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PROOF OF A DOUBLE INEQUALITY IN TRIANGLES

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Abstract. A double geometric inequality involving the side lengths, medians, angle bisectors and exradius of a triangle is proved by applying the "$R-r-s$" method in the theory of triangle inequalities. Several corollaries are obtained by using the main result and the other known inequalities.

1. Introduction

Let P be a point inside triangle ABC, let $PA=R_1$, $PB=R_2$, $PC=R_3$ and let r_1, r_2, r_3 denote the distances from P to the sides BC, CA, AB, respectively. Then the following beautiful linear inequality holds:

$$R_1+R_2+R_3 \geqslant 2(r_1+r_2+r_3). \tag{1}$$

This is the famous Erdoös-Mordell inequality. Some recent results on this subject can be found in [6, 12, 13, 14, 15, 16, 20]. In fact, the inequality (1) can be extended to

$$R_a+R_b+R_c \geqslant R_1+R_2+R_3 \geqslant 2(r_1+r_2+r_3), \tag{2}$$

where R_a, R_b, R_c are the circumradius of the triangles BPC, CPA, APB, respectively. It is interesting that the first inequality in (2) is actully equivalent to the Erdoös-Mordell inequality (see the fourth chapter of the author's monograph [17]).

In [9], the author and Chu established the following inequality for the sum of R_1, R_2, and R_3:

$$R_1+R_2+R_3 \geqslant \frac{1}{2}(m_a+m_b+m_c+3r), \tag{3}$$

where m_a, m_b, m_c are the medians of the triangle ABC and r is the inradius. We also further showed, via inequality (3), that

$$R_1+R_2+R_3 \geqslant \frac{1}{3}(m_a+m_b+m_c+h_a+h_b+h_c), \tag{4}$$

where h_a, h_b, h_c are the altitudes of the triangle ABC.

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Inspired and motivated by the first inequality in (2) and inequality (4), the author conjectures that the following inequality holds:

$$R_a + R_b + R_c \geqslant \frac{1}{3}(m_a + m_b + m_c + w_a + w_b + w_c), \tag{5}$$

where w_a, w_b, w_c are the lengths of angle bisectors of the triangle ABC. On the other hand, noticing that the following known result (see [17, p. 252] and [22]):

$$R_a + R_b + R_c \geqslant \sqrt{bc + ca + ab}, \tag{6}$$

where a, b, c are the lengths of the side of triangle ABC, the author also further conjectures that the following inequality holds:

$$m_a + m_b + m_c + w_a + w_b + w_c \leqslant 3\sqrt{bc + ca + ab}. \tag{7}$$

In addition, for any triangle ABC we have known that

$$m_a + m_b + m_c \leqslant r_a + r_b + r_c, \tag{8}$$

where r_a, r_b, r_c are the radii of excircles of the triangle ABC (see [1, inequality 8.20]). With the help of the computer, we find that if replacing $m_a + m_b + m_c$ by $r_a + r_b + r_c$ in (7), then we have the reverse inequality, i.e.,

$$r_a + r_b + r_c + w_a + w_b + w_c \geqslant 3\sqrt{bc + ca + ab}. \tag{9}$$

Of course, the above inequality can only be regarded as a conjecture before it is proved.

The aim of this article is to prove inequality (7) and inequality (9), i.e., the following double inequality which was given in [17, p. 253] as a conjecture.

THEOREM. *For any triangle ABC, the following inequality holds:*

$$m_a + m_b + m_c + w_a + w_b + w_c \leqslant 3\sqrt{bc + ca + ab} \leqslant w_a + w_b + w_c + r_a + r_b + r_c. \tag{10}$$

Both equalities in (10) hold if and only if triangle ABC is equilateral.

Clearly, the double inequality (10) can be written as

$$m_a + m_b + m_c \leqslant 3\sqrt{bc + ca + ab} - (w_a + w_b + w_c) \leqslant r_a + r_b + r_c, \tag{11}$$

which gives a refinement of the known inequality (8).

2. Lemmas

We shall apply the "$R - r - s$" method to prove the double inequality (10). This method has been proved to be effective for a number of symmetric triangle inequalities (cf. [2], [4], [5], [7], [10], [11], [23], [24]).

In what follows, we shall continue to use the previous symbols. Also, denote the semi-perimeter, the circumradius and the inradius of the triangle ABC by s, R, r, respectively. For the sake of simplicity, we shall occasionally use $\sum$ and $\prod$ to express cycle sums and products respectively.

LEMMA 1. *For any triangle ABC, the following inequality holds:*

$$\frac{(m_a+m_b+m_c)^2}{a^2+b^2+c^2} \leqslant 2+\frac{r^2}{R^2}. \tag{12}$$

Equality holds if and only if triangle ABC is equilateral.

The author has proved inequality (12) in [10]. Later, the author also gave two simpler proofs in [11].

Next, we apply inequality (12) to prove the following inequality, which was first given in [4] without proof.

LEMMA 2. *For any triangle ABC, the following inequality holds:*

$$m_a+m_b+m_c \leqslant \frac{2s^2}{\sqrt{bc+ca+ab}}. \tag{13}$$

Equality holds if and only if the triangle ABC is equilateral.

Proof. By inequality (12), to prove inequality (13) we need to prove

$$(a^2+b^2+c^2)\left(2+\frac{r^2}{R^2}\right) \leqslant \frac{4s^4}{bc+ca+ab},$$

i.e.,

$$4s^4R^2-(2R^2+r^2)(bc+ca+ab)(a^2+b^2+c^2) \geqslant 0.$$

Then, using the following known identities:

$$bc+ca+ab = s^2+4Rr+r^2, \tag{14}$$

$$a^2+b^2+c^2 = 2(s^2-4Rr-r^2), \tag{15}$$

the proof becomes

$$-2s^4r^2+2r^2(2R^2+r^2)(4R+r)^2 \geqslant 0.$$

Since we have the following known inequality (see [1, inequality 5.5]):

$$(4R+r)^2 \geqslant 3s^2, \tag{16}$$

thus we only need to prove that

$$-s^2+3(2R^2+r^2) \geqslant 0,$$

which can be written as

$$4R^2+4Rr+3r^2-s^2+2R(R-2r) \geqslant 0.$$

By Euler's inequality $R \geqslant 2r$ and Gerretsen's inequality (see [1, inequality 5.8])

$$g_2 \equiv 4R^2+4Rr+3r^2-s^2 \geqslant 0 \tag{17}$$

with equality iff ABC is equilateral, one sees that the claimed inequality is valid. Thus, inequality (13) is proved. It is easy to know that the equality in (13) holds if and only if the triangle is equilateral. This completes the proof of Lemma 2. □

LEMMA 3. *For any triangle ABC, the following inequality holds:*

$$\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\leqslant\frac{1}{2R}+\frac{3}{4r}. \tag{18}$$

Equality holds if and only if the triangle ABC is equilateral.

Inequality (18) was first proposed by the author in a Chinese paper [8]. A simple proof was given by Chu and the author in [4].

LEMMA 4. *For any triangle ABC, the following inequality holds:*

$$\frac{(w_a+w_b+w_c)^2}{bc+ca+ab}\leqslant\frac{s^4+2r(10R+9r)s^2+(4R+r)r^3}{(s^2+2Rr+r^2)^2}. \tag{19}$$

Equality holds if and only if the triangle ABC is equilateral.

Proof. By inequality (18), we have

$$(w_a+w_b+w_c)^2\leqslant w_a^2+w_b^2+w_c^2+w_aw_bw_c\left(\frac{1}{R}+\frac{3}{2r}\right).$$

Then using the following known identities:

$$w_aw_bw_c=\frac{16Rr^2s^2}{s^2+2Rr+r^2}, \tag{20}$$

$$w_a^2+w_b^2+w_c^2=\frac{s^6+3r^2s^4+(32R^2+40Rr+3r^2)r^2s^2+(4R+r)^2r^4}{s^4+2r(2R+r)s^2+r^2(2R+r)^2}, \tag{21}$$

we easily obtain

$$(w_a+w_b+w_c)^2\leqslant\frac{(s^2+4Rr+r^2)\left[s^4+2r(10R+9r)s^2+(4R+r)r^3\right]}{(s^2+2Rr+r^2)^2}.$$

Hence, inequality (19) follows immediately by using the previous identity (14). It is easy to know that the equality conditions of (19) is as mentioned in Lemma 4. □

LEMMA 5. *In the triangle ABC, if $B-C,\ C-A,\ A-B\neq 0$, then*

$$\sum\frac{1}{\cos(B-C)}=\frac{2R\left[(R+6r)s^2-(2R+r)(2R^2+7Rr+2r^2)\right]}{s^4-(6R^2+8Rr-2r^2)s^2+(2R^2+4Rr+r^2)(2R+r)^2}. \tag{22}$$

Proof. We first deduce the following identity (for any triangle ABC):

$$\prod\cos(B-C)=\frac{s^4-(6R^2+8Rr-2r^2)s^2+(2R^2+4Rr+r^2)(2R+r)^2}{8R^4}. \tag{23}$$

Since $\cos(B-C) = \cos B\cos C + \sin B\sin C$, it is easy to obtain

$$\prod\cos(B-C) = \prod\sin^2 A + \prod\cos^2 A + \prod\cos A\sum\sin B\sin C\cos A + \prod\sin A\sum\sin A\cos B\cos C. \tag{24}$$

Then using the following identities:

$$\sum\sin B\sin C\cos A = \frac{1}{2}\sum\sin^2 A, \tag{25}$$

$$\sum\sin A\cos B\cos C = \prod\sin A, \tag{26}$$

we get

$$\prod\cos(B-C) = 2\prod\sin^2 A + \prod\cos^2 A + \frac{1}{2}\prod\cos A\sum\sin^2 A. \tag{27}$$

Further, using the following known identities (cf. [19, p. 55]):

$$\prod\sin A = \frac{rs}{2R^2}, \tag{28}$$

$$\prod\cos A = \frac{s^2-(2R+r)^2}{4R^2}, \tag{29}$$

$$\sum\sin^2 A = \frac{1}{2R^2}(s^2-4Rr-r^2), \tag{30}$$

we obtain identity (23).

Now, we prove the following identity:

$$\sum\cos(C-A)\cos(A-B) = \frac{(R+6r)s^2-(2R+r)(2R^2+7Rr+2r^2)}{4R^3}. \tag{31}$$

Firstly, it is easy to get

$$\sum\cos(C-A)\cos(A-B) = \prod\sin A\sum\sin A + \prod\cos A\sum\cos A + \sum\sin B\sin C\cos A(\cos B+\cos C). \tag{32}$$

Using identity (25), we have

$$\begin{aligned}&\sum\sin B\sin C\cos A(\cos B+\cos C)\\ &= \sum\sin B\sin C\cos A\sum\cos A - \sum\sin B\sin C\cos^2 A\\ &= \frac{1}{2}\sum\sin^2 A\sum\cos A - \sum\sin B\sin C + \prod\sin A\sum\sin A,\end{aligned}$$

and then

$$\begin{aligned}&\sum\cos(C-A)\cos(A-B)\\ &= 2\prod\sin A\sum\sin A + \prod\cos A\sum\cos A - \sum\sin B\sin C\\ &\quad + \frac{1}{2}\sum\cos A\sum\sin^2 A.\end{aligned} \tag{33}$$

Thus, by using identities (28), (30) and the following identities (cf. [19, p. 55]):

$$\sum \sin A = \frac{s}{R}, \tag{34}$$

$$\sum \cos A = 1 + \frac{r}{R}, \tag{35}$$

$$\sum \sin B \sin C = \frac{s^2 + 4Rr + r^2}{4R^2}, \tag{36}$$

we obtain identity (31).

When $B-C$, $C-A$, $A-B \neq 0$, identity (22) follows from (23) and (31) immediately. Lemma 5 is proved.

□

3. Proof of the Theorem

3.1. Proof of the first inequality of the double inequality (10)

In this section, we prove the first inequality in (10), i.e., inequality (7).

Proof. According to Lemma 2, to prove inequality (7) we only need to prove

$$\frac{2s^2}{\sqrt{bc+ca+ab}} + w_a + w_b + w_c \leqslant 3\sqrt{bc+ca+ab},$$

that is

$$(w_a + w_b + w_c)\sqrt{bc+ca+ab} \leqslant 3(bc+ca+ab) - 2s^2.$$

Since $bc+ca+ab > s^2$ holds for any triangle ABC, we thus only consider to prove

$$(bc+ca+ab)(w_a+w_b+w_c)^2 \leqslant (3bc+3ca+3ab-2s^2)^2.$$

By Lemma 4, it is sufficient to prove

$$\begin{aligned} Q_0 \equiv & (s^2+2Rr+r^2)^2\left(3bc+3ca+3ab-2s^2\right)^2 \\ & -(bc+ca+ab)^2\left[s^4+2r(10R+9r)s^2+(4R+r)r^3\right] \geqslant 0. \end{aligned} \tag{37}$$

Substituting the previous identity (14) into Q_0, we get

$$Q_0 = 4r^2 Q_1, \tag{38}$$

where

$$\begin{aligned} Q_1 = & -3s^6 + (17R^2 - 12Rr - 4r^2)s^4 + r(4R+r)(22R^2+8Rr+r^2)s^2 \\ & + (9R^2+8Rr+2r^2)(4R+r)^2 r^2. \end{aligned}$$

Hence, we have to prove $Q_1 \geqslant 0$. Through analysis, we find the following identity (which is easily checked by expanding):

$$Q_1 = Q_2 + Q_3 + rQ_4, \tag{39}$$

where

$$\begin{aligned}
Q_2 &= (5R^2+2r^2)s^2(s^2-16Rr+5r^2),\\
Q_3 &= 3(s^2+24Rr)\left[-s^4+(4R^2+20Rr-2r^2)s^2-r(4R+r)^3\right],\\
Q_4 &= (72R^3-1267R^2r+224Rr^2-6r^3)s^2+(297R^2+80Rr+2r^2)(4R+r)^2r.
\end{aligned}$$

For any triangle ABC, we have Gerretsen's inequality (see [1, inequality 5.8])

$$g_1 \equiv s^2-16Rr+5r^2 \geqslant 0 \tag{40}$$

(with equality iff the triangle ABC is equilateral) and the fundamental Sondat's inequality (see [1, inequality 13.8] and [19, p. 2]):

$$t_0 \equiv -s^4+(4R^2+20Rr-2r^2)s^2-r(4R+r)^3 \geqslant 0, \tag{41}$$

with equality if and only if the triangle ABC is isosceles. Hence, we have $Q_2 \geqslant 0$ and $Q_3 \geqslant 0$. It remains to prove that $Q_4 \geqslant 0$ by (39). Obviously, if

$$72R^3-1267R^2r+224Rr^2-6r^3 \geqslant 0,$$

then $Q_4 > 0$. If the above inequality holds reversely, by the previous Gerretsen's inequality (17) we need to prove that

$$\begin{aligned}
&(72R^3-1267R^2r+224Rr^2-6r^3)(4R^2+4Rr+3r^2)\\
&+(297R^2+80Rr+2r^2)(4R+r)^2r \geqslant 0,
\end{aligned}$$

i.e.,

$$4(R-2r)(72R^4+137R^3r+199R^2r^2-92Rr^3+2r^4) \geqslant 0,$$

which is true by Euler's inequality $R \geqslant 2r$. Hence, we conclude that $Q_4 \geqslant 0$ holds for all triangles ABC. This completes the proof of the inequality (7). Also, it is easily known that the equality in (7) holds iff the triangle ABC is equilateral. □

3.2. Proof of the second inequality of the double inequality (10)

Next, we prove the second inequality in (10), i.e., inequality (9).

Proof. Firstly, we transform inequality (9) into an equivalent trigonometric inequality in the acute (non-obtuse) triangle ABC.

Let I be the incenter of the triangle ABC. Suppose that the line AI intersect BC at X, then $\angle AXC = B+\frac{1}{2}A = \frac{\pi+B-C}{2}$. Using $w_a = AI+IX$, we have

$$w_a = \frac{r}{\sin\frac{A}{2}} + \frac{r}{\cos\frac{B-C}{2}}, \tag{42}$$

so that

$$\sum w_a = r\sum\frac{1}{\sin\frac{A}{2}} + r\sum\frac{1}{\cos\frac{B-C}{2}}. \tag{43}$$

Therefore, by the following two known identities:

$$r_a + r_b + r_c = 4R + r, \tag{44}$$

$$r = 4R \prod \sin \frac{A}{2}, \tag{45}$$

we see that inequality (9) is equivalent to

$$4R \prod \sin \frac{A}{2} \left(\sum \frac{1}{\sin \frac{A}{2}} + \sum \frac{1}{\cos \frac{B-C}{2}} \right) + 4R \left(1 + \prod \sin \frac{A}{2} \right) \geqslant 3 \sqrt{\sum bc}.$$

Since $a = 2R \sin A$ etc., the above inequality is equivalent to the following trigonometric inequality:

$$2 \prod \sin \frac{A}{2} \left(\sum \frac{1}{\sin \frac{A}{2}} + \sum \frac{1}{\cos \frac{B-C}{2}} \right) + 2 \left(1 + \prod \sin \frac{A}{2} \right) \geqslant 3 \sqrt{\sum \sin B \sin C}. \tag{46}$$

Also, it is easily known that inequality (46) is equivalent to the following inequality in the acute triangle ABC:

$$2 \prod \cos A \left[\sum \frac{1}{\cos A} + \sum \frac{1}{\cos(B-C)} \right] + 2 \left(1 + \prod \cos A \right) \geqslant 3 \sqrt{\sum \sin 2B \sin 2C}. \tag{47}$$

Secondly, we prove that inequality (47) is valid for the acute triangle ABC.

We denote by L_0 the left hand of (47). Using identity (22) of Lemma 5, identity (29) and the following known identity (see [19, p. 56]):

$$\sum \frac{1}{\cos A} = \frac{4R^2 - r^2 - s^2}{4R^2 + 4Rr + r^2 - s^2}, \tag{48}$$

we easily obtain

$$L_0 = \frac{N_0}{M_0}, \tag{49}$$

where

$$\begin{aligned} M_0 =& R^2 \left[s^4 - (6R^2 + 8Rr - 2r^2)s^2 + (2R^2 + 4Rr + r^2)(2R + r)^2 \right], \\ N_0 =& s^6 - (7R^2 + 4Rr - 2r^2)s^4 + (12R^4 + 8R^3 r - 2R^2 r^2 - 4Rr^3 + r^4)s^2 \\ &+ (4R + r)(2R + r)^2 R^2 r. \end{aligned}$$

Also, it is easy to get

$$\sum \sin 2B \sin 2C = \frac{K_0}{4R^4}, \tag{50}$$

where

$$K_0 = s^4 - (4R^2 + 8Rr - 2r^2)s^2 + r(4R + r)(2R + r)^2.$$

Now, in view of the identities (49) and (50), to prove inequality (47) we need to prove

$$4R^4N_0^2-9K_0M_0^2 \geqslant 0. \tag{51}$$

With the help Maple, one easily obtains

$$4R^4N_0^2-9K_0M_0^2 = R^4X_0, \tag{52}$$

where

$$\begin{aligned}X_0 =& -5s^{12}+(88R^2+184Rr-38r^2)s^{10}-(608R^4+2592R^3r+1792R^2r^2\\ &-552Rr^3+111r^4)s^8+(2064R^6+13696R^5r+22672R^4r^2+9696R^3r^3\\ &-1392R^2r^4+336Rr^5-164r^6)s^6-(3456R^8+32960R^7r+90416R^6r^2\\ &+103296R^5r^3+47872R^4r^4+5536R^3r^5-640R^2r^6+464Rr^7+131r^8)s^4\\ &+(576R^8+7872R^7r+26992R^6r^2+39744R^5r^3+26640R^4r^4+7200R^3r^5\\ &-208R^2r^6-432Rr^7-54r^8)(2R+r)^2s^2-r(4R+r)(144R^6+704R^5r\\ &+1328R^4r^2+1152R^3r^3+504R^2r^4+108Rr^5+9r^6)(2R+r)^4.\end{aligned}$$

Hence, we have to prove that the inequality $X_0 \geqslant 0$ holds for the acute triangle ABC.

In the acute (non-obtuse) triangle ABC, we have the following two inequalities:

$$s_0 \equiv s^2-(2R+r)^2 \geqslant 0, \tag{53}$$

$$s_1 \equiv s^2-(2R^2+8Rr+3r^2) \geqslant 0. \tag{54}$$

Inequality (53) is equivalent to Ciamberlini's inequality ([3]):

$$s \geqslant 2R+r, \tag{55}$$

which follows from the previous identity (29). Inequality (54) was first presented by Walker in [21] (the author obtained a general generalization of this inequality in the recent paper [18]). Also, we have known that the equality in (53) holds iff the triangle ABC is a right triangle and the equality in (54) holds iff the triangle ABC is an isosceles right triangle or an equilateral triangle.

Next, we apply $s_0 \geqslant 0, s_1 \geqslant 0$, Gerretsen's inequality $g_2 \geqslant 0$ (i.e.,inequality (17)), and Sondat's inequality $t_0 \geqslant 0$ (i.e., inequality (41)) to prove that inequality $X_0 \geqslant 0$ holds for the acute triangle ABC.

Through analysis, we obtain the following identity:

$$X_0 = s_0\left[g_2t_0(5g_2s^2+s_0m_1+m_2)+s_0s_1m_3\right]+Y_0, \tag{56}$$

where

$$\begin{aligned}m_1 =&8R^2+24Rr-63r^2,\\ m_2 =&2r(144R^3-14R^2r-236Rr^2-145r^3),\\ m_3 =&16R^6-256R^5r+3280R^4r^2+4256R^3r^3-5944R^2r^4-2336Rr^5-800r^6,\end{aligned}$$

$$\begin{aligned}Y_0 =&(32R^8+1088R^7r-6848R^6r^2+15296R^5r^3+27504R^4r^4-21184R^3r^5\\&-18640R^2r^6-8896Rr^7-4928r^8)s^4-(256R^{10}+8960R^9r-47040R^8r^2\\&+63616R^7r^3+326912R^6r^4+84736R^5r^5-250304R^4r^6-228928R^3r^7\\&-110912R^2r^8-37888Rr^9-6144r^{10})s^2+16(8R^{10}+280R^9r-1502R^8r^2\\&+1796R^7r^3+10224R^6r^4+2952R^5r^5-7711R^4r^6-7028R^3r^7\\&-3063R^2r^8-836Rr^9-108r^{10})(2R+r)^2.\end{aligned}$$

By Euler's inequality $R \geqslant 2r$, one sees that $m_1 > 0$. If we set

$$e = R - 2r,$$

then $e \geqslant 0$ and it is easy to get

$$\begin{aligned}m_2 =&2r(144e^3+850e^2r+1436er^2+479r^3) > 0,\\ m_3 =&16e^4(e^2-4er+105r^2)+22816e^3r^3+81672e^2r^4+112512er^5+50112r^6 > 0.\end{aligned}$$

Thus, according to identity (56) and inequalities $g_2 \geqslant 0,\ t_0 \geqslant 0,\ s_0 \geqslant 0,\ s_1 \geqslant 0$, to prove inequality $X_0 \geqslant 0$ it remains to prove that $Y_0 \geqslant 0$ holds for the acute triangle ABC.

Putting

$$Y_0 = f(s^2),$$

then

$$\begin{aligned}f'(s^2) =&2(32R^8+1088R^7r-6848R^6r^2+15296R^5r^3+27504R^4r^4-21184R^3r^5\\&-18640R^2r^6-8896Rr^7-4928r^8)s^2-(256R^{10}+8960R^9r-47040R^8r^2\\&+63616R^7r^3+326912R^6r^4+84736R^5r^5-250304R^4r^6-228928R^3r^7\\&-110912R^2r^8-37888Rr^9-6144r^{10}).\end{aligned}$$

Since

$$\begin{aligned}&32R^8+1088R^7r-6848R^6r^2+15296R^5r^3+27504R^4r^4-21184R^3r^5\\&-18640R^2r^6-8896Rr^7-4928r^8\\&=32e^8+1600e^7r+11968e^6r^2+38848e^5r^3+110064e^4r^4\\&\quad+381632e^3r^5+882992e^2r^6+971520er^7+371968r^8 > 0,\end{aligned}$$

thus by inequality (55) we have for the acute triangle ABC that

$$\begin{aligned}f'(s^2) \geqslant& 2(32R^8+1088R^7r-6848R^6r^2+15296R^5r^3+27504R^4r^4-21184R^3r^5\\&-18640R^2r^6-8896Rr^7-4928r^8)(2R+r)^2+(-256R^{10}-8960R^9r\\&+47040R^8r^2-63616R^7r^3-326912R^6r^4-84736R^5r^5+250304R^4r^6\\&+228928R^3r^7+110912R^2r^8+37888Rr^9+6144r^{10})\\=&32e^8+704e^7r+6328e^6r^2+31024e^5r^3+91425e^4r^4+164970e^3r^5\\&+173409e^2r^6+89424er^7+11752r^8 > 0.\end{aligned}$$

Hence, $f(s^2)$ is increasing. According to this conclusion, we next divide two cases to prove that $Y_0 \geqslant 0$ holds for the non-obtuse triangle ABC.

Case 1. R and r satisfy the inequality $R^2-2Rr-r^2>0$.

In this case, to prove $Y_0>0$ we need to prove $f((2R+r)^2)>0$ (since $f(s^2)$ is increasing). It is easy to check that

$$f((2R+r)^2)=256(R^2-2Rr-r^2)(R^2+4Rr+2r^2)(2R+r)^4r^4. \tag{57}$$

Thus, by the hypothesis we have $f((2R+r)^2)>0$ and inequality $Y_0>0$ is proved.

Case 2. R and r satisfy the inequality $R^2-2Rr-r^2 \leqslant 0$.

Since $f(s^2)$ is increasing, by Walker's inequality (54), to prove $Y_0 \geqslant 0$ in the above case we only need to prove that $f(2R^2+8Rr+3r^2) \geqslant 0$. With the help of the Maple software, it is easy to obtain

$$f(2R^2+8Rr+3r^2)=-64(R-2r)(R^2-2Rr-r^2)Z_0, \tag{58}$$

where

$$\begin{aligned}Z_0=&-2R^9-68R^8r+462R^7r^2-628R^6r^3-1499R^5r^4+2224R^4r^5+3533R^3r^6\\&+2242R^2r^7+1036Rr^8+216r^9.\end{aligned}$$

Thus, by Euler's inequality $R \geqslant 2r$, we need to prove $Z_0>0$ under the hypothesis. In deed, we can rewrite Z_0 as follows:

$$\begin{aligned}Z_0=&-(R^2-2Rr-r^2)(2e^7+8122r^6e+3372r^5e^2+1863r^4e^3+1788r^3e^4+716r^2e^5\\&+100re^6+1560r^7)+6(1801R+746r)r^8,\end{aligned} \tag{59}$$

where $e=R-2r \geqslant 0$. Hence, by the hypothesis $R^2-2Rr-r^2 \leqslant 0$ we have $Z_0>0$ and $Y_0 \geqslant 0$ is proved. Also, in the second case, the equality in $Y_0 \geqslant 0$ holds if and only if $(R-2r)(r^2+2Rr-R^2)=0$, and we further know that the equality holds if and only if the triangle ABC is an equilateral triangle or an isosceles right triangle.

Combining the discussions of the above two cases, we complete the proof of $Y_0 \geqslant$ for the acute triangle ABC and conclude that the equality in $Y_0 \geqslant 0$ holds if and only if the triangle ABC is an equilateral triangle or an isosceles right triangle. Thus, we complete the proofs of inequality $X_0 \geqslant 0$ and inequality (47).

Finally, making the substitutions $A \to (\pi-A)/2$ etc., in (47), then inequality (46) follows immediately. Therefore, inequality (9) is proved. According to identity (56) and the equality conditions of (17), (41), (53) and (54), it is easy to conclude that the equality condition of $X_0 \geqslant 0$ is the same as that of $Y_0 \geqslant 0$. Furthermore, we easily know that both equalities of (46) and (9) hold if and only if the triangle ABC is equilateral. This completes the proof of the theorem. □

4. Corollaries and Conjectures

In this section, we give several corollaries of the theorem and present a few related interesting conjectures as open problems.

Since $3(bc+ca+ab) \leqslant (a+b+c)^2$, by inequality (7) we obtain the following linear inequality:

COROLLARY 1. *For any triangle ABC, the following inequality holds:*

$$m_a+m_b+m_c+w_a+w_b+w_c \leqslant 2\sqrt{3}s. \tag{60}$$

REMARK 1. Inspired by the above inequality, the author has proved the following inequality:

$$3w_a+m_a+m_b+m_c \leqslant \frac{3\sqrt{3}}{2}(b+c). \tag{61}$$

Clearly, we also have the other two similar relations. Adding up these three inequalities, we obtain inequality (60).

By the previous inequality (7) and the following equivalent form of the Gerretsen inequality (17) (see [1, inequality 5.17]), i.e.,

$$bc+ca+ab \leqslant 4(R+r)^2, \tag{62}$$

we obtain the following linear inequality.

COROLLARY 2. *For any triangle ABC, the following inequality holds:*

$$m_a+m_b+m_c+w_a+w_b+w_c \leqslant 6(R+r). \tag{63}$$

The above inequality is obviously stronger than the following known result (see [1, inequality 6.14]):

$$w_a+w_b+w_c \leqslant 3(R+r). \tag{64}$$

In the triangle ABC, we have known that

$$m_a w_a \geqslant s(s-a). \tag{65}$$

Thus, by inequality (7) and the simplest arithmetic-geometric mean inequality, we can obtain the following inequality involving the side lengths of the triangle ABC.

COROLLARY 3. *For any triangle ABC, the following inequality holds:*

$$\sqrt{b+c-a}+\sqrt{c+a-b}+\sqrt{a+b-c} \leqslant 3\sqrt{\frac{bc+ca+ab}{a+b+c}}. \tag{66}$$

In the acute triangle ABC, we have the following inequality (see [17, p. 252]):

$$bc+ca+ab \geqslant \frac{9}{4}(R+2r)^2. \tag{67}$$

Thus, by the previous inequality (9) and identity (44) we can obtain the following inequality:

COROLLARY 4. *For the acute triangle ABC, the following inequality holds:*

$$w_a+w_b+w_c \geqslant \frac{1}{2}R+8r. \tag{68}$$

In any triangle ABC, we have the known inequality $w_a \leqslant \sqrt{r_b r_c}$. Thus, using the Cauchy-Schwarz inequality we have

$$(w_a + r_a)^2 \leqslant (r_c + r_a)(r_a + r_b). \tag{69}$$

Consequently, by the previous inequality (9) one obtains the following corollary:

COROLLARY 5. *For any triangle ABC, the following inequality holds:*

$$\sqrt{(r_c + r_a)(r_a + r_b)} + \sqrt{(r_a + r_b)(r_b + r_c)} + \sqrt{(r_b + r_c)(r_c + r_a)} \geqslant 3\sqrt{bc + ca + ab}. \tag{70}$$

Next, we introduce a conjecture related to $\sqrt{bc + ca + ab}$:

CONJECTURE 1. *For any triangle ABC, the following inequality holds:*

$$m_a + w_b + w_c \leqslant \frac{3}{2}\sqrt{bc + ca + ab}. \tag{71}$$

REMARK 2. If the above inequality holds, then by the previous inequality (6) we can deduce the following geometric inequality:

$$R_a + R_b + R_c \geqslant \frac{2}{3}(m_a + w_b + w_c). \tag{72}$$

Also, if (71) is true then we can deduce the following three inequalities:

$$m_a + w_b + w_c \leqslant \sqrt{3}s, \tag{73}$$

$$m_a + w_b + w_c \leqslant 3(R + r), \tag{74}$$

$$2m_a + w_b + w_c \leqslant w_a + r_a + r_b + r_c. \tag{75}$$

However, the above four inequalities have not been proved at the present.

CONJECTURE 2. *For any triangle ABC, the following inequality holds:*

$$m_a + m_b + m_c \leqslant \frac{w_a^2}{m_b + m_c} + \frac{w_b^2}{m_c + m_a} + \frac{w_c^2}{m_a + m_b} \leqslant r_a + r_b + r_c. \tag{76}$$

Finally, for the Erdös-Mordell inequality we present a new sharpened version:

CONJECTURE 3. *Let $k \geqslant 9/4$ be a real number, then for any point P inside triangle ABC it holds:*

$$\frac{R_1 + R_2 + R_3}{r_1 + r_2 + r_3} \geqslant 2 \cdot \frac{ka^2 + m_a w_a}{ka^2 + r_b r_c}. \tag{77}$$

The equivalent form $m_a w_a \geqslant r_b r_c$ of inequality (65) shows that the value of the right hand of (77) is at least 2.

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