac{1}{2}A^{1/2}YB^{1/2}.$$ 

Proof. Letting $Y = -Y$, $K_2 = -C_2$ and $L_2 = -D_2$ in Theorem 2.7, we give inequality (2.3). □

Remark 2.11. Inequality (1.8) can be obtained by letting $X = Y$ in inequality (2.3).

Remark 2.12. Inequality (1.4) can be given by letting $X = Y = I$ in inequality (2.3).

Corollary 2.13. Let $A, B, X, Y \in \mathbb{M}_n$. Then

$$\|AX + YB\| \leq \max \{\|C\|, \|D\|\}, \quad (2.4)$$

where $C$ and $D$ are given in Corollary 2.10.

Proof. Inequality (2.4) is a direct consequence of inequality (2.3) by applying the spectral norm. □

Remark 2.14. Inequality (1.5) can be obtained by letting $X = Y = I$ in inequality (2.4).

Corollary 2.15. Let $A, B, X, Y \in \mathbb{M}_n$. Then

$$\|AX + YB\|_p \leq \left(\|C\|_p^p + \|D\|_p^p\right)^{1/p}, \quad (2.5)$$

where $C$ and $D$ are given in Corollary 2.10.
Proof. Apply the Schatten $p$-norms on inequality (2.3), we give inequality (2.5).

\[ \square \]

Remark 2.16. Inequality (1.6) can be obtained by letting $X = Y = I$ in inequality (2.5).

A generalization of the generalized anticommutator will now be given.

Theorem 2.17. Let $A, B, X, Y \in \mathbb{M}_n$ and let $f$ be a nonnegative increasing concave function on $[0, \infty)$. Then

\[
\left\| \left\| f\left(\left\| (AX + YB) \oplus 0\right\|\right) \right\| \leq \left\| Z \oplus W \right\|, \tag{2.6}
\]

where

\[ Z = f(K_1) + f(|K_2|), \]

\[ K_1 = \frac{1}{2} A + \frac{1}{2} A^{1/2} |X^*|^2 A^{1/2}, \]

\[ K_2 = \frac{1}{2} B^{1/2} X^* A^{1/2} + \frac{1}{2} B^{1/2} Y^* A^{1/2}, \]

\[ W = f(L_1) + f(|L_2|), \]

\[ L_1 = \frac{1}{2} B + \frac{1}{2} B^{1/2} |Y|^2 B^{1/2} \]

and

\[ L_2 = \frac{1}{2} A^{1/2} X B^{1/2} + \frac{1}{2} A^{1/2} Y B^{1/2}. \]

Proof. Let

\[ S = \begin{bmatrix} A^{1/2} & Y B^{1/2} \\ 0 & 0 \end{bmatrix}, \]

\[ T = \begin{bmatrix} X^* A^{1/2} & B^{1/2} \\ 0 & 0 \end{bmatrix}, \]

\[ E = \begin{bmatrix} A & A^{1/2} Y B^{1/2} \\ B^{1/2} Y^* A^{1/2} & B^{1/2} |Y|^2 B^{1/2} \end{bmatrix} \]

and

\[ F = \begin{bmatrix} A^{1/2} |X^*|^2 A^{1/2} & A^{1/2} X B^{1/2} \\ B^{1/2} X^* A^{1/2} & B \end{bmatrix}. \]
Then, we have

\[ s_j(f(||(AX + YB) \oplus 0||)) = s_j(f(||ST^*||)) \]

\[ = f(s_j(ST^*)) \]

\[ \leq f \left( \frac{1}{2} s_j(S^*S + T^*T) \right) \quad \text{(by Lemma 2.1)} \]

\[ = f \left( \frac{1}{2} s_j(E + F) \right) \]

\[ = s_j \left( f \left( \frac{1}{2} E + \frac{1}{2} F \right) \right) \quad \text{(by Lemma 2.2)} \]

\[ = s_j \left( f \left( \left[ \begin{array}{cc} K_1 & L_2 \\ K_2 & L_1 \end{array} \right] \right) \right). \]

This implies that,

\[ ||f(||(AX + YB) \oplus 0||)|| \leq \left| f \left( \left[ \begin{array}{cc} K_1 & L_2 \\ K_2 & L_1 \end{array} \right] \right) \right| \]

\[ = \left| f \left( \left[ \begin{array}{cc} K_1 & 0 \\ 0 & L_1 \end{array} \right] + \left[ \begin{array}{cc} 0 & L_2 \\ K_2 & 0 \end{array} \right] \right) \right| \]

\[ \leq \left| f \left( \left[ \begin{array}{cc} K_1 & 0 \\ 0 & L_1 \end{array} \right] \right) \right| + \left| f \left( \left[ \begin{array}{cc} 0 & L_2 \\ K_2 & 0 \end{array} \right] \right) \right|, \]

(by Lemma 2.3),

\[ \leq \left| f \left( \left[ \begin{array}{cc} K_1 & 0 \\ 0 & L_1 \end{array} \right] \right) \right| + f \left( \left[ \begin{array}{cc} |K_2| & 0 \\ 0 & |L_2| \end{array} \right] \right| \]

\[ = \left| \left[ \begin{array}{cc} f(K_1) & 0 \\ 0 & f(L_1) \end{array} \right] \right| + \left| \left[ \begin{array}{cc} f(|K_2|) & 0 \\ 0 & f(|L_2|) \end{array} \right] \right| \]

\[ = \left| \left[ \begin{array}{cc} Z & 0 \\ 0 & W \end{array} \right] \right| \]

\[ = ||Z \oplus W||, \]

which is precisely inequality (2.6). \( \Box \)

**Corollary 2.18.** Let \( A, B, X, Y \in \mathbb{M}_n \). Then

\[ ||AX + YB|| \leq \max \{ ||K_1 + |K_2||, ||L_1 + |L_2|| \}, \quad (2.7) \]

where \( K_1, K_2, L_1 \) and \( L_2 \) are given in Theorem 2.17.

**Proof.** Inequality (2.7) is a direct consequence of Theorem 2.17 by considering \( ||.|| \) and letting \( f(t) = t \). \( \Box \)

**Remark 2.19.** Inequality (1.5) can be obtained by letting \( X = Y = I \) in Corollary 2.18.


**Corollary 2.20.** Let \( A, B, X, Y \in \mathbb{M}_n \). Then for \( 1 \leq p \leq \infty \),

\[
\|AX + YB\|_p \leq \left( \|K_1 + |K_2|\|_p^p + \|L_1 + |L_2|\|_p^p \right)^{1/p},
\]

(2.8)

where \( K_1, K_2, L_1 \) and \( L_2 \) are given in Theorem 2.17.

*Proof.* Inequality (2.8) is a direct consequence of Theorem 2.17 by considering \( \| \cdot \|_p \) and letting \( f(t) = t \). \( \square \)

**Remark 2.21.** Inequality (1.6) can be obtained by letting \( X = Y = I \) in Corollary 2.20.

Some applications of Theorem 2.17 will now be given.

**Corollary 2.22.** Let \( A, B, X, Y \in \mathbb{M}_n \). Then

\[
\||\log((AX + YB) + I)|| \leq ||M \oplus N||,
\]

(2.9)

where

\[
M = (\log(K_1 + I) + \log(|K_2| + I))
\]

and

\[
N = (\log(L_1 + I) + \log(|L_2| + I)).
\]

\( K_1, K_2, L_1 \) and \( L_2 \) are given in Theorem 2.17.

*Proof.* Inequality (2.9) is a direct consequence of Theorem 2.17 by letting \( f(t) = \log(t + 1) \). \( \square \)

**Corollary 2.23.** Let \( A, B, X, Y \in \mathbb{M}_n \). Then, for \( r \in (0, 1] \), we have

\[
\||((AX + YB)^r)|| \leq ||P \oplus Q||,
\]

(2.10)

where

\[
P = (K_1^r + |K_2|^r) \text{ and } Q = (L_1^r + |L_2|^r).
\]

\( K_1, K_2, L_1 \) and \( L_2 \) are given in Theorem 2.17.

*Proof.* Inequality (2.10) is a direct consequence of Theorem 2.17 by letting \( f(t) = t^r \) and \( r \in (0, 1] \). \( \square \)

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